

Scale Invariant Limit and Emergence of Complexity: Applications to Traffic Flow

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Abstract- Some novel applications of recently developed analytic formalism involving a nonlinear variations in the usual definition of limit are studied in the context of some simple classically non smooth systems. A Gell-mann-Low type renormalization group equation is derived indicating inherent scale invariance as well as an effective running of scales. A few elementary models of traffic flow are examined as simple prototypes of intelligent systems.

Key Words- Scale invariance, Ultrametric, Renormalization group, Traffic flow.

I. Introduction

Consider the well known limiting equality in the ordinary calculus: $\lim_{x \rightarrow 0} x = 0$. This paper deals not with the equality but explores rather the possibility of a variation that might be triggered into the flow (motion) of the real variable x as it approaches 0 [1]. Classically, of course, the answer is trivial: x moves toward 0 linearly with uniform rate 1. Previously, we have mostly discussed non classical but analytic consequences of such a variation [1,2,3,4]. Here, we present a few applications of new non classical results to some simple problems in traffic flow [6,7].

Let us begin by noting that, the dimensionless variable x , introduced above, as, in fact, measured in unit (scale) of (the uniform rate) 1. The ordinary calculus formulated on the basis of the test book (C, δ) definition of limit is therefore scale dependent. More importantly, real values (numbers) assumed (represented) by x also depend on the choice of this a priori scale 1, leading to the supposedly indistinguishable nature of a real number and its value. A somewhat counterintuitive, but nevertheless, simple fact is the variability in the values of a particular real number as the associated scale is varied. For instance, the number 2 has the value 2 in the scale of 1 unit, but would have value $\frac{1}{2}$ in the scale of 4 unit. Such a simple scale variation might indeed arise when a dynamical variable undergoes linear motion following the scale invariant equation

$$t \frac{dx}{dt} = x \quad (1)$$

When vanishing of acceleration may be violated, for instance, at a finite number of moments (or a countable set, at the most). For instance, assuming an impulsive force acting momentarily at $t = 1$, (1) may be said to support the piecewise continuous solution

$$x = at, \quad 0 < t < 1, \text{ and } x = bt, \quad t > 1 \quad (2)$$

Clearly, x changes at uniform rate a till $t \leq 1$, and then jumps suddenly over to the rate b at an instant $t \geq 1$ due to an application of an externally applied impulsive force. Further, the scale change in the motion (2) is made admissible here because of the scale invariance of equation (1). Conventionally, however, one would model this motion as due to a delta function induced impulsive force, viz.,

$$\frac{d^2x}{dt^2} = (b - a) \delta(t - 1), \quad x(0) = 0, \quad \dot{x}(0) = a \quad (3)$$

Which can equivalently be recast as

$$\frac{dx}{dt} = a + V(t), \quad x(0) = 0 \quad (4)$$

Where V denotes the discontinuous jump in the velocity

$$V(t) = a, \quad 0 < t < 1, \quad \text{and} \quad V(t) = b - a, \quad t > 1 \quad (5)$$

Classically, the discontinuous function (2) indeed is a solution, in a rigorous analytic sense, to the equation (1) only when the said equation is assumed to be valid on the deleted set $R \setminus \{1\}$. Our aim here is to realize the above solution instead as a smooth solution of (1) under a *Principle of inversion induced incremental variability* for a variable, for instance, time t [5]. In that extended analytic formalism (1), along with the classically discontinuous solution (2), now would be well defined everywhere on R . This would then lead to some interesting ramifications.

The advantage of the scale invariant approach over the conventional one based on linear but nevertheless singular ODEs of the form (3) and (4) are indeed manifold. The primary advantage is, of course, the elimination of explicit presence of singularity. The singularities and/ or discontinuities in the original linear approach are hidden in the scale invariant equation if one could succeed proving the existence of *piecewise continuous/ smooth solutions* of the form (2) in the new formalism. Such a solution has been constructed under various setting previously (see [1] and references therein). Further, the scale invariance itself can be interpreted as to offer a mechanism of giving away with the (integrated) forcing term V to appear explicitly in (4). The *precise nature of the forcing term can therefore be considered as of secondary importance; the primary information with regard to a motion is formation of discontinuities and singularities of various kinds on the solution curve as the motion is unfolded*. Piecewise continuous solution of the form (2) indicates application of impulsive force at an instant in an otherwise force free uniform motion. The nature of impulsive force may be of two types: (i) externally applied Newtonian forces of usual classical mechanics, and (ii) non-Newtonian or internal forces that might arise instantaneously in the motion out of certain decision making processes, a novel source of force fields that appears to have been available to animated and/ or intelligent living systems. The scale invariant equation (1) appears to have been ideally suited to handle this class of non-Newtonian forces where exact specification of the imposed force field is difficult and/or irrelevant. We observe that similar shift of emphasis from force field to geometry was considered in the theory of general relativity. The distinctive feature of the present approach is, however, scale invariance and injection of nonlinearity even at the level of differential increments of elementary calculus (see, for instance, [1]).

To give an example of the precise sense of scale invariance that we are talking about, let us cite the case of motor driving on a highway. Two cars left alone on a straight high way in uniform motion from opposite directions would definitely collide. However, when driven intelligently would certainly make instantaneous decisions to cross each other and avoid collisions safely. Exactly in a similar vain columns of cars do avoid collisions by creating and maintaining variable size gaps in between two cars. Such instantaneous decision makings do introduce variations in the uniform velocity gradients of the cars thus mimicking instantaneous internally generated impulsive forces. It is apparent that such internally generated forces and the associated variations in the concerned variable must be independent of a specific scale. In other words, such variations are inherently scale free. More importantly, such impulsive forces have a smoothening effect on the motion of cars rather than engendering singular jerky accidents (jerky breaks means bad driving!). The scale free equation (2) obviously offers an elegant modeling of such intelligent flows, by eliminating the need for introducing delta function like singular forces.

Let us now investigate afresh how a piecewise smooth function of the form (2) is realized as a smooth function in an extended scale invariant setting. Suppose, a dynamical variable x , physically the position of a ball, say, moves from the origin (the reference point) to a point x_0 in time $t_- = 1 - \eta$, $\eta > 0$ at an uniform rate a until there appears a hole at $x = a$ [$x = at, t = 1, x = a$] on the path of the ball. According to the Newtonian law the ball will roll down to the pit and the uniform motion is violated. Supposing on the other hand that the ball is replaced by an animated object (sufficiently rich (complex) intelligent living system), it is very likely that noticing the hole in front of its chosen linear path, the system instantaneously decides to make a jump and cross it quite smoothly. Such a change in motion, in realistic situations, is not at all catastrophic in nature; the only possibility being that the system might simply readjust the original uniform motion to a new uniform rate b , say. Here the

instantaneous impulsive change in the motion is realized in a smooth manner making use of a *freedom of creating an extra amount of space dynamically* that is available in the scale invariant formalism.

In the following two sections, we present the precise analytical formalism leading to the promised scale invariance along with associated non classical consequences [1,2,3,4,5]. In Sec.4, we initiate in a more systematic way applications of the formalism to traffic flow [6, 7] which we consider as a prototype for an intelligent system.

II. Dynamical Creation of Space

We explain the actual sense of space creation dynamically through following example.

Suppose, at first, the (intelligent) ball is moving with uniform velocity 1:

$$\frac{dx}{dt} = 1, \quad x(0) = 0 \quad (6)$$

This is the simplest possible initial value problem (IVP) which is being studied under various guise (for instance, in [5]) with an aim of constructing a class of so called *generalized solutions*. In the present context, $x = t - 1$ is the distance of the ball from the hole, so that hole is reached at time $t = 1$. Using translation invariance, we replace $t - 1$ by t and assume $t \rightarrow 0^+$, thus establishing the relevance of the IVP in the present context. We denote the presence of a singularity at 0 because of the hole. The motion now obviously has the scale invariant form (1), defined originally on the deleted neighbourhood of $I = (-1,1)/\{0\}$, say. Since, $t = 0$ is unattainable, and the size of the pinched hole of I has no positive lower bound, it might be imagined to have the shape of a totally disconnected Cantor set having a countable number of disjoint gaps (open intervals) of arbitrarily small lengths. Indeed, exploiting scale invariance, (1) may be rewritten as

$$t_1 \frac{dx}{dt_1} = x \quad (7)$$

Where $t_1 = \frac{t}{\delta}$ and $0 < \delta < t$ so that as $t \rightarrow 0^+$, $\delta \rightarrow 0^+$ in such a manner that $t_1 = \delta^{-v(t^\sim(t))}$ goes to zero at a much slower rate. However, v denotes an ultrametric valuation defined over a class of infinitesimals $t^\sim(t)$ those are assumed to reside in $(0, \delta)$ [1,2,4] (see below for more details). We simply note here that the scale δ may be identified with an accuracy level beyond which ie, elements of $(0, \delta)$, are practically invisible (undetactable) relative to the real variable $t > \delta$.

Let $T := \log_{\delta^{-1}} t_1 = v(t^\sim)$. We call T as a dressed (deformed) value for the original linear time t relative to the scale δ . Exactly in a similar manner we also write $X := \log_{\delta^{-1}} x_1$ where $x_1 = \frac{x}{\delta}$.

We now have an important observation. The limit $\delta \rightarrow 0^+$ realizes a nonclassical extension of the ordinary linear neighbourhood of 0 of the form $(-\delta, \delta)$ into a topologically inequivalent ultrametric neighbourhood U^\sim so that the singleton set $\{0\}$ of \mathbb{R} is extended into a Cantor like set C^\sim (more precisely, the gaps of C^\sim) accommodating infinitesimals t^\sim . The valuation $v(t^\sim)$ now replaces the ordinary linear measure t for the closed interval $[0, t]$ by the nonlinear, but nevertheless, smooth measure $dT = v(t^\sim)$ defined instead over an infinitesimal neighbourhood of $C^\sim \subset (-\delta, \delta)$, as $\delta \rightarrow 0^+$. Clearly, the original initial value $\dot{x}(0) = 1$ for (6) gets extended smoothly over to a differential equation over C^\sim in the form

$$\frac{dX}{dT} = 1, T \neq 0 \Rightarrow t \frac{\partial x}{\partial t} = x \quad (8)$$

Using the deformed variables X and T , thereby replacing the original ODE (6) on, say, the deleted neighbourhood $(-1,1)/\{0\}$, onto an infinitesimally small (deleted) neighbourhood C^\sim of 0. Accordingly, the original $t - x$ plane in the neighbourhood of 0 is extended over to a $T - X$ plane (realized as a subset of the product space $C^\sim \times C^\sim$), which is nothing but the $t - x$ plane in the log-log scale. In [1], the valuation v is shown to represent a multifractal measure defined by the scaling law $v(t^\sim(t)) = \alpha \delta (1 + \sum \beta_i \delta^{s_i})$ where $0 < \alpha, |\beta_i| < 1$, $0 < s_{i+1} < s_i < 1$ and $0 < \delta < t$.

Notice that (8) defines a nontrivial iteration process for the original equation (6) on $\mathbb{R}/\{0\}$ over to smaller and smaller infinitesimal neighbourhoods successively over various higher order logarithmic scales. This process of replicating (6) over finer and finer logarithmic scales, because of continuous rescaling symmetry of (7) (say) in the limit $\delta \rightarrow 0^+$, constitutes an explicit model of topological dynamics leading to generation of space out of nothing. The limit $t \rightarrow 0^+$ simply signifies the vanishing of the linear measure t only, but the freedom of invoking the continuous spectrum of rescalings, as depicted in (8) in conjunction with the inversion mediated transitions of the form $t^\sim(t) \propto \delta^2$ for each fixed δ , opens up a new level of realizing multiplicatively a nonlinear multifractal measure leading to a novel phenomenon of growth of measure introduced in [2]. We give below another very interesting interpretation of this multifractal measure.

Utilizing this mechanism of dynamic generation of an extra piece of space, an intelligent ball, now, would cross the hole smoothly by one or many smaller scale jumps and then would continue to move linearly at the same pace, as if there had not been any hole (obstruction) on its chosen trajectory. Obviously, we experience such smooth manoeuvrings quite often in real world phenomena which in classical scenario would definitely lead to unmanageable singularities and/or discontinuities. Before leading this section, let us remark for later reference, that the above space creation mechanism is actually realized by a class of generalized solution of (6), represented here as $x = \lambda t^{1-v(t^\sim)}$, which coincides with the standard solution only in the trivial case $v(t^\sim) = 0$ (corresponding to $\lambda = 1$, the initial condition being $x(0) = 0$). Notice that such a solution can nontrivially exist only on an ultrametric extension of \mathbb{R} , allowing locally constant functions satisfying $\frac{d}{dt} t^{-v(t^\sim(t))} = 0$ on a set like C^\sim . As an effect, x happens to be an exact solution of the scale free equation (1).

III. Infinitesimals and Valuation

The theory of infinitesimals and the corresponding concept of an ultrametric valuation is being developed in reference [1,2,3,4]. For an easy reading of this paper, we, however, give a brief resume of key concept here. The symbol $t^\sim(t)$, introduced above, represents a class of infinitesimals, defined by the inequality $0 < t^\sim < \delta < t$ along with the inversion rule $\frac{t^\sim}{\delta} \propto \frac{\delta}{t}$. As a consequence, we have the representations $t = \delta \delta^{-v(t^\sim)}$ and $t^\sim(t) = \lambda(\delta) \delta \delta^{v(t^\sim)}$ for constants (in λ): $0 < \lambda(\delta) < 1$ for a given fixed $\delta > 0$. The positive definite function $v(t^\sim) := \lim_{\delta \rightarrow 0^+} \log_{\delta^{-1}} \frac{\delta}{t^\sim}$ denotes the ultrametric valuation defined over the (uncountable) set of infinitesimals t^\sim . Eliminating the explicit presence of the scale we do have more explicit formula (these are new results, being reported first time here) : $t^\sim = \lambda(t) t^{s(t)}$, where $s(t) = \frac{1+v(t^\sim(t))}{1-v(t^\sim(t))}$ and $\lambda(t)$ is either a pure constant or a slowly varying function of t in the sense that $\frac{\log \lambda}{\log t} \rightarrow 0$ as $t \rightarrow 0$. In that limit assuming $v(t^\sim)$ attains a finite nonzero value

($0 < v_0 < 1$ (say) we have $s(t) \rightarrow s_0 > 1$, a constant. But, as shown above, a constant such as s_0 cannot be a pure constant; s_0 could indeed be a locally constant function in the induced ultrametric topology, with explicit variability in higher order, for example, double logarithmic scale $\log \log t^{-1}$ ie. $\frac{ds_0}{dt} = 0$, but

$$\log t^{-1} \frac{ds^\sim(t)}{d \log t^{-1}} = -s^\sim(t), s = 1 + s^\sim \quad (9)$$

in the limit $t \rightarrow 0^+$.

Notice that $s = 1$ when $v = 0$, telling that infinitesimals $t^\sim = \lambda t$ reduce to the trivial infinitesimal 0 of \mathbb{R} . Nontrivial infinitesimals t^\sim actually live and vary in the countable number of gaps $(-\delta, \delta)/C^\sim$ of an ultrametric Cantor subset C^\sim of the interval $(-\delta, \delta)$. As will become clear choice of the Cantor set C^\sim decides the actual functional behavior of the valuation (or weight) $v(t^\sim)$ assigned to the set of infinitesimals. The variable scaling exponent $s(t) > 1$ tells that the set of (positive) infinitesimals t^\sim lying in an open subset of $(0, t)$ at a certain scale t gets further crumbled into a positive measure Cantor like fractal set as $t \rightarrow 0^+$ through smaller and smaller scales (ie, by inversion, higher and higher order logarithmic scales). As a consequence, $v(t^\sim)$ actually represents a multifractal measure defined over $(0, t)$ in the limit $t \rightarrow 0^+[1]$.

We also have an interesting new interpretation not yet reported elsewhere. Recalling the explicit formula $t^\sim(t) = \lambda(t)t^{(s(t^\sim))}$ for infinitesimals in conjunction with the variability equation (9), we now see the connection of the multifractal valuation with the renormalization group β function

$$\log t^{-1} \frac{dt^\sim}{d \log t^{-1}} = \beta(t^\sim) \quad (10)$$

where $\beta(t^\sim) = t^\sim v(t^\sim)(1 - v(t^\sim))$. The equation (10) is the Gell-mann and Low renormalization group equation [8] and explains the precise mechanism of realizing nontrivial scale invariance at the heart of real number system and thus leading to spontaneous running (flow) of time (real) variable over infinitely smaller and/or larger scales. Let us recall that exact scale invariance is achieved when $\beta(t^\sim) = 0$. On ordinary real number system $t^\sim = 0$, trivially, leading to trivial scale invariance of the scale free equation (1) for any finite non zero $\delta > 0$. The analysis of sec.2 now reveals a new nontrivial rescaling symmetry even in the limit $\delta \rightarrow 0^+$. We now see that $\beta(t^\sim)$ can indeed vanish effectively in the vicinity of nontrivial (infinitely small) scales of the form δ when t is allowed to approach closer and closer to δ . This follows from the logarithmic factor appearing in the β function in the conjunction with the inversion rule $t^\sim t = \lambda \delta^2$. Below we reinterpret this effective running of scales in the context of traffic flow.

IV. Traffic Flow: Basic Building Blocks

Traffic flow, in general, and motion of a car on road, in particular, represents a model of what we call flow (motion) of an *intelligent system*. This, in turn, means that flow (motion) of the system is driven by an intelligent agent. A moving car, without a (human/intelligent) driver is a hazard on the road. However, when driven by an intelligent (expert) driver could be a source of joy and freedom for the driver. The art of scale driving rests on a principle of managing gaps between two consecutive cars as well as on the skill of manoeuvring space around cars. In sec.2 we have seen how an obstacle could be avoided smoothly by exploiting the possibility of creating extra piece of space dynamically while in motion.

By a slight variation of the problem defined by (6), we may also offer an explanation how a manned car might avoid collision which would otherwise appear immanent. Suppose two cars are approaching each other at uniform (and also equal) rate from opposite sides of an otherwise lonely lane, so that the separation x goes to zero (ie, cars come just happen to touch bonnet to bonnet) at a time $t = 1$. Translating time t again to 1, and rewriting $t-1$ by t , (6), again is realized in this new setting involving two cars heading from opposite directions. Two well regulated cars without drivers are known to collide for sure, very much in conformity with the classical mechanics. Cars driven by healthy expert drivers are also known to avoid accidents. Such avoidance of accidents are also very natural in the present extended scale invariant mechanics.

We realize that as the two cars close each other, an instantaneous danger signal passes through independently through each of the drivers mind, making them alert, and thus releasing an instinctive flow of intelligent force, of sort, towards reducing speed sufficiently, together with a little manoeuvring skill to create extra amount of space to avoid accident by smoothly cruising past each other, and then again releasing back to their original flow. Mathematically, as $t \rightarrow 0$, (6) gets replaced by its replica (8), in a very small neighbourhood of the two advancing bonnets of the cars, so that we can now realize (8) in the form

$$\frac{dx}{dt} = 1 \Rightarrow \frac{dX}{dT} = \frac{dv}{dT} \Rightarrow \log t^{-1} \frac{d \log x}{d \log t^{-1}} = -v(1 - v) \quad (11)$$

when we recall the relation between v and s and use the variability equation (9) (together with the inherent reparametrization invariance of the 2nd equation in (11)). The β function like equation (11) now gives the precise mechanism of avoidance of head on collision. By invoking the large resource of variability in the multifractal valuation, the classical limit $x = 0$ of two cars separation is screened by transferring the ordinary flow $t \rightarrow 0$ over infinitely large logarithmic scales $\log \log t^{-1}$. The art of intelligent manoeuvring skills thus relate to such infinitely large scales. More interestingly, the flow of intelligent force is subtly encoded into the analytic structure of the multifractal measure $v(t^\sim(t)) = \alpha t(1 + \sum \beta_i t^{s_i})$ where $0 < \alpha, |\beta_i| < 1, 0 < s_{i+1} < s_i < 1$. As a consequence, the parameters α, β_i and exponents s_i actually relate to driver's intelligent response quotients and may perhaps be identified and measured in appropriate controlled experiments on car driving. However, we leave this topic for more detailed investigations in future.

Before closing, we, however, would like to comment that the above mechanisms might also have parallels in a macroscopic formulation to traffic flow. For instance, we consider the simplest model of traffic flow on a one lane highway when cars are moving uniformly (and unidirectionally) with equal speed 1 and also keeping a fixed bonnet to bonnet gap l . However, on a highway traffic, keeping a fixed gap l is hardly realistic; instead l , indeed is realized as a locally constant function, with smooth variability over microscopic Cantor sets those might arise dynamically with the flow. Because of variability in drivers' responses as encoded into the multifractal measure ν , l will continue to oscillate in a highly unpredictable, nonlinear manner leading naturally to occasional jam like conditions without any apparent external reasons when the flow continues over quite a long time. The unpredictable speeding down of a car because of a sudden fluctuation in l of the front car would create a backward flow of slowing down cars resulting in an ultimate almost zero speed car. According to the driver of that zero speed car, the fluctuating gap size l has already touched an expected lower bound of safe driving.

More complex traffic flow models will be considered elsewhere.

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