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Existence and Stability of Non-Collinear Librations Points in the Restricted Problem with Poynting Robertson Light Drag Effect

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ABSTRACT

This paper deals with the existence and stability of libration points in restricted problem under the effect of dissipative force *i.e.* Poynting Robertson Light drag. We have determined the equations of motion of infinitesimal mass and investigated the stability of non-collinear libration points in the linear sense and found that there exist only two non-collinear libration points which are unstable for all mass parameter μ .

Keywords: Restricted three body problem, Dissipative Force, Poynting Robertson light drag, libration points, Linear stability.

I. Introduction

In 1994, C. D. Murray has obtained the components of force under the effect of Poynting Robertson light drag in case of point masses. i.e.

$$(F_x, F_y) = -\frac{k}{r_1^2}\begin{pmatrix} \dot{x} - y + \frac{x}{r_1^2}(x\dot{x} + y\dot{y}), \\ \dot{y} + x + \frac{y}{r_1^2}(x\dot{x} + y\dot{y}) \end{pmatrix}$$

where the first component is referred as drag component and second component represents the Doppler shift of the solar radiation.

Using the methodology of C. D. Murray, we have studied the effect of Poynting Robertson light drag on non-collinear libration points in the circular restricted three body problem. In the connection of dissipative forces there are lots of papers published in recent years. The effect of the Poynting Robertson drag in the restricted three body problem has been studied by Schuerman (1980). He discussed the position as well as the stability of the Lagrangian equilibrium points. Ishwar B. and Kushvah B.S. (2006) have examined the linear stability of triangular equilibrium points in the generalized photo gravitational restricted three body problem with Poynting Robertson drag. After considering the smaller primary as an oblate body and bigger one as radiating they have concluded that the triangular equilibrium points are unstable. By considering smaller primary as an oblate body and bigger one as radiating, Kushvah B.S., Sharma J.P. and Ishwar B. (2007) have discussed the non-linear stability in the generalized restricted three body problem with Poynting Robertson drag. They have proved that the triangular points are stable in non-linear sense. Furthermore, Jagdish Singh and Joel John Taura (2014) have extended the work to understand various issues related to the dynamics of a particle around radiating primaries. Singh Jagadish and Emmanuel A.B. (2014) have discussed the stability of triangular equilibrium points in photo gravitational circular restricted three body problem with Poynting Robertson drag and a smaller triaxial primary. They have proved that the parameters involved in the problem (radiation pressure, oblateness and Poynting Robertson drag) influence the position and linear stability of triangular points. In the presence of Poynting Robertson drag triangular points are unstable and in the absence of Poynting Robertson drag these points are conditionally stable.

The classical three body problem has five Lagrangian points. Their location and stability properties are well known. The three collinear points are unstable for every value of the mass parameter and non-collinear points are stable for $\mu < 0.03852$ as in [8].

In the present paper, we want to study the existence and stability of the non-collinear libration points in the restricted three body problem with drag force.

II. Equations Of Motion

Let there be three masses m_1, m_2, m_3; $(m_1 \geq m_2)$ such that the bodies with masses m_1 and m_2 revolve with the same angular velocity *n* (say) in circular orbits without rotation about their centre of mass *O*. m_3 is an infinitesimal mass which is moving in the plane of motion of m_1 and m_2 and is being influenced by their motion but not influencing them. The line joining m_1 and m_2 is taken as *X*- axis and '*O*' their center of mass as origin and the line passing through *O* and perpendicular to *OX* and lying in the plane of motion of m_1 and m_2 is the *Y*-axis. We consider a synodic system of coordinates *O (xyz)*; initially coincident with the inertial system *O (XYZ)*, rotating with the angular velocity *n* about *Z*-axis; (the *z*-axis is coincident with *Z*-axis), (Fig.1).

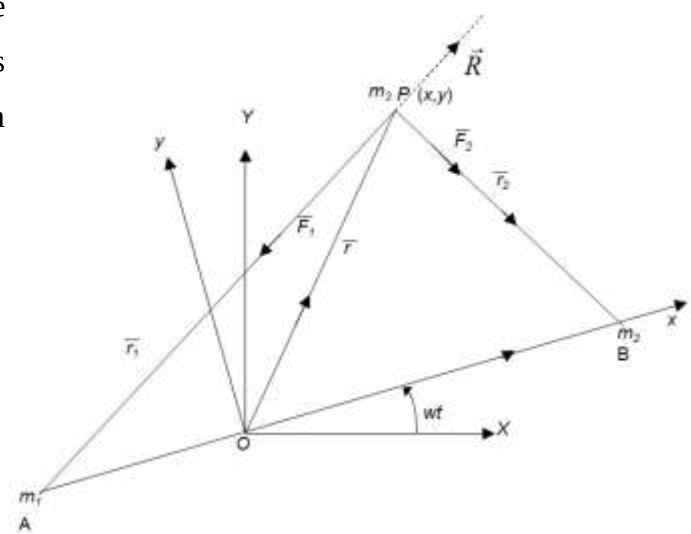

Fig.1 Configuration of the restricted three body problem with Poynting Robertson Drag $\vec{R}$.

In the synodic axes the equation of motion of m_3 of the restricted three body problem with Poynting Robertson Drag $\vec{R}$ is

$$m_3\left(\frac{\partial^2\vec{r}}{\partial t^2}+2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t}+\frac{\partial\vec{\omega}}{\partial t}\times\vec{r}+\vec{\omega}\times(\vec{r}\times\vec{\omega})\right)=\vec{F} \quad (1)$$

where

$$\vec{F}=\vec{F}_1+\vec{F}_2+\vec{R},$$

$\vec{F}_1$ = Gravitational Force acting on m_3 due to m_1

$$=G\frac{m_3m_1}{\bar{r}_1^2}\hat{r}_1,$$

$\vec{F}_2$ = Gravitational Force acting on m_3 due to m_2

$$=G\frac{m_3m_2}{\bar{r}_2^2}\hat{r}_2,$$

$\vec{R}$ = Poynting Robertson Drag Force acting on m_3 due to m_1 along $\overline{AP}$.

Its components along the synodic axes (x, y) are

$$R_x=\frac{k}{r_1^2}\left(\dot{x}-y+\frac{x}{r_1^2}(x\dot{x}+y\dot{y})\right) \text{ and}$$

$$R_y=\frac{k}{r_1^2}\left(\dot{y}+x+\frac{y}{r_1^2}(x\dot{x}+y\dot{y})\right).$$

where

$$\vec{r}=\overline{OP}=xi+yj,$$

$\bar{\omega}=n\mathrm{K}$ =Angular velocity of the axes

$$O(x\,y)=\text{constant},$$

$k\in(0,1)$ is the dissipative constant.

The equations of motion of m_3 in Cartesian coordinates (x, y) are

$$\ddot{x}-2n\dot{y}-n^2x=-Gm_1\frac{(x-x_1)}{r_1^3}-Gm_2\frac{(x-x_2)}{r_2^3}-G\frac{k}{r_1^2}\left(\dot{x}-y+\frac{x}{r_1^2}(x\dot{x}+y\dot{y})\right)$$

$$\ddot{y}+2n\dot{x}-n^2y=-Gm_1\frac{y}{r_1^3}-Gm_2\frac{y}{r_2^3}-G\frac{k}{r_1^2}\left(\dot{y}+x+\frac{y}{r_1^2}(x\dot{x}+y\dot{y})\right)$$

where

n = Mean motion, G =Gravitational constant,

$(x_1,0)$ & $(x_2,0)$ = coordinates of A and B in the synodic system.

We shall adopt the notation and terminology of Szebehely (1967). As a consequence the distance between the primaries does not change and is taken equal to one; the sum of the masses of the primaries is also taken as one. The unit of time is chosen so as to make the gravitational constant unity. . The equations of motions of the infinitesimal mass m_3 in the synodic coordinate system (x, y) and using dimensionless variables are given by

$$\ddot{x}-2\dot{y}=\Omega_x-\frac{k}{r_1^2}\left(\dot{x}-y+\frac{x}{r_1^2}(x\dot{x}+y\dot{y})\right), \quad (2)$$

$$\ddot{y}+2\dot{x}=\Omega_y-\frac{k}{r_1^2}\left(\dot{y}+x+\frac{y}{r_1^2}(x\dot{x}+y\dot{y})\right) \quad (3)$$

where

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2}$$

$$r_1^2 = (x+\mu)^2 + y^2, \qquad (4)$$

$$r_2^2 = (x+\mu-1)^2 + y^2, \qquad (5)$$

$$\mu = \frac{m_2}{m_1 + m_2} \le \frac{1}{2} \Rightarrow m_1 = 1-\mu \; ; \; m_2 = \mu,$$

The Robertson drag effect is of the order of $k = 10^{-5}$. (generally $k \in (0,1)$ as stated above)

III. Stationary Solutions (Libration Points)

The solutions (x, y) of equations (2) and (3) with $\ddot{x} = 0, \ddot{y} = 0, \dot{x} = 0, \dot{y} = 0$ are given by

$$x - (1-\mu)\frac{(x+\mu)}{r_1^3} - \mu\frac{(x+\mu-1)}{r_2^3} + \frac{k}{(r_1)^2}y = 0, \qquad (6)$$

and

$$y(1 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}) - \frac{k}{(r_1)^2}x = 0 \qquad (7)$$

Here, if we take $k = 0,$ then it will be the classical case of the restricted three body problem and the solutions of these equations are just the five classical Lagrangian equilibrium points L_i (i =1, 2, 3, 4, 5). The L_i (i =1, 2, 3) are three collinear libration points which lie along the x-axis and L_i (i =4, 5) are the two non-collinear libration points which make the equilateral triangles with the primaries. Due to the presence of the Poynting Robertson light drag force; it is clear from equations (6) and (7) that collinear equilibrium solution does not exist. Since there is a possibility of non collinear libration points under the effect of drag forces, now we restrict our analysis to these points. Their locations are

$$L_{4,5}\left[x_0 = \frac{1}{2} - \mu, \quad y_0 = \pm\frac{\sqrt{3}}{2}\right] \text{ as in [1]}$$

Now, we suppose that the solution of the equations (6) and (7) when $k \neq 0$ and $y \neq 0$ are given by

$$\bar{x} = x_0 + \pi_1, \qquad \bar{y} = y_0 + \pi_2 \;, \qquad \pi_1, \pi_2 << 1$$

Making the above substitutions in the equations (6) and (7), and applying Taylors series expansion around the libration points by using that (x_0, y_0) is a solution of these equations when $k = 0$, we can get a linear set of equations.

$$\pi_1\left[1+(1-\mu)\frac{3(x_0+\mu)^2}{\left\{(x_0+\mu)^2+{y_0}^2\right\}^{\frac{5}{2}}}-\frac{1}{\left\{(x_0+\mu)^2+{y_0}^2\right\}^{\frac{3}{2}}}+\mu\frac{3(x_0+\mu-1)^2}{\left\{(x_0+\mu-1)^2+{y_0}^2\right\}^{\frac{5}{2}}}-\frac{1}{\left\{(x_0+\mu-1)^2+{y_0}^2\right\}^{\frac{3}{2}}}\right]$$
$$+\pi_2\left[(1-\mu)\frac{3(x_0+\mu)y_0}{\left\{(x_0+\mu)^2+{y_0}^2\right\}^{\frac{5}{2}}}+\mu\frac{3(x_0+\mu-1)y_0}{\left\{(x_0+\mu-1)^2+{y_0}^2\right\}^{\frac{5}{2}}}\right]$$
$$+k\frac{y_0}{\left[(x_0+\mu)^2+{y_0}^2\right]^2}=0 \tag{8}$$

and

$$\pi_2\left[1+(1-\mu)\frac{3{y_0}^2}{\left\{(x_0+\mu)^2+{y_0}^2\right\}^{\frac{5}{2}}}-\frac{1}{\left\{(x_0+\mu)^2+{y_0}^2\right\}^{\frac{3}{2}}}+\mu\frac{3{y_0}^2}{\left\{(x_0+\mu-1)^2+{y_0}^2\right\}^{\frac{5}{2}}}-\frac{1}{\left\{(x_0+\mu-1)^2+{y_0}^2\right\}^{\frac{3}{2}}}\right]$$
$$+\pi_1\left[(1-\mu)\frac{3(x_0+\mu)y_0}{\left\{(x_0+\mu)^2+{y_0}^2\right\}^{\frac{5}{2}}}+\mu\frac{3(x_0+\mu-1)y_0}{\left\{(x_0+\mu-1)^2+{y_0}^2\right\}^{\frac{5}{2}}}\right]$$
$$-k\frac{x_0}{\left[(x_0+\mu)^2+{y_0}^2\right]^2}=0 \tag{9}$$

After substituting the values of the constants x_0 and y_0 in the above equations and rejecting the second and higher order terms in π_1 and π_2, we get the values of π_1 and π_2 as

$$\pi_1=-\frac{1}{\sqrt{3}}\mu k,$$

$$\pi_2=\frac{5}{9}\mu k.$$

Hence, putting the values of π_1 and π_2, the displaced equilibrium points are given by

$$L_{4,5}\left[\bar{x}=\frac{1}{2}-\mu-\frac{1}{\sqrt{3}}\mu k\ ,\ \bar{y}=\pm\left\{\frac{\sqrt{3}}{2}+\frac{5}{9}\mu k\right\}\right] \qquad (10)$$

Here, the shifts in L_4 and L_5 are of $O(k/\mu)$. Now we calculate $(\bar{x},\bar{y})$ numerically, taking $k=10^{-5}$ for different values of μ (Table 1). Also figure (*i*) and (*ii*) indicates the relationship among the values of $\mu,\bar{x}$ and $\mu,\bar{y}$. Here we observe that while using Poynting Robertson drag, as far as the μ values increases corresponding $\bar{x}$ values decreases and the $\bar{y}$ values increases.

Table 1

μ	$k=0$		$k=10^{-5}$	
	$\bar{x}$	$\bar{y}$	$\bar{x}$	$\pm\ \bar{y}$
0.01	0.49	0.866025	0.49	0.866025
0.02	0.48	0.866025	0.48	0.866026
0.03	0.47	0.866025	0.47	0.866026
0.04	0.46	0.866025	0.46	0.866026
0.05	0.45	0.866025	0.45	0.866026
0.06	0.44	0.866025	0.44	0.866026
0.07	0.43	0.866025	0.43	0.866026
0.08	0.42	0.866025	0.42	0.866026
0.09	0.41	0.866025	0.409999	0.866026
0.1	0.4	0.866025	0.399999	0.866026
0.2	0.3	0.866025	0.299999	0.866027
0.3	0.2	0.866025	0.199998	0.866027
0.4	0.1	0.866025	0.0999977	0.866028
0.5	0	0.866025	-2.88675×10^{-6}	0.866028

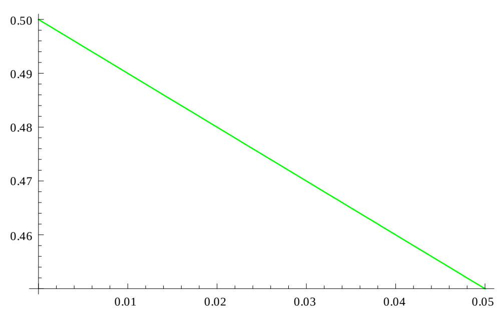

Fig. (*i*) μ Vs $\bar{x}$

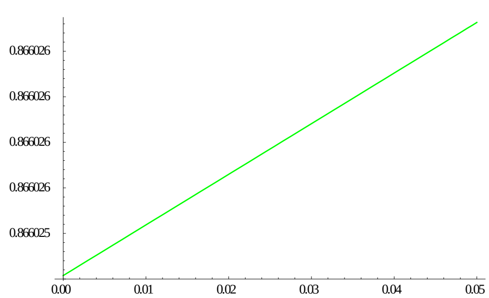

Fig. (*ii*) μ Vs $\bar{y}$

IV. Stability Of $L_{4,5}$

We can write the variational equations by putting $x=\bar{x}+\xi$ and $y=\bar{y}+\eta$ in the equations of motion (2) and (3), where $(\bar{x},\bar{y})$ are the coordinates of the libration points under consideration.

Now, the variational equations are

$$\ddot{\xi}-2\dot{\eta}=f(\bar{x}+\xi,\bar{y}+\eta),$$

$$\ddot{\eta}+2\dot{\xi}=g(\bar{x}+\xi,\bar{y}+\eta), \qquad \xi,\eta<<1.$$

Where

$$f(\bar{x},\bar{y})=\Omega_x-\frac{k}{r_1^2}\left(\dot{x}-\bar{y}+\frac{\bar{x}}{r_1^2}(\bar{x}\dot{x}+\bar{y}\dot{y})\right),$$

$$g(\bar{x},\bar{y})=\Omega_y-\frac{k}{r_1^2}\left(\dot{y}+\bar{x}+\frac{\bar{y}}{r_1^2}(\bar{x}\dot{x}+\bar{y}\dot{y})\right).$$

Therefore, expanding $f(\bar{x},\bar{y})$ and $g(\bar{x},\bar{y})$ by Taylors Theorem, we get

$$\ddot{\xi}-2\dot{\eta}=\Omega_x(\bar{x},\bar{y})+\xi\left[\begin{array}{l}1-\frac{\mu}{(\bar{r}_2)^3}+\frac{3\mu(\bar{x}+\mu-1)^2}{(\bar{r}_2)^5}\\ +\frac{3(1-\mu)(\bar{x}+\mu)^2}{(\bar{r}_1)^5}-\frac{(1-\mu)}{(\bar{r}_1)^3}-\frac{k}{\bar{r}_1^2}-k\frac{\bar{x}^2}{\bar{r}_1^4}\\ -\frac{k}{\bar{r}_1^2}\left(\frac{-2\bar{x}(\dot{x}\bar{x}+\dot{y}\bar{y})(\bar{x}+\mu)}{\bar{r}_1^4}+\frac{\dot{x}\bar{x}}{\bar{r}_1^2}+\frac{\dot{x}\bar{x}+\dot{y}\bar{y}}{\bar{r}_1^2}\right)\\ +\frac{2k(\bar{x}+\mu)}{\bar{r}_1^4}\left((\dot{x}-\bar{y})+\frac{\bar{x}(\dot{x}\bar{x}+\dot{y}\bar{y})}{\bar{r}_1^2}\right)\end{array}\right]+\eta\left[\begin{array}{l}\frac{3\bar{y}\mu(\bar{x}+\mu-1)}{(\bar{r}_2)^5}+\frac{3\bar{y}(1-\mu)(\bar{x}+\mu)}{(\bar{r}_1)^5}\\ -k\frac{\bar{x}\bar{y}}{\bar{r}_1^4}-\frac{k}{\bar{r}_1^2}\left(-1-\frac{2\bar{x}\bar{y}(\dot{x}\bar{x}+\dot{y}\bar{y})}{\bar{r}_1^4}+\frac{\dot{y}\bar{x}}{\bar{r}_1^2}\right)\\ +\frac{2k\bar{y}}{\bar{r}_1^4}\left(\dot{x}-\bar{y}+\frac{\bar{x}(\dot{x}\bar{x}+\dot{y}\bar{y})}{\bar{r}_1^2}\right)\end{array}\right] \quad (11)$$

$$\ddot{\eta}+2\dot{\xi}=\Omega_y(\bar{x},\bar{y})+\xi\left[\begin{array}{l}\frac{3\bar{y}\mu(\bar{x}+\mu-1)}{(\bar{r}_2)^5}+\frac{3\bar{y}(1-\mu)(\bar{x}+\mu)}{(\bar{r}_1)^5}\\ -k\frac{\bar{x}\bar{y}}{\bar{r}_1^4}-\frac{k}{\bar{r}_1^2}\left(1-\frac{2\bar{y}(\dot{x}\bar{x}+\dot{y}\bar{y})(\bar{x}+\mu)}{\bar{r}_1^4}+\frac{\dot{x}\bar{y}}{\bar{r}_1^2}\right)\\ +\frac{2k(\bar{x}+\mu)}{\bar{r}_1^4}\left(\dot{y}+\bar{x}+\frac{\bar{y}(\dot{x}\bar{x}+\dot{y}\bar{y})}{\bar{r}_1^2}\right)\end{array}\right]+\eta\left[\begin{array}{l}1+\frac{3\bar{y}^2\mu}{(\bar{r}_2)^5}-\frac{\mu}{(\bar{r}_2)^2}+\frac{3(1-\mu)\bar{y}^2}{(\bar{r}_1)^5}\\ -\frac{(1-\mu)}{(\bar{r}_1)^3}-\frac{k}{\bar{r}_1^2}-k\frac{\bar{y}^2}{\bar{r}_1^4}\\ -\frac{k}{\bar{r}_1^2}\left(\frac{-2\bar{y}^2(\dot{x}\bar{x}+\dot{y}\bar{y})}{\bar{r}_1^4}+\frac{\dot{y}\bar{y}}{\bar{r}_1^2}+\frac{\dot{x}\bar{x}+\dot{y}\bar{y}}{\bar{r}_1^2}\right)\\ +\frac{2k\bar{y}}{\bar{r}_1^4}\left((\dot{y}+\bar{x})+\frac{\bar{y}(\dot{x}\bar{x}+\dot{y}\bar{y})}{\bar{r}_1^2}\right)\end{array}\right] \quad (12)$$

where ξ_0 and η_0 are constants and λ is a complex constant. Then we have

$$\lambda^2 \xi_0 e^{\lambda t} - 2\lambda \eta_0 e^{\lambda t} = \xi_0 e^{\lambda t}\left[\begin{array}{l} 1-\dfrac{\mu}{(\bar r_2)^3}+\dfrac{3\mu(\bar x+\mu-1)^2}{(\bar r_2)^5} \\ +\dfrac{3(1-\mu)(\bar x+\mu)^2}{(\bar r_1)^5}-\dfrac{(1-\mu)}{(\bar r_1)^3} \\ -\lambda\dfrac{k}{\bar r_1^2}\left(1+\dfrac{\bar x^2}{\bar r_1^2}\right) \\ -\dfrac{k}{\bar r_1^2}\left(\dfrac{-2\bar x(\dot x\bar x+\dot y\bar y)(\bar x+\mu)}{\bar r_1^4}+\dfrac{\dot x\bar x}{\bar r_1^2}+\dfrac{\dot x\bar x+\dot y\bar y}{\bar r_1^2}\right) \\ +\dfrac{2k(\bar x+\mu)}{\bar r_1^4}\left((\dot x-\bar y)+\dfrac{\bar x(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right) \end{array}\right] + \eta_0 e^{\lambda t}\left[\begin{array}{l} \dfrac{3\bar y\mu(\bar x+\mu-1)}{(\bar r_2)^5}+\dfrac{3\bar y(1-\mu)(\bar x+\mu)}{(\bar r_1)^5} \\ -\lambda k\dfrac{\bar x\bar y}{\bar r_1^4}-\dfrac{k}{\bar r_1^2}\left(-1-\dfrac{2\bar x\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^4}+\dfrac{\dot y\bar x}{\bar r_1^2}\right) \\ +\dfrac{2k\bar y}{\bar r_1^4}\left(\dot x-\bar y+\dfrac{\bar x(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right) \end{array}\right] \quad (13)$$

$$\lambda^2 \eta_0 e^{\lambda t} + 2\lambda \xi_0 e^{\lambda t} = \xi_0 e^{\lambda t}\left[\begin{array}{l} \dfrac{3\bar y\mu(\bar x+\mu-1)}{(\bar r_2)^5}+\dfrac{3\bar y(1-\mu)(\bar x+\mu)}{(\bar r_1)^5} \\ -\lambda k\dfrac{\bar x\bar y}{\bar r_1^4}-\dfrac{k}{\bar r_1^2}\left(1-\dfrac{2\bar y(\dot x\bar x+\dot y\bar y)(\bar x+\mu)}{\bar r_1^4}+\dfrac{\dot x\bar y}{\bar r_1^2}\right) \\ +\dfrac{2k(\bar x+\mu)}{\bar r_1^4}\left(\dot y+\bar x+\dfrac{\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right) \end{array}\right] + \eta_0 e^{\lambda t}\left[\begin{array}{l} 1+\dfrac{3\bar y^2\mu}{(\bar r_2)^5}-\dfrac{\mu}{(\bar r_2)^2}+\dfrac{3(1-\mu)\bar y^2}{(\bar r_1)^5} \\ -\dfrac{(1-\mu)}{(r_1)^3}-\lambda\dfrac{k}{\bar r_1^2}\left(1+\dfrac{\bar y^2}{\bar r_1^2}\right) \\ -\dfrac{k}{\bar r_1^2}\left(\dfrac{-2\bar y^2(\dot x\bar x+\dot y\bar y)}{\bar r_1^4}+\dfrac{\dot y\bar y}{\bar r_1^2}+\dfrac{\dot x\bar x+\dot y\bar y}{\bar r_1^2}\right) \\ +\dfrac{2k\bar y}{\bar r_1^4}\left((\dot y+\bar x)+\dfrac{\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right) \end{array}\right] \quad (14)$$

Now, from equations (13) and (14), we derive the following simultaneous linear equations

$$\xi\left\{\begin{aligned}&\lambda^2+\frac{1-\mu}{(\bar r_1)^3}\left(1-\frac{3(\bar x+\mu)^2}{(\bar r_1)^2}\right)\\&+\frac{\mu}{(\bar r_2)^3}\left(1-\frac{3(\bar x+\mu-1)^2}{(\bar r_2)^2}\right)-1\\&+\lambda\frac{k}{\bar r_1^2}\left(1+\frac{\bar x^2}{\bar r_1^2}\right)+\frac{k}{\bar r_1^2}\left(\frac{-2\bar x(\dot x\bar x+\dot y\bar y)(\bar x+\mu)}{\bar r_1^4}+\frac{\dot x\bar x}{\bar r_1^2}+\frac{\dot x\bar x+\dot y\bar y}{\bar r_1^2}\right)\\&-\frac{2k(\bar x+\mu)}{\bar r_1^4}\left((\dot x-\bar y)+\frac{\bar x(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right)\end{aligned}\right\}$$

$$+\eta\left\{\begin{aligned}&-2\lambda-\frac{3\bar y\mu(\bar x+\mu-1)}{(\bar r_2)^5}-\frac{3\bar y(1-\mu)(\bar x+\mu)}{(\bar r_1)^5}\\&+\lambda k\frac{\bar x\bar y}{\bar r_1^4}+\frac{k}{\bar r_1^2}\left(-1-\frac{2\bar x\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^4}+\frac{\dot y\bar x}{\bar r_1^2}\right)\\&-\frac{2k\bar y}{\bar r_1^4}\left(\dot x-\bar y+\frac{\bar x(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right)\end{aligned}\right\}=0 \qquad (15)$$

and

$$\xi\left\{\begin{aligned}&2\lambda-\frac{3\bar y\mu(\bar x+\mu-1)}{(\bar r_2)^5}\\&-\frac{3\bar y(1-\mu)(\bar x+\mu)}{(\bar r_1)^5}+\lambda k\frac{\bar x\bar y}{\bar r_1^4}\\&+\frac{k}{\bar r_1^2}\left(1-\frac{2\bar y(\dot x\bar x+\dot y\bar y)(\bar x+\mu)}{\bar r_1^4}+\frac{\dot x\bar y}{\bar r_1^2}\right)\\&-\frac{2k(\bar x+\mu)}{\bar r_1^4}\left(\dot y+\bar x+\frac{\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right)\end{aligned}\right\}$$

$$+\eta\left\{\begin{aligned}&\lambda^2+\frac{1-\mu}{(\bar r_1)^3}\left(1-\frac{3\bar y^2}{(\bar r_1)^2}\right)+\frac{\mu}{(\bar r_2)^3}\left(1-\frac{3\bar y^2}{(\bar r_2)^2}\right)\\&-1+\lambda\frac{k}{\bar r_1^2}\left(1+\frac{\bar y^2}{\bar r_1^2}\right)+\frac{k}{\bar r_1^2}\left(\frac{-2\bar y^2(\dot x\bar x+\dot y\bar y)}{\bar r_1^4}+\frac{\dot y\bar y}{\bar r_1^2}+\frac{\dot x\bar x+\dot y\bar y}{\bar r_1^2}\right)\\&-\frac{2k\bar y}{\bar r_1^4}\left((\dot y+\bar x)+\frac{\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right)\end{aligned}\right\}=0 \qquad (16)$$

The simultaneous linear Equations (15) and (16) can be written as

$$\xi\left(\lambda^2+e-h-1-\lambda k_{x,\dot x}-k_{x,x}\right)+\eta\left(-2\lambda-g-\lambda k_{x,\dot y}-k_{x,y}\right)=0 \qquad (17)$$

$$\xi\left(2\lambda-g-\lambda k_{y,\dot x}-k_{y,x}\right)+\eta\left(\lambda^2+e-f-1-\lambda k_{y,\dot y}-k_{y,y}\right)=0 \qquad (18)$$

Where

$$e=\frac{1-\mu}{(\bar r_1)^3}+\frac{\mu}{(\bar r_2)^3}, \qquad (19)$$

$$f=3\left[\frac{1-\mu}{(\bar r_1)^5}+\frac{\mu}{(\bar r_2)^5}\right]\bar y^2, \quad (20)$$

$$g=3\left[\frac{(1-\mu)(\bar x+\mu)}{(\bar r_1)^5}+\frac{\mu(\bar x+\mu-1)}{(\bar r_2)^5}\right]\bar y, \quad (21)$$

$$h=3\left[\frac{(1-\mu)(\bar x+\mu)^2}{(\bar r_1)^5}+\frac{\mu(\bar x+\mu-1)^2}{(\bar r_2)^5}\right]. \quad (22)$$

and

$$k_{x,x}=\left(\frac{\partial R_x}{\partial x}\right)_- =-\frac{k}{\bar r_1^2}\left(\frac{-2\bar x(\dot x\bar x+\dot y\bar y)(\bar x+\mu)}{\bar r_1^4}+\frac{\dot x\bar x}{\bar r_1^2}+\frac{\dot x\bar x+\dot y\bar y}{\bar r_1^2}\right)+\frac{2k(\bar x+\mu)}{\bar r_1^4}\left((\dot x-\bar y)+\frac{\bar x(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right),$$

$$k_{x,\dot x}=\left(\frac{R_x}{\partial\dot x}\right)_- =\frac{k}{\bar r_1^2}\left(1+\frac{\bar x^2}{\bar r_1^2}\right),$$

$$k_{x,y}=\left(\frac{\partial R_x}{\partial y}\right)_- =\frac{k}{\bar r_1^2}\left(-1-\frac{2\bar x\,\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^4}+\frac{\dot y\bar x}{\bar r_1^2}\right)-\frac{2k\,\bar y}{\bar r_1^4}\left(\dot x-\bar y+\frac{\bar x(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right),$$

$$k_{x,\dot y}=\left(\frac{\partial R_x}{\partial\dot y}\right)_- =k\frac{\bar x\,\bar y}{\bar r_1^4},$$

$$k_{y,x}=\left(\frac{\partial R_y}{\partial x}\right)_- =\frac{k}{\bar r_1^2}\left(1-\frac{2\,\bar y(\dot x\bar x+\dot y\bar y)(\bar x+\mu)}{\bar r_1^4}+\frac{\dot x\bar y}{\bar r_1^2}\right)-\frac{2k(\bar x+\mu)}{\bar r_1^4}\left(\dot y+\bar x+\frac{\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right),$$

$$k_{y,\dot x}=\left(\frac{\partial R_y}{\partial\dot x}\right)_- =k\frac{\bar x\,\bar y}{\bar r_1^4},$$

$$k_{y,y}=\left(\frac{\partial R_y}{\partial y}\right)_- =\frac{k}{\bar r_1^2}\left(\frac{-2\bar y^2(\dot x\bar x+\dot y\bar y)}{\bar r_1^4}+\frac{\dot y\bar y}{\bar r_1^2}+\frac{\dot x\bar x+\dot y\bar y}{\bar r_1^2}\right)-\frac{2k\,\bar y}{\bar r_1^4}\left((\dot y+\bar x)+\frac{\bar y(\dot x\bar x+\dot y\bar y)}{\bar r_1^2}\right),$$

$$k_{y,\dot y}=\left(\frac{\partial R_y}{\partial\dot y}\right)_- =\frac{k}{\bar r_1^2}\left(1+\frac{\bar y^2}{\bar r_1^2}\right). \quad (23)$$

Neglecting terms of $O(k^2)$, the condition for the determinant of the linear equations defined by the equations (17) and (18) to be zero is

$$\begin{aligned}\lambda^4-(k_{x,\dot x}+k_{y,\dot y})\lambda^3+[2(1+e)-f-h-k_{x,x}\\ +2(k_{x,\dot y}-k_{y,\dot x})-k_{y,y}]\lambda^2\\ +[(1-e+f)k_{x,\dot x}+(1-e+h)k_{y,\dot y}\\ +2(k_{x,y}-k_{y,x})-g(k_{x,\dot y}+k_{y,\dot x})]\lambda\\ +[(e-h-1)(e-f-1)-(g)^2+(1-e+f)k_{x,x}\\ +(1-e+h)k_{y,y}-g(k_{x,y}+k_{y,x})]=0\end{aligned} \quad (24)$$

This quadratic equation (24) has the general form

$$\lambda^4+\sigma_3\lambda^3+(\sigma_{20}+\sigma_2)\lambda^2+\sigma_1\lambda+(\sigma_{00}+\sigma_0)=0 \quad (25)$$

where

$$\sigma_0=(1-e+f)k_{x,x}+(1-e+h)k_{y,y}-g(k_{x,y}+k_{y,x}),$$

$$\sigma_1 = (1-e+f)k_{x,\dot{x}} + (1-e+h)k_{y,\dot{y}} + 2(k_{x,y} - k_{y,x}) - g(k_{x,\dot{y}} + k_{y,\dot{x}}),$$

$$\sigma_2 = -k_{y,y} - k_{x,x},$$

$$\sigma_3 = -k_{x,\dot{x}} - k_{y,\dot{y}},$$

$$\sigma_{20} = 2(1+e) - f - h,$$

$$\sigma_{00} = (e-h-1)(e-f-1) - g^2.$$

Here σ_{00}, σ_{20} and $\sigma_i (i = 0,1,2,3)$ can be derived by evaluating e, f, g and h defined earlier. The value of the coefficient in the zero drag case is denoted by adding additional subscript 0. If we neglect product of powers of μ with any of the constants defined in equation (23), we obtain

$$\sigma_{00} = \frac{27}{4}\mu, \quad \sigma_{20} = 1, \quad \sigma_0 = 0, \quad \sigma_1 = 3k, \quad \sigma_2 = 0, \quad \sigma_3 = -3k. \qquad (26)$$

By assuming σ_i to be small, we investigate the stability of the non zero drag case. We can use the classical solutions of the zero drag case (i.e. when k =0).The equation (25) reduces to

$$\lambda^4 + \sigma_{20}\lambda^2 + \sigma_{00} = 0 \qquad (27)$$

The four classical solutions for L_4 and L_5 to $O(\mu)$ are given by the pair of values

$$L_{4,5}: \quad \lambda_{1,2} = \pm\sqrt{-1+\frac{27}{4}\mu}, \quad \lambda_{3,4} = \pm\sqrt{-\frac{27}{4}\mu} \qquad (28)$$

Since we are primarily interested in the stability of L_4 and L_5 under the effects of a drag force, we restrict our analysis to these points. The four roots of the classical characteristic equation can be written as

$$\lambda_n = \pm \mathrm{T} i \qquad (n = 1,\dots,4) \qquad (29)$$

where

$$\mathrm{T} = \sqrt{\frac{\sigma_{20} \pm \sqrt{\sigma_{20}^{\ 2} - 4\sigma_{00}}}{2}} \qquad (30)$$

is a real quantity for L_4 and L_5.Using the values of σ_{00} and σ_{20} given in Equations(26) we have

$$\mathrm{T}^2 = 1 - \frac{27}{4}\mu \quad or \quad \mathrm{T}^2 = \frac{27}{4}\mu \qquad (31)$$

With the introduction of drag we assume a solution of the form

$$\lambda = \lambda_n(1+\rho+\upsilon i) = [\mp\upsilon \pm (1+\rho)i]\mathrm{T} \qquad (32)$$

where ρ and υ are small real quantities. To lowest order we have

$$\lambda^2 = [-(1+2\rho) - 2\upsilon i)]\mathrm{T}^2 \qquad (33)$$

$$\lambda^3 = [\pm 3\upsilon \mp (1+3\rho)i)]\mathrm{T}^3 \qquad (34)$$

$$\lambda^4 = [(1+4\rho)i + 4\upsilon i)]\mathrm{T}^4 \qquad (35)$$

Substituting these in equation (25), and neglecting products of ρ or υ with σ_i, and solving the real and imaginary parts of the resulting simultaneous equations for ρ or υ we get

$$\upsilon = \frac{\pm\sigma_3 T^2 \mp \sigma_1}{2T(2T^2 - \sigma_{20})} \qquad (36)$$

$$\rho = \frac{(\sigma_{00} + \sigma_0) - (\sigma_{20} + \sigma_2)T^2 + T^4}{2T^2(\sigma_{20} - 2T^2)} \qquad (37)$$

(i) The stability of L_4

For L_4, we have

$$\upsilon = \frac{\sigma_3 T^2 - \sigma_1}{2T(2T^2 - \sigma_{20})} \qquad (38)$$

$$\rho = \frac{(\sigma_{00} + \sigma_0) - (\sigma_{20} + \sigma_2)T^2 + T^4}{2T^2(\sigma_{20} - 2T^2)} \qquad (39)$$

On putting the values of σ_i, in equations (38) and (39) from equation (26) and also taking, $T^2 = \frac{27}{4}\mu$, we have

$$\upsilon = -\frac{k(4+27\mu)}{2\sqrt{3\mu}(-2+27\mu)},$$

$$\rho = \frac{27\mu}{8-108\mu}$$

Now, putting these values of ρ and υ in equation (35), and neglecting the terms of $O(k\mu)$, we get the characteristic equation as

$$\lambda^4 - \frac{729\mu^2}{16-216\mu} = 0$$

Whose roots are

$$\lambda_1 = -\frac{3\sqrt{3\mu}}{2^{\frac{3}{4}}(2-27\mu)^{\frac{1}{4}}}, \qquad \lambda_2 = -\frac{3i\sqrt{3\mu}}{2^{\frac{3}{4}}(2-27\mu)^{\frac{1}{4}}},$$

$$\lambda_3 = \frac{3i\sqrt{3\mu}}{2^{\frac{3}{4}}(2-27\mu)^{\frac{1}{4}}}, \qquad \lambda_4 = \frac{3\sqrt{3\mu}}{2^{\frac{3}{4}}(2-27\mu)^{\frac{1}{4}}}.$$

Also on taking $T^2 = 1 - \frac{27}{4}\mu$ in equations (38) and (39) from equation (26), we get the characteristic equation as

$$\lambda^4 + \frac{(-4+27\mu)(-4+81\mu)}{8(-2+27\mu)} + 12ik = 0$$

whose roots are

$$\lambda_1 = -\frac{(-16+432\mu)^{\frac{1}{4}} - (192ik)^{\frac{1}{4}}}{2^{\frac{3}{4}}(-2+27\mu)^{\frac{1}{4}}},$$

$$\lambda_2 = -\frac{(-192k)^{\frac{1}{4}} - i(16-432\mu)^{\frac{1}{4}}}{2^{\frac{3}{4}}(-2+27\mu)^{\frac{1}{4}}},$$

$$\lambda_3 = \frac{(-192k)^{\frac{1}{4}} - i(16-432\mu)^{\frac{1}{4}}}{2^{\frac{3}{4}}(-2+27\mu)^{\frac{1}{4}}},$$

$$\lambda_4 = \frac{(-16+432\mu)^{\frac{1}{4}} + (192ik)^{\frac{1}{4}}}{2^{\frac{3}{4}}(-2+27\mu)^{\frac{1}{4}}}.$$

If $\upsilon \neq 0$,

According to Murray (1994), the resulting motion of a particle is asymptotically stable only when all the real parts of λ are negative and the condition for asymptotically stable under the arbitrary drag force is given by

$$0 < \sigma_1 < \sigma_3 \qquad (40)$$

where σ_1 and σ_3 are defined in equation (26).But we see that the linear stability of triangular equilibrium points does not depend on the value of $k_{x,x}$ and $k_{y,y}$. Therefore the condition $\sigma_3 > 0$ can only be satisfied when k is positive and the drag force is a function of $\dot{x}$ and $\dot{y}$.

But here in our case of Poynting Robertson drag $\sigma_1 = 3k, \sigma_3 = -3k$ and therefore $\sigma_1 > \sigma_3$ and hence L_4 is not asymptotically stable. Further one of the roots of λ i.e. λ_4 has positive real root. Therefore L_4 is not stable. Thus we conclude that L_4 is neither stable nor asymptotically stable and hence linearly unstable.

Similarly, we conclude that L_5 is neither stable nor asymptotically stable and hence linearly unstable.

Conclusion

Using Poynting Robertson light drag in the restricted three body problem, we have studied the existence of Libration points and their linear stability. It is well known that Poynting Robertson force has two components, the drag component and the Doppler shift effect. By considering both of the components, we have shown that there exist two non-collinear stationary points $L_4(\bar{x}, \bar{y})$ and $L_5(\bar{x}, -\bar{y})$ [Eq.10].

If we put $k = 0$, the above results agree with the classical restricted three body problem.

In the classical case i.e. when $k = 0$, we observe that as the value of μ increases, the abscissa $\bar{x}$ of L_4 decreases and the ordinate $\bar{y}$ of L_4 remains constant, while in our case, when $k = 10^{-5}$, we observe that, the abscissa $\bar{x}$ of L_4 decreases and the ordinate $\bar{y}$ of L_4 changes slightly. The results are shown in Table (1). We have also shown this result graphically in Fig. (i) and (ii).

In the case of Poynting Robertson drag, we have derived a set of linear equations in terms of ξ and η, [Eq.17 and 18], which involves the components of the Poynting Robertson light drag force evaluated at the libration points [Eq.19-23]. From these we derive a characteristic equation having the general form [Eq.25].

Further we have derived the approximate expressions for $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_{00}$ and σ_{20} occurring in the above characteristic equation. These expressions are given in terms of the partial derivatives of the Poynting Robertson light drag, evaluated at the libration points.

Using the Murray terminology, in the case of drag force, we assume a solution of the form [Eq.32].Where υ and ρ are small real quantities and $\lambda_n = \pm \mathrm{T} i \quad (n = 1, \ldots.. 4)$ is a real quantity for L_4 and L_5 in the classical case. After substituting the values of λ, λ^2, λ^3 and λ^4 in the characteristic equation, we get values of υ and ρ [Eq.36] [Eq.37].

Further to investigate the stability of the shifted points, by using Murray terminology, the resulting motion of a particle is asymptotically stable only when all the real parts of λ are negative. Also the condition for asymptotical stability under the drag force is given by [Eq.40].

The condition $\sigma_3 > 0$ can only be satisfied when $k > 0$. In the case of Poynting Robertson light drag $\sigma_1 = 3k$ and $\sigma_3 = -3k$ therefore the equation (40) is not satisfied. Therefore L_4 and L_5 are not asymptotically stable. Further we have seen that one of the roots of λ i.e. λ_4 has positive real root, thus L_4 and L_5 are not stable. Hence due to Poynting Robertson light drag, L_4 and L_5 are neither stable nor asymptotically stable but unstable whereas in the classical case L_4 and L_5 are stable for the mass ratio $\mu < 0.03852$, as in [2].

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