Some New Super Mean Graphs

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Abstract - Let G be a graph and $f : V(G) \to \{1, 2, 3, \ldots, p + q\}$ be an injection. For each edge $e = uv$, the induced edge labeling f^* is defined as follows:

$$f^*(e) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

Then f is called super mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, \ldots, p+q\}$. A graph that admits a super mean labeling is called super mean graph. In this paper, we establish the supermeanness of the graphs $H_n \odot mK_1$, $T_n \odot K_1$, $Q_n \odot K_1$, $C_n + v_1v_3$ $(n \geq 5)$, $T_n(C_m)$ and slanting ladder SL_n for $n \geq 2$, $n \neq 3t + 1$, $t \geq 1$.

Key Words. super mean graph, super mean labeling.

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1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let G(V, E) be a graph with p vertices and q edges. For notations and terminology we follow [1].

The path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. A triangular snake is obtained from a path by identifying each edge of the path with an edge of the cycle C_3. The graph $C_n + v_1v_3$ is obtained from the cycle $C_n : v_1v_2 \ldots v_nv_1$ by adding an edge between the vertices v_1 and v_3. The balloon of the triangular snake $T_n(C_m)$ is the graph obtained from C_m by identifying an end vertex of the basic path in T_n at a vertex of C_m. A quadrilateral snake is obtained from a path by identifying each edge of the path with an edge of the cycle C_4. If m number of pendant vertices is attached at each vertex of G, then the resultant graph obtained from G is the graph $G \odot mK_1$. When m = 1, $G \odot K_1$ is the corona of G. The H-graph of a path P_n, denoted by H_n is the graph obtained from two copies of P_n

with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even. The slanting ladder SL_n is a graph obtained from two paths $u_1u_2 \ldots u_n$ and $v_1v_2 \ldots v_n$ by joining each u_i with v_{i+1}, $1 \le i \le n-1$.

The concept of mean labeling was introduced and studied by S.Somasundaram and R. Ponraj [4]. Some new families of mean graphs are discussed in [9, 10]. The concept of super mean labeling was introduced and studied by D. Ramya et al. [3]. Further some more results on super mean graphs are discussed in [2, 5, 6, 7, 8].

In this paper, we establish the supermeanness of the graphs $H_n \odot mK_1$, $T_n \odot K_1$, $Q_n \odot K_1$, $C_n + v_1v_3$, $T_n(C_m)$ and slanting ladder SL_n for $n \geq 2$, $n \neq 3t + 1$, $t \geq 1$.

A vertex labeling of G is an assignment $f : V(G) \to \{1, 2, 3, \ldots, p + q\}$ be an injection. For a vertex labeling f, the induced edge labeling $f^*(e = uv)$ is defined by

$$f^*(e) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

Then f is called super mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, \ldots, p+q\}$. Clearly f^* is injective. A graph that admits a super mean labeling is called super mean graph.

A super mean labeling of the graph $K_{2,4}$ is shown in Figure 1.

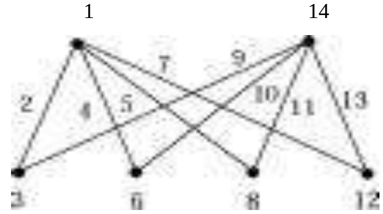


Figure 1.

2 Super Mean Graphs

Theorem 2.1. The graph $H_n \odot mK_1$ is a super mean graph for all positive integers m and n.

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the path of length n – 1. Let $x_{i,k}$ and $y_{i,k}$, $1 \le k \le m$ be the pendant vertices at u_i and v_i respectively, for $1 \le i \le n$. The graph $H_n \odot mK_1$ has 2n(m+1) vertices and 2n(m+1)-1 edges.

Define $f: V(H_n \odot mK_1) \to \{1, 2, 3, \ldots, p + q = 4n(m+1) - 1\}$ as follows:

For $1 \le i \le n$,

$$f(u_i) = \begin{cases} 2(m+1)(i-1)+1, & i \text{ is odd} \\ 2(m+1)i-1, & i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} f(u_i)+2n(m+1)+2m, & i \text{ is odd and } n \text{ is odd} \\ f(u_i)+2n(m+1)-2m, & i \text{ is even and } n \text{ is odd} \\ f(u_i)+2n(m+1), & n \text{ is even.} \end{cases}$$

For $1 \le i \le n$ and $1 \le k \le m$,

$$f(x_{i,k}) = \begin{cases} 2(m+1)(i-1)+4k-1, & i \text{ is odd} \\ 2(m+1)(i-2)+4k+1, & i \text{ is even} \end{cases}$$

$$f(y_{i,k}) = \begin{cases} f(x_{i,k})+2n(m+1)-2m, & i \text{ is odd and } n \text{ is odd} \\ f(x_{i,k})+2n(m+1)+2m, & i \text{ is even and } n \text{ is odd} \\ f(x_{i,k})+2n(m+1), & n \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows:

For $1 \le i \le n-1$,

$$f^*(u_i u_{i+1}) = 2i(m+1) \text{ and}$$

$$f^*(v_i v_{i+1}) = f^*(u_i u_{i+1}) + 2n(m+1).$$

For $1 \le i \le n$ and $1 \le k \le m$,

$$f^*(u_i x_{i,k}) = 2(m+1)(i-1)+2k$$

$$f^*(v_i y_{i,k}) = f^*(u_i x_{i,k})+2n(m+1)$$

$$f^*(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}) = 2n(m+1) \qquad \text{if } n \text{ is odd}$$

$$f^*(u_{\frac{n+2}{2}} v_{\frac{n}{2}}) = 2n(m+1) \qquad \text{if } n \text{ is even.}$$

Thus, f is a super mean labeling of $H_n \odot mK_1$. Hence $H_n \odot mK_1$ is a a super mean graph for all positive integers m and n.

For example, a super mean labeling of $H_5 \odot 4K_1$ and $H_4 \odot 5K_1$ are shown in Figure 2.

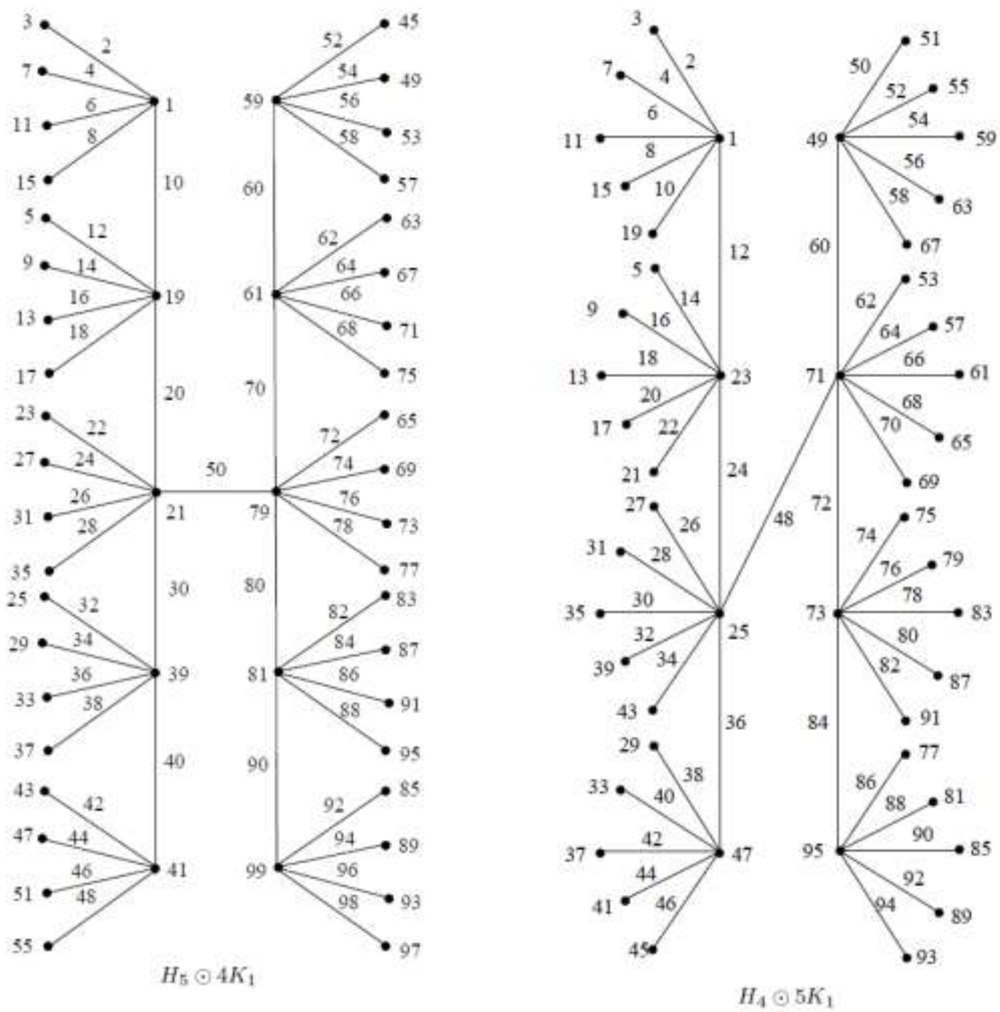


Figure 2.

□

Theorem 2.2. The graph $T_n \odot K_1$ is a super mean graph, for n ≥1.

Proof. Let $u_1, u_2, \ldots, u_n, u_{n+1}$ be the vertices on the path of length n in T_n and let v_i, $1 \le i \le n$ be the vertices of T_n in which v_i is adjacent to u_i and u_{i+1}. Let $v_i'v_i$ be the path attached at each v_i, $1 \le i \le n$ and $u_i'u_i$ be the path attached at each u_i, $1 \le i \le n+1$. The graph $T_n \odot K_1$ has 4n + 2 vertices and 5n +1 edges.

Define $f : V(T_n \odot K_1) \rightarrow \{1, 2, 3, \ldots, p + q = 9n + 3\}$ as follows:

$f(u_i) = 9i - 6, \quad 1 \leq i \leq n+1$

$f(v_i) = 9i - 4, \quad 1 \leq i \leq n$

$f(v_i') = 9i - 2, \quad 1 \leq i \leq n$

$f(u_i') = 9i - 8, \quad 1 \leq i \leq n+1.$

For the vertex labeling f, the induced edge labeling f^* is given as follows:

$f^*(u_iu_{i+1}) = 9i - 1, \quad 1 \leq i \leq n$

$f^*(u_iv_i) = 9i - 5, \quad 1 \leq i \leq n$

$f^*(v_iu_{i+1}) = 9i, \quad 1 \leq i \leq n$

$f^*(v_iv_i') = 9i - 3, \quad 1 \leq i \leq n$

$f^*(u_iu_i') = 9i - 7, \quad 1 \leq i \leq n+1.$

Thus, f is a super mean labeling and hence $T_n \odot K_1$ is a super mean graph, for $n \geq 1$.

For example, a super mean labeling of $T_5 \odot K_1$ is shown in Figure 3.

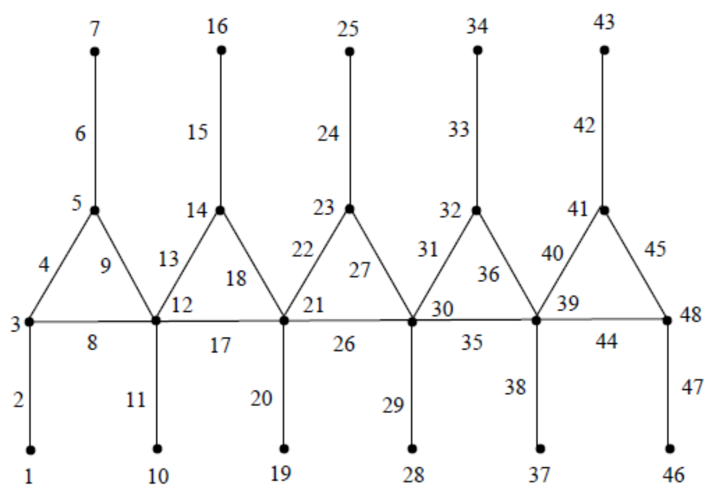


Figure 3.

□

Theorem 2.3. The graph $Q_n \odot K_1$ is a super mean graph for $n \geq 1$.

Proof. Let $u_1, u_2, \ldots, u_n, u_{n+1}$ be the vertices on the path of length n in Q_n and let v_i and w_i be the vertices of Q_n in which v_i is adjacent to u_i and w_i is adjacent to u_{i+1}, for each i, $1 \leq i \leq n$. Let $v_i'v_i$, $w_i'w_i$ be the path attached at each v_i, w_i respectively for each i, $1 \leq i \leq n$ and $u_i'u_i$ be the path attached at each u_i, $1 \leq i \leq n+1$.

The graph $Q_n \odot K_1$ has 6n + 2 vertices and 7n + 1 edges.

Define $f : V(Q_n \odot K_1) \rightarrow \{1, 2, 3, \ldots, p + q = 13n + 3\}$ as follows:

$f(u_1) = 3$; $f(u_i) = 13i - 14$, $2 \leq i \leq n+1$

$f(v_1) = 5$; $f(v_i) = 13i - 5$, $2 \leq i \leq n$

$f(w_1) = 14$; $f(w_i) = 13i - 3$, $2 \leq i \leq n$

$f(u_1') = 1$; $f(u_2') = 9$; $f(u_i') = 13i - 10$, $3 \leq i \leq n+1$

$f(v_1') = 7$; $f(v_i') = 13i - 8$, $2 \leq i \leq n$

$f(w_1') = 16$; $f(w_i') = 13i + 2$, $2 \leq i \leq n$.

For the vertex labeling f, the induced edge labeling f^* is given as follows:

$f^*(u_i v_i) = 13i - 9$, $1 \leq i \leq n$

$f^*(v_1 w_1) = 10$

$f^*(v_i w_i) = 13i - 4$, $2 \leq i \leq n+1$

$f^*(w_1 u_2) = 13$

$f^*(w_i u_{i+1}) = 13i - 2$, $2 \leq i \leq n$

$f^*(u_1 u_2) = 8$

$f^*(u_i u_{i+1}) = 13i - 7$, $2 \leq i \leq n$

$f^*(u_1' u_1) = 2$

$f^*(u_2' u_2) = 11$

$f^*(u_i' u_i) = 13i + 1$, $3 \leq i \leq n+1$

$f^*(v_1' v_1) = 6$

$f^*(v_i' v_i) = 13i - 6$, $2 \leq i \leq n$

$f^*(w_1' w_1) = 15$

$f^*(w_i' w_i) = 13i$, $2 \leq i \leq n$.

Hence, f is a super mean labeling and hence $Q_n \odot K_1$ is a super mean graph for $n \geq 1$.

For example, a super mean labeling of $Q_5 \odot K_1$ is shown in Figure 4.

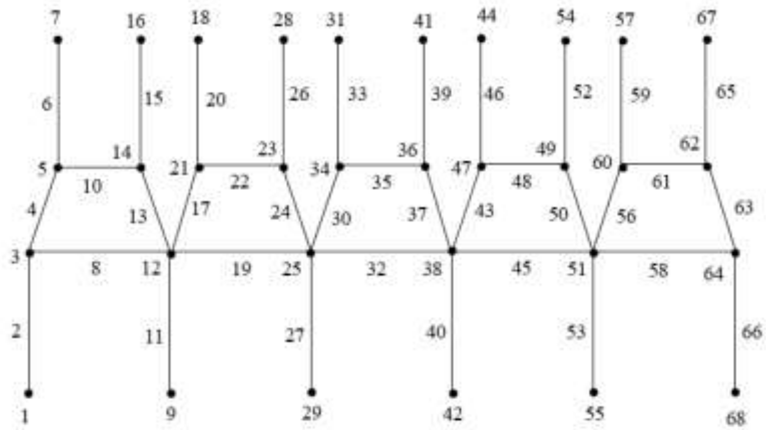


Figure 4.

□

Theorem 2.4. The graph $C_n + v_1v_3$ is a super mean graph for $n \geq 5$.

Proof. Let C_n be a cycle with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_n$ and e' be the edges joining v_1v_3. The graph $C_n + v_1v_3$ has n vertices and n +1 edges.

Define $f : V(C_n + v_1v_3) \to \{1, 2, 3, \ldots, p + q = 2n + 1\}$ as follows:

Case (i) : n is odd.

$$f(v_i) = \begin{cases} 6, & i = 1 \\ 1, & i = 2 \\ 3, & i = 3 \\ 10, & i = 4 \\ 4i - 5, & 5 \leq i \leq \dfrac{n+3}{2} \\ 4(n-i) + 8, & \dfrac{n+3}{2} + 1 \leq i \leq n - 1 \\ 9, & i = n \end{cases}$$

For the vertex labeling *f*, the induced edge labels are obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 4, & i=1 \\ 2, & i=2 \\ 7, & i=3 \\ 4i-3, & 4 \le i \le \frac{n+1}{2} \\ 4(n-i)+6 & \frac{n+3}{2} \le i \le n-2 \\ 11, & i=n-1 \end{cases}$$

$$f^*(v_1v_3)=5 \quad and$$
$$f^*(v_nv_1)=8.$$

Case (ii): n is even.

$$f(v_i) = \begin{cases} 6, & i=1 \\ 1, & i=2 \\ 3, & i=3 \\ 4i-6, & 4 \le i \le \frac{n+2}{2} \\ 4(n-i)+9 & \frac{n+4}{2}+1 \le i \le n. \end{cases}$$

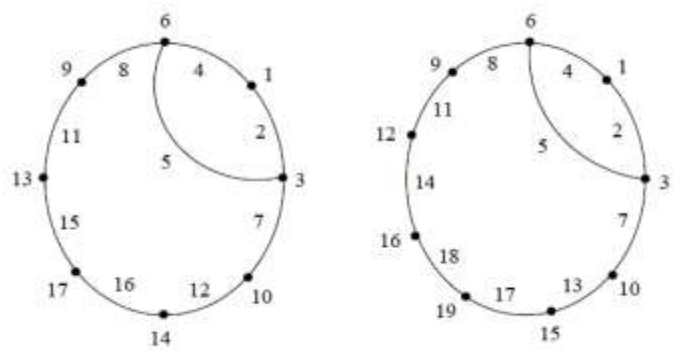
For the vertex labeling *f*, the induced edge labels are obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 4, & i=1 \\ 2, & i=2 \\ 7, & i=3 \\ 4i-4, & 4 \le i \le \frac{n+2}{2} \\ 4(n-i)+7 & \frac{n+4}{2} \le i \le n-1 \end{cases}$$

$$f^*(v_1v_3)=5 \quad and$$
$$f^*(v_nv_1)=8.$$

Hence, f is a super mean labeling and hence $C_n + v_1v_3$ is a super mean graph for n $\geq$5.

For example, a super mean labeling of $C_8 + v_1v_3$ and $C_9 + v_1v_3$ are shown in Figure 5.



$C_8 + v_1v_3$ $C_9 + v_1v_3$

Figure 5.

□

Theorem 2.5. If C_m is a super mean graph, then $T_n(C_m)$ is also a super mean graph for $n \geq 1$.

Proof. Let f be a super mean labeling of $C_m (m \neq 4)$ with vertices $u_1, u_2, \ldots, u_m$. Let $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, v_{n+1}$ and $w_1, w_2, \ldots, w_n$ be the vertices of $T_n(C_m)$. The graph $T_n(C_m)$ has m + 2n vertices and m + 3n edges.

We define $g : V(T_n(C_m)) \rightarrow \{1, 2, 3, \ldots, p + q = 2m + 5n\}$ as follows:

$g(u_i) = f(u_i)$, $1 \leq i \leq m$

$g(v_i) = 2m + 5(i - 1)$, $1 \leq i \leq n + 1$

$g(w_i) = 2(m + 1) + 5(i - 1)$, $1 \leq i \leq n$

For the vertex labeling g, the induced edge labeling g^* is defined as follows:

$g^*(u_iu_{i+1}) = f^*(u_iu_{i+1})$, $1 \leq i \leq m$

$g^*(v_iv_{i+1}) = 2m + 5i - 2$, $1 \leq i \leq n$

$g^*(v_iw_i) = 2m + 5i - 4$, $1 \leq i \leq n$

$g^*(w_iv_{i+1}) = 2m + 5i - 1$, $1 \leq i \leq n$

It can be easily verified that g is a super mean labeling and hence $T_n(C_m)$ is a super mean graph for $n \geq 1$, $m \geq 3$ and $m \neq 4$.

For example, a super mean labeling of C_9, C_{10}, $T_4(C_9)$, $T_4(C_{10})$ are shown in Figure 6.

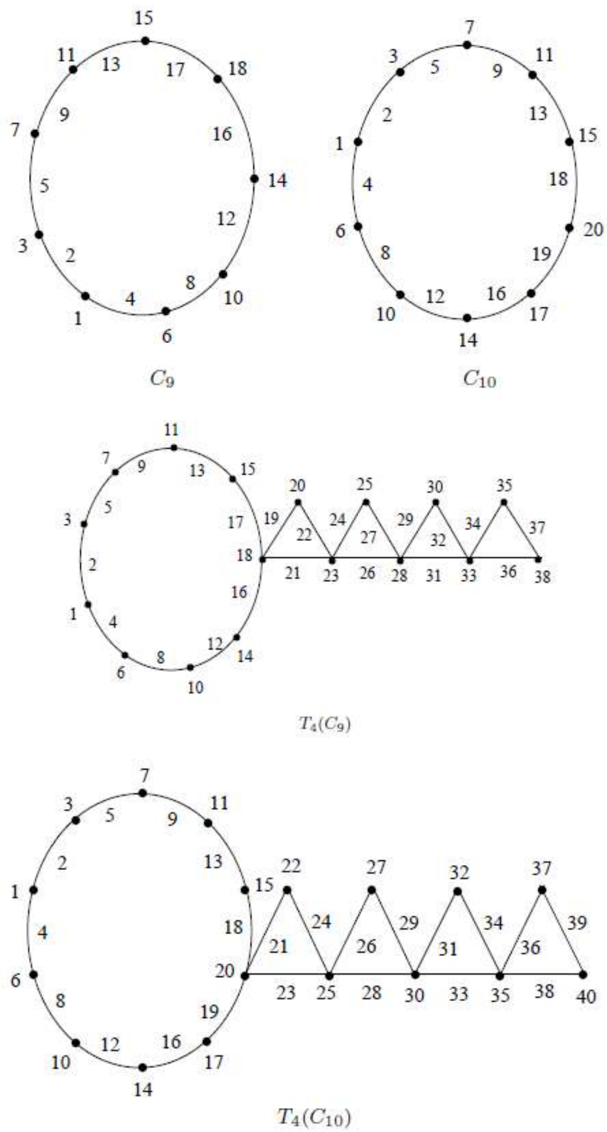


Figure 6.

□

Remark 2.6. C_4 is not a super mean graph, but $T_n(C_4)$ is a super mean graph for $n \geq 1$.

A super mean labeling of $T_3(C_4)$ is shown in Figure 7.

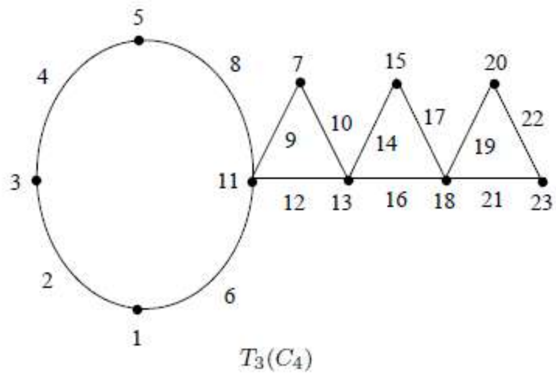


Figure 7.

□

Theorem 2.7. The slanting ladder SL_n is a super mean graph, for $n \geq 2$ and $n \neq 3t + 1$, $t \geq 1$.

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the paths of length n–1. The graph SL_n has 2n vertices and 3(n–1) edges.

We define $f: V(SL_n) \rightarrow \{1, 2, \ldots, p + q = 5n - 3\}$ as follows:

$f(u_{3i-2}) = 15i - 10,\quad 1 \leq i \leq n - 2$

$f(u_{3i-4}) = 15i - 23,\quad 2 \leq i \leq n - 1$

$f(u_{3i-6}) = 15i - 34,\quad 3 \leq i \leq n$

$f(v_1) = 1$

$f(v_{3i-4}) = 15i - 27,\quad 2 \leq i \leq n - 1$

$f(v_{3i-6}) = 15i - 33,\quad 3 \leq i \leq n$

$f(v_{3i-8}) = 15i - 45,\quad 4 \leq i \leq n - 2.$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$f^*(u_{3i-2}u_{3i-1}) = 15i - 9,\quad 1 \leq i \leq n - 2$

$f^*(u_{3i-4}u_{3i-3}) = 15i - 21,\quad 2 \leq i \leq n - 1$

$f^*(u_{3i-6}u_{3i-5}) = 15i - 29,\quad 3 \leq i \leq n - 3$

$f^*(v_1v_2) = 2$

$f^*(v_{3i-4}v_{3i-3}) = 15i - 22,\quad 2 \leq i \leq n - 1$

$f^*(v_{3i-6}v_{3i-5}) = 15i - 31,\quad 3 \leq i \leq n - 3$

$f^*(v_{3i-8}v_{3i-7}) = 15i - 43,\quad 4 \leq i \leq n - 2$

$f^*(u_1v_2) = 4$

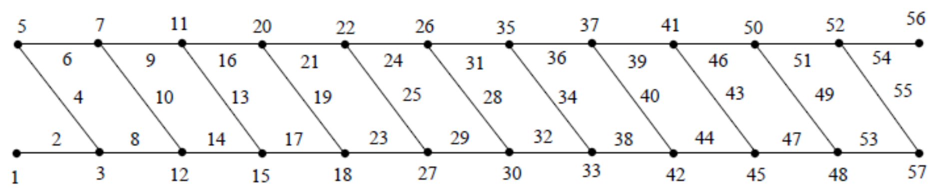
$f^*(u_{3i-4}v_{3i-3}) = 15i - 20, \quad 2 \leq i \leq n - 1$

$f^*(u_{3i-6}v_{3i-5}) = 15i - 32, \quad 3 \leq i \leq n - 3$

$f^*(u_{3i-8}v_{3i-7}) = 15i - 41, \quad 4 \leq i \leq n - 2.$

Thus, f is a super mean labeling and hence SL_n is a super mean graph, for $n \geq 2$ and $n \neq 3t + 1, t \geq 1$.

For example, a super mean labeling of SL_{12} is shown in Figure 8.



SL_{12}

Figure 8.

□

Problem 2.8. Super meanness of SL_n for $n = 3t + 1, t \geq 1$ is to be discussed.

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