Uniform Location-Scale Model: An Equivariant Estimation Approach

T. Leo Alexander
Associate Professor, Department of Statistics, LoyolaCollege, Chennai

Abstract

A detailed discussion on equivariant estimation of the parameters of location, scale and location-scale models are given by LEHMANN AND CASELLA (1998). EDWIN PRABAKARAN and CHANDRASEKAR (1994) developed simultaneous equivariant estimation approach and illustrated the method with examples. In this paper, uniform Location-Scale model is considered and Q_A– MRE(Quadratic type MRE) estimator(s) of the parameters based on type II censored samples are obtained.

Key words: Censored sampling, Equivariant estimation,Location-scale model, Q_A– MREand Uniform model.

1. Introduction

Equivariance is a desirable property used for restricting the class of estimators whenever the model possesses symmetry. ZACKS (1971) and LEHMANN AND CASELLA (1998) provide a detailed study of the problem of equivariant estimation for certain models. In the case of location-scale model, LEHMANN AND CASELLA (1998) develops marginal Equivariant procedure for estimating the parameters. EDWIN PRABAKARAN and CHANDRASEKAR (1994) have proposed a simultaneous Equivariant estimation for estimating the parameters of a location-scale model. For a detailed discussion on simultaneous equivariant estimation and related results the reader is referred to EDWIN PRABAKARAN (1995). Contributions to simultaneous Equivariant estimation based on censored samples studied in Leo Alexander(2000).

In this paper, we consider uniform location-scale model and obtain $Q_A - MRE$ estimators for the vector parameters $(\xi,\tau)'$ *and* $(\xi,\tau^2)'$ based on the type II right censored samples. Further MRE estimator of $(\xi,\tau)'$ is obtained with respect to the Linex type loss function.

Let $\mathbf{X} = (X_1, X_2,..., X_N)'$ have joint pdf

$$g(x;\xi,\tau) = (1/\tau^N)\, f\left(\frac{x_1-\xi}{\tau},\ldots,\frac{x_N-\xi}{\tau}\right), \qquad \ldots(1.1)$$

where f is known and $\theta=(\xi,\tau)'$ is unknown, $\xi\in R$, $\tau>0$. We wish to derive the MRE estimator of $(\xi,\tau^m)'$ based on type II right censored sample when the loss function is invariant and has the form

$$L(\theta,d)=a_{11}\{\{d_1-\xi)/\tau\}^2+2a_{12}\{(d_1-\xi)/\tau\}\{(d_2/\tau^m)-1\}+a_{22}\{(d_2/\tau^m)-1\}^2. \quad ...(1.2)$$

We also derive the MRE estimator of $(\xi,\tau)'$ with respect to the invariance Linex loss function of the form

$$L(\xi,\tau;\delta)=e^{a(\delta_1-\xi)/\tau}-a(\delta_1-\xi)/\tau-1+e^{b(\delta_2/\tau-1)}-b(\delta_2/\tau-1). \quad ...(1.3)$$

1.1 Preliminaries

Suppose N randomly selected units were placed on a test simultaneously, the failure times of the first n units to fail were observed. Thus the number of completely determined life spans is n and the number of censored ones is (N-n). let$X_{i:N}$, i=1,2,…,n denote the failure times of the completely observed items. Then the joint probability density function (pdf) of $(X_{1:N}, X_{2:N},..., X_{n:N})$ (BAIN, 1978) is

$$g_\theta(x_1,x_2,...,x_n)=\frac{N!}{(N-n)!}\prod_{i=1}^{n} f_\theta(x_i)\ 1-F_\theta(x_i)\ ^{N-n} \quad ...(1.4)$$

Here f_θ and F_θ denote the common pdf and the distribution function of the failure times of the units selected randomly, which are put to test. Further n is assumed to be known in advance.

2 uniform location – scale model

In this case, the common pdf is taken to be

$$f_\theta(x)=\begin{cases}1/\tau\ , & \xi\le x\le\xi+\tau\ ;\ \xi\in R,\ \tau>0\\ 0\ , & otherwise\end{cases}$$

Note that $\theta=(\xi,\tau)'$.Thus (1.4) reduces to

$$g_\theta(x_{1:N},\ldots,x_{n:N})=\frac{N!}{(N-n)!}\frac{1}{\tau^n}\left\{1-\frac{(x_{n:N}-\xi)}{\tau}\right\}^{N-n}, \quad \ldots(2.1)$$

$$\xi\le x_{1:N}\le x_{n:N}\le\xi+\tau;\xi\in R,\tau>0.$$

Note that the above pdf belongs to a location – scale model.

Case (i): We are interested in obtaining $Q_A - MRE$ estimator for vector parameters $(\xi,\tau)'$ *and* $(\xi,\tau^2)'$ based on the Type-II right censosed sample. Following Edwin Prabakaran and Chandrasekar (1994), we obtain the MRE estimator of $(\xi,\tau)'$ *and* $(\xi,\tau^2)'$. Let us discuss the problem of estimating $(\xi,\tau)'$ *and* $(\xi,\tau^2)'$. In order to obtain MRE estimator of $(\xi,\tau)'$, take

$$\delta_{01}(\mathbf{X}) = X_{1:N} \text{ and } \delta_{02}(\mathbf{X}) = X_{n:N} - X_{1:N} .$$

Here $\delta_0(X)$ is an equivariant estimator and $(X_{1:N}, X_{n:N})'$ is a sufficient statistic. Since we are interested in the evaluation of conditional distribution under $(\xi,\tau)' = (0,1)'$, we take $(\xi,\tau)' = (0,1)'$ in (2.1), in order to find $(w_1^*, w_2^*)'$,

where

$$\begin{aligned} w_1^* = [&a_{11}a_{22}E(\delta_{02}{}^2 \,|z)E(\delta_{01}g\,|z) - a_{12}{}^2E(\delta_{01}g\,|z)E(\delta_{01}\delta_{02}\,|z) \\ &-a_{12}a_{22}\{E(\delta_{02}{}^2 \,|z)E(g\,|z) - E(\delta_{02}g\,|z)E(\delta_{02}\,|z)\}]/ \\ &\{a_{11}a_{22}E(g^2\,|\,z)E(\delta_{02}{}^2\,|\,z) - a_{12}{}^2E^2(\delta_{02}g\,|\,z\} \end{aligned} \qquad \ldots(2.2)$$

and

$$\begin{aligned} 1/w_2^* = [&-a_{11}a_{12}\{E(g^2\,|z)E(\delta_{01}\delta_{02}\,|z) - E(\delta_{02}g\,|\,z)E(\delta_{02}g\,|\,z)\} \\ &+a_{11}a_{22}\{E(g^2\,|z)E(\delta_{02}\,|z) - a_{12}{}^2E(\delta_{02}g\,|z)E(g\,|z)\}]/ \\ &\{a_{11}a_{22}E(g^2\,|\,z)E(\delta_{02}{}^2\,|\,z) - a_{12}{}^2E^2(\delta_{02}g\,|\,z\} \quad . \end{aligned} \qquad \ldots(2.3)$$

Consider the transformation

$$Z_1 = X_{1:N}\ ,\ Z_2 = X_{n:N} - X_{1:N} \ \ and\ \ Z_i = \frac{X_{i-1:N} - X_{1:N}}{X_{n:N} - X_{1:N}}, \quad i = 3,4,\ldots, n.$$

Then

$$X_{1:N} = Z_1,\ \ X_{n:N} = Z_1 + Z_2\ \ , X_{i-1:N} = Z_1 + Z_2 Z_i\,, \qquad i = 3,4,\ldots, n$$

and the Jacobian of the transformation is $J = Z_2{}^{n-2}$.

Thus the joint pdf of $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ is given by

$$h(z_1,\ldots,z_n) = \frac{N!}{(N-n)!} z_2{}^{n-2}(1 - z_1 - z_2)^{N-n},$$

$$0 < z_3 < z_4 < \ldots < z_n < 1, 0 < z_1 + z_2 < 1.$$

Then the marginal pdf of $(Z_3, \ldots, Z_n)$ is given by

$$h_1(z_3,\ldots,z_n)=\frac{N!}{(N-n)!}\int_0^1\int_0^{1-z_2} z_2^{\,n-2}(1-z_1-z_2)^{N-n}\,dz_1\,dz_2$$
$$=(n-2)!,\qquad 0<z_3<z_4<\ldots<z_n<1.$$

So, the conditional pdf of (Z_1,Z_2) given $(Z_3,\ldots,Z_n)$ is given by

$$h_2((z_1,z_2)\,|\,z_3,\ldots,z_n)=\frac{N!}{(N-n)!(n-2)!}z_2^{\,n-2}(1-z_1-z_2)^{N-n},\qquad \ldots(2.4)$$
$$,0<z_1+z_2<1.$$

and the conditional pdf of Z_2 given $(Z_3,\ldots,Z_n)$ is given by

$$h_3(z_2\,|\,z_3,\ldots,z_n)=\frac{N!}{(N-n+1)!(n-2)!}z_2^{\,n-2}(1-z_2)^{N-n+1}\qquad ,0<z_2<1.$$

Therefore

$$E(\delta_{01}\delta_{02}\,|\,\mathbf{z})=E(Z_1Z_2\,|\,\mathbf{z})$$
$$=\frac{N!}{(N-n)!(n-2)!}\int_0^1\int_0^{1-z_2} z_1z_2^{\,n-1}(1-z_1-z_2)^{N-n}\,dz_1\,dz_2.$$

Put $z_1=u(1-z_2)$, so that

$$E(\delta_{01}\delta_{02}\,|\,\mathbf{z})=\frac{(n-1)}{(N+1)(N+2)}\qquad \ldots(2.5)$$

and

$$E(\delta_{02}^{\;2}\,|\,\mathbf{z})=E(z_2^{\;2}\,|\,\mathbf{z})$$
$$=\frac{N!}{(N-n+1)!(n-2)!}\int_0^1 z_2^{\,n+1-1}(1-z_2)^{N-n+1}\,dz_2=\frac{n(n-1)}{(N+1)(N+2)}.\qquad \ldots(2.6)$$

Similarly

$$E(\delta_{02}\,|\,\mathbf{z})=E(Z_2\,|\,\mathbf{z})=\frac{n-1}{N+1}\qquad \ldots(2.7)$$

Thus, in view of (2.5),(2.6)and(2.7), we have

$$w_1^{\,*}=1/n\quad and\quad 1/w_2^{\,*}=(N+2)/n\quad .$$

Therefore the MRE estimator $\delta^*=(\delta_1^{\,*},\delta_2^{\,*})'$ *of* $(\xi,\tau)'$ is given by

$$\delta_1^*(\mathbf{X}) = \delta_{01} - \delta_{02}/n \text{ and } \delta_2^*(\mathbf{X}) = (N+2)/n \ \ \delta_{02}$$

Remark 2.1 If n=N then the estimator of $(\xi, \tau)'$ reduce to

$$\delta_1^*(\mathbf{X}) = \delta_{01} - \delta_{02}/N \text{ and } \delta_2^*(\mathbf{X}) = (N+2)/N \ ,$$

which is same as the complete sample case (Edwin Prabakaran, 1994).

Now let us consider the problem of estimating $(\xi, \tau^2)'$. define

$$\delta_0(\mathbf{X}) = (\delta_{01}(\mathbf{X}), \delta_{02}(\mathbf{X}))',$$

where $\delta_{01}(\mathbf{X}) = X_{1:N}$ *and* $\delta_{02}(\mathbf{X}) = (X_{n:N} - X_{1:N})^2$.

Taking $g(X) = (X_{n:N} - X_{1:N})$, from equations (2.2) and (2.3), we obtain

$$w_1^* = [a_{11}a_{22}(n+2)(N+3) - a_{12}^2(n-1)(N+4) - a_{12}a_{22}\{(n+2)(N+2)(N+3) - n(N+4)\}]/\alpha$$

and

$$w_2^* = (N+3)(N+4)[a_{11}a_{22}\, n(N+3)/(n+1) - a_{12}^2(N+2)^2 + a_{11}a_{12}/(n+1)]/\alpha,$$

where $\alpha = n[a_{11}a_{22}(n+2)(N+3) - a_{12}^2(n+1)(N+4)]$.

Therefore the MRE estimator $\delta^* = (\delta_1^*, \delta_2^*)'$ *of* $(\xi, \tau^2)'$ is given by

$$\delta_1^* = \delta_{01}(\mathbf{X}) - g(X)\, w_1^* \qquad \text{and} \qquad \delta_2^* = \delta_{02}(\mathbf{X})/w_2^* \ ,$$

where w_1^* and w_2^* are as given above. It may be verified that the sufficient conditions of Edwin Prabakaran and Chandrasekar (1994) are satisfied for $(\delta_1^*, \delta_2^*)'$. Since the calculation is routine, we omit the details.

Case (ii): Consider the location – scale invariant Linex loss function (Varian, 1975)

$$L(\xi, \tau; \delta) = e^{a(\delta_1 - \xi)/\tau} - a(\delta_1 - \xi)/\tau - 1 + e^{b(\delta_2/\tau - 1)} - b(\delta_2/\tau - 1).$$

In order to find $(w_1^*, w_2^*)'$, take $\delta_{01}(\mathbf{X}) = X_{1:N}$ and $\delta_{02}(\mathbf{X}) = X_{n:N} - X_{1:N}$., consider

$$R(\delta\,|\,\mathbf{z}) = e[\{e^{-a\delta_1} - a\delta_1 - 1 + e^{b(\delta_2 - 1)} - b(\delta_2 - 1) - 1\}\,|\,\mathbf{z}].$$

Then

$$R(\delta \mid \mathbf{z}) = E[(e^{a\delta_{01}-aw_1\delta_{02}}) \mid \mathbf{z}] - aE(\delta_{01} \mid \mathbf{z}) + aw_1 E(\delta_{02} \mid \mathbf{z}) - 1 + e^{-b}E(e^{b/w_2\,\delta_{02}} \mid \mathbf{z}) + b - b/w_2 E(\delta_{02} \mid \mathbf{z}) - 1$$

$$= \frac{N!}{(N-n)!(n-2)!}\int_0^1\int_0^{1-z_2} e^{a(z_1-w_1z_2)} z_2^{\,n-2}(1-z_1-z_2)^{N-n}\, dz_1\, dz_2$$

$$-a \quad \frac{N!}{(N-n)!(n-2)!}\int_0^{1-z_2} z_1 z_2^{\,n-2}(1-z_1-z_2)^{N-n}\, dz_2 + aw_1\frac{(n-1)}{(N+1)} - 1$$

$$+e^{-b}\frac{N!}{(N-n)!(n-2)!}\int_0^1 e^{b/w_2z_2)} z_2^{\,n-2} z_2^{\,N-n}\, dz_2 + b - \frac{b}{w_2}\frac{(n-1)}{(N+1)} - 1 \quad ,$$

in view of (2.4) and (2.7).

Thus $(w_1^{\,*}, w_2^{\,*})'$ is to be obtained as the value of $(w_1, w_2)'$ minimizing $R(\delta \mid \mathbf{z})$.

Therefore the MRE estimator of $(\xi, \tau)'$ is given by

$$\delta_1^{\,*}(\mathbf{X}) = X_{1:N} - (X_{n:N} - X_{1:N})\, w_1^{\,*} \qquad \text{and} \qquad \delta_2^{\,*}(\mathbf{X}) = (X_{n:N} - X_{1:N}) / w_2^{\,*}\ .$$

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