Near Rough Connected Topologized Approximation Spaces

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Abstract

The purpose of this paper is to introduce various levels of connectedness in approximation spaces using some classes of near open sets. Moreover, proved results, examples and counter examples are provided.

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1. Preliminaries

This section presents a review of some fundamental notions of topological spaces and rough set theory.

A topological space [12] is a pair (X,τ) consisting of a set X and a family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.
(T2) τ is closed under arbitrary union.
(T3) τ is closed under finite intersection.

Throughout this paper (X,τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space, and the family of all closed subsets of X is denoted by τ^*. The family τ of open subsets of X is also called a topology on X. A subset A of X in a topological space (X,τ) is said to be clopen if it is both open and closed in (X,τ).

A family $\mathrm{B} \subseteq \tau$ is called a basis for (X,τ) iff every nonempty open subset of X can be represented as a union of subfamily of B. Clearly, a topological space can have many bases. A family $\mathsf{S} \subseteq \tau$ is called a subbasis for (X,τ) iff the family of all finite intersections of S is a basis for (X,τ).

The $\tau-$interior of a subset A of X is denoted by A° and it is defined by $A^\circ = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$. Evidently, A° is the largest open subset of X which contained in A. Note that A is open iff $A = A^\circ$. The $\tau-$closure of a subset A of X is denoted by A^- and it is defined by $A^- = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in \tau^*\}$. Evidently, A^- is the smallest closed subset of X which contains A. Note that A is closed iff $A = A^-$.

Some forms of near open sets which are essential for our present study are introduced in the following definition.

Definition 1.1. Let (X,τ) be a topological space. The subset A of X is called:

i) Semi-open [14] (briefly $s-$open) if $A\subseteq A^{\circ -}$.

ii) Pre-open [16] (briefly $p-$open) if $A\subseteq A^{-\circ}$.

iii) $\gamma-$open [7] (b-open [6]) if $A\subseteq A^{\circ -}\cup A^{-\circ}$.

iv) $\alpha-$open [17] if $A\subseteq A^{\circ -\circ}$.

v) $\beta-$open [1] (Semi-pre-open [5]) if $A\subseteq A^{-\circ -}$.

The complement of an $s-$open (resp. $p-$open, $\gamma-$open, $\alpha-$open and $\beta-$open) set is called $s-$closed (resp. $p-$closed, $\gamma-$closed, $\alpha-$closed and $\beta-$closed) set. The family of all $s-$open (resp. $p-$open, $\gamma-$open, $\alpha-$open and $\beta-$open) sets of (X,τ) is denoted by $SO(X)$ (resp. $PO(X)$, $\gamma O(X)$, $\alpha O(X)$ and $\beta O(X)$). The family of all $s-$closed (resp. $p-$closed, $\gamma-$closed, $\alpha-$closed and $\beta-$closed) sets of (X,τ) is denoted by $SC(X)$ (resp. $PC(X)$, $\gamma C(X)$, $\alpha C(X)$ and $\beta C(X)$).

The near interior (resp. near closure) of a subset A of X is denoted by $A^{j\circ}$ (resp. A^{j-}) and it is defined by

$$A^{j\circ}=\cup\{G\subseteq X : G\subseteq A,\ G \text{ is a } j-\text{open set}\}$$

(resp. $A^{j-}=\cap\{H\subseteq X : A\subseteq H,\ H \text{ is a } j-\text{closed set}\}$), where

$j\in\{s,p,\gamma,\alpha,\beta\}$. Evidently, $A^{j\circ}$ is the largest j-open subset of X which contained in A. Note that A is j-open iff $A=A^{j\circ}$. Also, A^{j-} is the smallest j-closed subset of X which contains A. Note that A is j-closed iff $A=A^{j-}$.

From known results [1, 7], we have the following remark.

Remark 1.1. Let (X,τ) be a topological space. Then

i) $\tau\subseteq\alpha O(X)\subseteq SO(X)\,(PO(X))\subseteq\gamma O(X)\subseteq\beta O(X)$.

ii) $\tau^{*}\subseteq\alpha C(X)\subseteq SC(X)\,(PC(X))\subseteq\gamma C(X)\subseteq\beta C(X)$.

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K=(X,R)$, where X is a set called the universe and R is an equivalence relation [15, 18]. The equivalence classes of R are also known as the granules, elementary sets or blocks. We shall use R_x to denote the equivalence class containing $x\in X$, and X/R to denote the set of all elementary sets of R. In the approximation space $K=(X,R)$, the lower (resp. upper) approximation of a subset A of X is given by

$$\underline{R}A=\{x\in X : R_x\subseteq A\}\ \left(\text{resp. } \overline{R}A=\{x\in X : R_x\cap A\neq\phi\}\right).$$

Pawlak noted [18] that the approximation space $K=(X,R)$ with equivalence relation R defines a uniquely topological space (X,τ) where τ is the family of all clopen sets in (X,τ) and X/R is a basis for τ. Moreover, the lower (resp. upper) approximation of any subset A of X is exactly the interior (resp. closure) of A.

If R is a general binary relation, then the approximation space $K=(X,R)$ defines a uniquely topological space (X,τ_K) such that τ_K is the topology on X generated by the subbasis $S=\{xR: x\in X\}$, where $xR=\{y\in X: xRy\}$ [4, 13].

Definition 1.2 [4]. Let $K=(X,R)$ be an approximation space with general binary relation R and τ_K be the topology on X generated by the subbasis $S=\{xR: x\in X\}$, where $xR=\{y\in X: xRy\}$. Then the triple $\kappa=(X,R,\tau_K)$ is called a topologized approximation space.

Definition 1.3 [4]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and let A be a subset of X. The lower (resp. upper) approximation of A is denoted by $\underline{R}A$ (resp. $\overline{R}A$) and it is defined by

$$\underline{R}A=A^{\circ} \text{ (resp. } \overline{R}A=A^{-}).$$

The following general definition is given to introduce the near lower and near upper approximations in a topologized approximation space $\kappa=(X,R,\tau_K)$.

Definition 1.4 [4]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and let A be a subset of X. The near lower (briefly *j*-lower) (resp. near upper (briefly *j*-upper)) approximation of A is denoted by $\underline{R}_jA$ (resp. $\overline{R}_jA$) and it is defined by

$$\underline{R}_jA=A^{j\circ} \text{ (resp. } \overline{R}_jA=A^{j-}), \text{ where } j\in\{s,p,\gamma,\alpha,\beta\}.$$

Proposition 1.1 [2]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and let A be a subset of X. Then

i) $\underline{R}A\subseteq\underline{R}_\alpha A\subseteq\underline{R}_s A\left(\underline{R}_p A\right)\subseteq\underline{R}_\gamma A(X)\subseteq\underline{R}_\beta A$.

ii) $\overline{R}_\beta A\subseteq\overline{R}_\gamma A\subseteq\overline{R}_s A\left(\overline{R}_p A\right)\subseteq\overline{R}_\alpha A\subseteq\overline{R}A$.

Proposition 1.2 [4]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If $A\subseteq X$, then

$$\underline{R}A\subseteq\underline{R}_jA\subseteq A\subseteq\overline{R}_jA\subseteq\overline{R}A,\quad\forall\, j\in\{s,p,\gamma,\alpha,\beta\}.$$

Proposition 1.3 [4]. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If A and B are two subsets of X, then

i) $\underline{R}_j\phi=\overline{R}_j\phi=\phi$ and $\underline{R}_jX=\overline{R}_jX=X$.

ii) If $A\subseteq B$, then $\underline{R}_jA\subseteq\underline{R}_jB$.

iii) If $A\subseteq B$, then $\overline{R}_jA\subseteq\overline{R}_jB$.

iv) $\underline{R}_j(X-A)=X-\overline{R}_jA$.

v) $\overline{R}_j(X-A)=X-\underline{R}_jA$.

Where $j\in\{s,p,\gamma,\alpha,\beta\}$.

2. Near rough connected topologized approximation spaces

The present section is devoted to introduce various levels of connectedness in approximation spaces with general binary relations using some classes of near open sets.

Definition 2.1 [4]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and let A be a subset of X. Then

i) A is called totally $R-$definable (exact) set if $\underline{R}A = A = \overline{R}A$,

ii) A is called internally $R-$definable set if $A = \underline{R}A$,

iii) A is called externally $R-$definable set if $A = \overline{R}A$,

iv) A is called $R-$indefinable (rough) set if $A \neq \underline{R}A$ and $A \neq \overline{R}A$.

Remark 2.1. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and let A be a subset of X.

- If A is exact set, then it is both internally $R-$definable and externally $R-$definable set.
- $\underline{R}A$ is the largest internally $R-$definable set contained in A.
- $\overline{R}A$ is the smallest externally $R-$definable set contains A.

Proposition 2.1 [9]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and let A be a subset of X. Then

i) A is exact set if and only if $X - A$ is exact.

ii) A is internally $R-$definable (resp. externally $R-$definable) set if and only if $X - A$ is externally $R-$definable (resp. internally $R-$definable) set.

Definition 2.2 [3]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and let A be a subset of X. Then

i) A is called totally j-definable (j-exact) set if $\underline{R}_j A = A = \overline{R}_j A$,

ii) A is called internally j-definable set if $A = \underline{R}_j A$,

iii) A is called externally j-definable set if $A = \overline{R}_j A$,

iv) A is called j-indefinable (j-rough) set if $A \neq \underline{R}_j A$ and $A \neq \overline{R}_j A$.

Where $j \in \{s, p, \gamma, \alpha, \beta\}$.

Remark 2.2. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and let A be a subset of X.

- If A is j-exact set, then it is both internally j-definable and externally j-definable set
- $\underline{R}_j A$ is the largest internally j-definable set contained in A.
- $\overline{R}_j A$ is the smallest externally j-definable set contains A.

Where $j \in \{s, p, \gamma, \alpha, \beta\}$.

Proposition 2.2. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and let A be a subset of X. Then

i) A is j-exact set if and only if $X - A$ is j-exact.

ii) A is internally j-definable (resp. externally j-definable) set if and only if $X - A$ is externally j-definable (resp. internally j-definable) set.

Where $j \in \{s, p, \gamma, \alpha, \beta\}$.

Proof. By using Proposition 1.3, the proof is obvious. □

Definition 2.3 [9]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. Then κ is said to be rough disconnected if there are two nonempty subsets A and B of X such that

$$A \cup B = X \quad \text{and} \quad A \cap \overline{R}B = \overline{R}A \cap B = \phi .$$

The space $\kappa = (X, R, \tau_K)$ is said to be rough connected if it is not rough disconnected [9].

Definition 2.4 [10]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. Then κ is said to be semi-rough (briefly s-rough) disconnected if there are two nonempty subsets A and B of X such that

$$A \cup B = X \quad \text{and} \quad A \cap \overline{R}_s B = \overline{R}_s A \cap B = \phi .$$

The space $\kappa = (X, R, \tau_K)$ is said to be s-rough connected if it is not s-rough disconnected [10].

Definition 2.5 [11]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. Then κ is said to be pre-rough (briefly p-rough) disconnected if there are two nonempty subsets A and B of X such that

$$A \cup B = X \quad \text{and} \quad A \cap \overline{R}_p B = \overline{R}_p A \cap B = \phi .$$

The space $\kappa = (X, R, \tau_K)$ is said to be p-rough connected if it is not p-rough disconnected [11].

Theorem 2.1 [9]. A topologized approximation space $\kappa = (X, R, \tau_K)$ is rough disconnected if and only if there exists a nonempty exact proper subset of X.

Theorem 2.2 [10]. A topologized approximation space $\kappa = (X, R, \tau_K)$ is s-rough disconnected if and only if there exists a nonempty s-exact proper subset of X.

Theorem 2.3 [11]. A topologized approximation space $\kappa = (X, R, \tau_K)$ is p-rough disconnected if and only if there exists a nonempty p-exact proper subset of X.

The following general definition introduces the concept of j-rough disconnected topologized approximation space for all $j \in \{\gamma, \alpha, \beta\}$.

Definition 2.6. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. Then κ is said to be j-rough disconnected for all $j \in \{\gamma, \alpha, \beta\}$ if there are two nonempty subsets A and B of X such that

$$A \cup B = X \quad \text{and} \quad A \cap \overline{R}_j B = \overline{R}_j A \cap B = \phi .$$

The space $\kappa = (X, R, \tau_K)$ is said to be j-rough connected for all $j \in \{\gamma, \alpha, \beta\}$ if it is not j-rough disconnected.

Theorem 2.4. A topologized approximation space $\kappa = (X, R, \tau_K)$ is j-rough disconnected for all $j \in \{\gamma, \alpha, \beta\}$ if and only if there exists a nonempty j-exact proper subset of X.

Proof. We shall prove this theorem in the case of $j = \beta$ and the other cases can be proved similarly.

Let $\kappa = (X, R, \tau_K)$ be a β-rough disconnected topologized approximation space. Then there exist two nonempty subsets A and B of X such that $A \cup B = X$ and $A \cap \overline{R}_\beta B = \overline{R}_\beta A \cap B = \phi$. But $A \subseteq \overline{R}_\beta A$, hence $A \cap B = \phi$. Thus $A = X - B$. Also $A = X - \overline{R}_\beta B$, since $A \cap \overline{R}_\beta B = \phi$ and $A \cup \overline{R}_\beta B \supseteq A \cup B = X$. Hence $A = \underline{R}_\beta A$ and $B = \overline{R}_\beta B$. Similarly $B = \underline{R}_\beta B$ and $A = \overline{R}_\beta A$. Therefore there exists a nonempty β-exact proper subset A of X.

Conversely, Suppose that A is a nonempty β-exact proper subset of X. Then by Proposition 2.2, we get $B = X - A$ is also a nonempty β-exact proper subset of X. Hence $A \cup B = X$ and $A \cap \overline{R}_\beta B = A \cap B = \overline{R}_\beta A \cap B = \phi$.

Thus $\kappa = (X, R, \tau_K)$ is β-rough disconnected. □

Example 2.1. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space such that $X = \{a, b, c, d\}$ and $R = \{(a,a),(b,b),(d,d)(a,b),(b,a)\}$. Then $aR = \{a,b\} = bR$, $cR = \phi$ and $dR = \{d\}$. Hence

$$\mathrm{S} = \{\phi, \{d\}, \{a,b\}\},\ \mathrm{B} = \{X, \phi, \{d\}, \{a,b\}\},\ \tau_K = \{X, \phi, \{d\}, \{a,b\}, \{a,b,d\}\},$$

$$\beta O(X) = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\},$$

and

$$\beta C(X) = \{\phi, X, \{b,c,d\}, \{a,c,d\}, \{a,b,c\}, \{c,d\}, \{b,d\}, \{b,c\}, \{a,d\}, \{a,c\}, \{a,b\}, \{d\}, \{c\}, \{b\}, \{a\}\}.$$

Since $A = \{a,b,c\}$ is a nonempty β-exact proper subset of X, then the space $\kappa = (X, R, \tau_K)$ is β-rough disconnected.

Proposition 2.3. The implications between rough disconnected and j – rough disconnected topologized approximation spaces for all $j \in \{p, s, \gamma, \alpha, \beta\}$ are given by the following diagram.

rough disconnected
$\Downarrow$
α - rough disconnected $\Rightarrow$ s - rough disconnected
$\Downarrow$ $\Downarrow$
p - rough disconnected $\Rightarrow$ γ - rough disconnected
$\Downarrow$
β - rough disconnected

Proof. Let $\kappa = (X, R, \tau_K)$ be s – rough disconnected topologized approximation space. Then by Theorem 2.2, there exists a nonempty s-exact proper subset A of X. Hence $\underline{R}_s A = A = \overline{R}_s A$. By Proposition 1.1 and Proposition 1.2, we get

$$A = \underline{R}_s A \subseteq \underline{R}_\gamma A \subseteq A \subseteq \overline{R}_\gamma A \subseteq \overline{R}_s A = A.$$

Then $\underline{R}_\gamma A = A = \overline{R}_\gamma A$. So A is a nonempty γ-exact proper subset of X. Therefore $\kappa = (X, R, \tau_K)$ is γ – rough disconnected.

The other cases can be proved similarly. □

The converse of Proposition 2.3 does not hold, in general, as shown in the following example.

Example 2.2. Let $\kappa=(X,R,\tau_K)$ be the topologized approximation space which is given in Example 2.1. Then

$$\tau_K=\{X,\phi,\{d\},\{a,b\},\{a,b,d\}\},\ \tau_K^*=\{X,\phi,\{a,b,c\},\{c,d\},\{c\}\},$$
$$\beta O(X)=\{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},$$
$$\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\},$$
and
$$\beta C(X)=\{\phi,X,\{b,c,d\},\{a,c,d\},\{a,b,c\},\{c,d\},\{b,d\},\{b,c\},\{a,d\},$$
$$\{a,c\},\{a,b\},\{d\},\{c\},\{b\},\{a\}\}.$$

So κ is β-rough disconnected, since $A=\{a,b,c\}$ is a nonempty β-exact proper subset of X. But κ is not rough disconnected, because there is not any exact proper subset of X.

Definition 2.7 [8]. Let $\kappa=(X,R_1,\tau_K)$, $\mathrm{Q}=(Y,R_2,\tau_Q)$ be two topologized approximation spaces. Then a mapping $f:\kappa\to\mathrm{Q}$ is called near rough (briefly j-rough) continuous for all $j\in\{p,s,\gamma,\alpha,\beta\}$ if $f^{-1}(\underline{R_2}V)\subseteq \underline{R_1}_j f^{-1}(V)$ for every subset V of Y in Q.

In Definition 2.7, f^{-1} does not mean the inverse function, but it means the inverse image.

Theorem 2.5. Let $f:\kappa\to\mathrm{Q}$ be a mapping from a topologized approximation space $\kappa=(X,R_1,\tau_K)$ to a topologized approximation space $\mathrm{Q}=(Y,R_2,\tau_Q)$. Then for all $j\in\{\gamma,\alpha,\beta\}$ the following statements are equivalent.

i) f is j-rough continuous.

ii) The inverse image of each internally R_2-definable set in Q is internally j-definable set in κ.

iii) The inverse image of each externally R_2-definable set in Q is externally j-definable set in κ.

iv) $f(\overline{R_1}_j A)\subseteq \overline{R_2}\,f(A)$ for every subset A of X in κ.

v) $\overline{R_1}_j f^{-1}(B)\subseteq f^{-1}(\overline{R_2}B)$ for every subset B of Y in Q.

Proof. We shall prove this theorem in the case of $j=\beta$ and the other cases can be proved similarly.

(i)$\Rightarrow$(ii) Let f be β-rough continuous and let V be an internally R_2-definable set in Q. Then $\underline{R_2}V=V$ and $f^{-1}(V)$ is a subset of X in κ. By (i), we get $f^{-1}(V)=f^{-1}(\underline{R_2}V)\subseteq \underline{R_1}_\beta f^{-1}(V)$. Then $f^{-1}(V)\subseteq \underline{R_1}_\beta f^{-1}(V)$. But $\underline{R_1}_\beta f^{-1}(V)\subseteq f^{-1}(V)$. Hence $f^{-1}(V)=\underline{R_1}_\beta f^{-1}(V)$. Therefore $f^{-1}(V)$ is internally β-definable set in κ.

(ii)$\Rightarrow$(i) Let A be a subset of Y in Q. Since $\underline{R_2}A\subseteq A$, then $f^{-1}(\underline{R_2}A)\subseteq f^{-1}(A)$. But $\underline{R_2}A$ is internally R_2-definable set in Q, then by (ii), we get $f^{-1}(\underline{R_2}A)$ is internally β-

definable set in κ contained in $f^{-1}(A)$. Hence $f^{-1}\left(\underline{R_2}\,A\right)\subseteq \underline{R_1}_{\beta}\, f^{-1}(A)\subseteq f^{-1}(A)$, since $\underline{R_1}_{\beta}\, f^{-1}(A)$ is the largest internally β-definable set contained in $f^{-1}(A)$. Thus $f^{-1}\left(\underline{R_2}\,A\right)\subseteq \underline{R_1}_{\beta}\, f^{-1}(A)$ for every subset A of Y in $\mathtt{Q}$. Therefore f is β-rough continuous.

(ii)$\Rightarrow$(iii) Let F be an externally R_2 – definable set in $\mathtt{Q}$, then by Proposition 2.1, we get $Y-F$ is internally R_2 – definable. Thus by (ii), we have $f^{-1}(Y-F)$ is internally β-definable set in κ. Since $f^{-1}(Y-F)=X-f^{-1}(F)$, then $X-f^{-1}(F)$ is internally β-definable set in κ. Hence $f^{-1}(F)$ is externally β-definable set in κ.

Similarly we can prove (iii)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iv) Let A be a subset of X in κ, then $\overline{R_2}\, f(A)$ is an externally R_2 – definable set in $\mathtt{Q}$. Hence $Y-\overline{R_2}\, f(A)$ is internally R_2 – definable set in $\mathtt{Q}$. Thus by (ii), we get $f^{-1}\left(Y-\overline{R_2}\, f(A)\right)=X-f^{-1}\left(\overline{R_2}\, f(A)\right)$ is internally β-definable set in κ, and so $f^{-1}\left(\overline{R_2}\, f(A)\right)$ is externally β-definable set containing A in κ. Thus $A\subseteq \overline{R_1}_{\beta}\, A\subseteq f^{-1}\left(\overline{R_2}\, f(A)\right)$, since $\overline{R_1}_{\beta}\, A$ is the smallest externally β-definable set containing A in κ. Hence

$$f\left(\overline{R_1}_{\beta}\, A\right)\subseteq f\left[f^{-1}\left(\overline{R_2}\, f(A)\right)\right]\subseteq \overline{R_2}\, f(A).$$

Therefore $f\left(\overline{R_1}_{\beta}\, A\right)\subseteq \overline{R_2}\, f(A)$ for every subset A in κ.

(iv)$\Rightarrow$(v) Let B be a subset of Y in $\mathtt{Q}$. Let $A=f^{-1}(B)$, then A is a subset of X in κ. By (iv), we get

$$f\left(\overline{R_1}_{\beta}\, A\right)\subseteq \overline{R_2}\, f(A)=\overline{R_2}\, f\left(f^{-1}(B)\right)\subseteq \overline{R_2}\, B.$$

Hence $\overline{R_1}_{\beta}\, A\subseteq f^{-1}\left(\overline{R_2}\, B\right)$. Thus $\overline{R_1}_{\beta}\, A=\overline{R_1}_{\beta}\, f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}\, B\right)$.
Therefore $\overline{R_1}_{\beta}\, f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}\, B\right)$ for every subset B of Y in $\mathtt{Q}$.

(v)$\Rightarrow$(ii) Let G be an internally R_2 – definable set in $\mathtt{Q}$, then $B=Y-G$ is externally R_2 – definable set in $\mathtt{Q}$. Thus by (v), we get

$$\overline{R_1}_{\beta}\, f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}\, B\right).$$

Since B is externally R_2 – definable set, then $f^{-1}\left(\overline{R_2}\, B\right)=f^{-1}(B)$. Thus $\overline{R_1}_{\beta}\, f^{-1}(B)\subseteq f^{-1}(B)$. But $f^{-1}(B)\subseteq \overline{R_1}_{\beta}\, f^{-1}(B)$, then $\overline{R_1}_{\beta}\, f^{-1}(B)=f^{-1}(B)$. Hence $f^{-1}(B)$ is externally β-definable set in κ.
Since $f^{-1}(B)=f^{-1}(Y-G)=X-f^{-1}(G)$, then $X-f^{-1}(G)$ is externally β-definable set in κ. Therefore $f^{-1}(G)$ is internally β-definable set in κ. □

Example 2.3. Let $\kappa=(X,R_1,\tau_K)$, $\mathtt{Q}=(Y,R_2,\tau_Q)$ be two topologized approximation spaces such that $X=\{a,b,c,d\}$, $Y=\{y_1,y_2,y_3,y_4\}$,
$R_1=\{(a,a),(b,b),(d,d)(a,b),(b,a)\}$ and $R_2=\{(y_1,y_1),(y_4,y_4),(y_1,y_2)\}$. Then

$\tau_K = \{X, \phi, \{d\}, \{a,b\}, \{a,b,d\}\}$ and $\tau_Q = \{Y, \phi, \{y_4\}, \{y_1, y_2\}, \{y_1, y_2, y_4\}\}$. Hence

$$\beta O(X) = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\},$$
$$\{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}.$$

Define a mapping $f : \kappa \to \mathrm{Q}$ such that

$$f(a) = y_1,\ f(b) = y_2,\ f(c) = y_4 \text{ and } f(d) = y_3.$$

Then f is not a β-rough continuous mapping, since $V = \{y_4\}$ is an internally R_2 – definable set in Q, but $f^{-1}(V) = \{c\}$ is not an internally β-definable set in κ.

Lemma 2.1 [10]. Let $\kappa = (X, R_1, \tau_K)$ and $\mathrm{Q} = (Y, R_2, \tau_Q)$ be two topologized approximation spaces. If $f : \kappa \to \mathrm{Q}$ is a s-rough continuous mapping, then the inverse image of each exact set in Q is s-exact set in κ.

Lemma 2.2 [11]. Let $\kappa = (X, R_1, \tau_K)$ and $\mathrm{Q} = (Y, R_2, \tau_Q)$ be two topologized approximation spaces. If $f : \kappa \to \mathrm{Q}$ is a p-rough continuous mapping, then the inverse image of each exact set in Q is p-exact set in κ.

Lemma 2.3. Let $\kappa = (X, R_1, \tau_K)$ and $\mathrm{Q} = (Y, R_2, \tau_Q)$ be two topologized approximation spaces. If $f : \kappa \to \mathrm{Q}$ is a j-rough continuous mapping for all $j \in \{\gamma, \alpha, \beta\}$, then the inverse image of each exact set in Q is j-exact set in κ.

Proof. We shall prove this lemma in the case of $j = \alpha$ and the other cases can be proved similarly.

Let A be an exact set in Q, then A is both internally and externally R_2 – definable set in Q. Hence by Theorem 2.5, we get $f^{-1}(A)$ is both internally and externally α-definable set in κ. Therefore $f^{-1}(A)$ is an α-exact set in κ. □

Theorem 2.6. Let $\kappa = (X, R_1, \tau_K)$, $\mathrm{Q} = (Y, R_2, \tau_Q)$ be two topologized approximation spaces and let $f : \kappa \to \mathrm{Q}$ be a j-rough continuous mapping of X onto Y for all $j \in \{p, s, \gamma, \alpha, \beta\}$. If $\kappa = (X, R_1, \tau_K)$ is j-rough connected, then $\mathrm{Q} = (Y, R_2, \tau_Q)$ is rough connected.

Proof.

We shall prove this theorem in the case of $j = \gamma$ and the other cases can be proved similarly.

Assume that $\mathrm{Q} = (Y, R_2, \tau_Q)$ is rough disconnected topologized approximation space. Then by Theorem 2.1, there exists a nonempty exact proper subset A of Y in Q. Since f is γ-rough continuous mapping from X onto Y, then by Lemma 2.3, we get $f^{-1}(A)$ is a nonempty γ-exact proper subset of X in κ. Thus κ is γ-rough disconnected, but this is a contradiction.

Therefore $\mathrm{Q} = (Y, R_2, \tau_Q)$ is rough connected. □

3. Conclusions

In this paper, we used some classes of near open sets to introduce various levels of connected topologized approximation spaces.

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