Sum of Squares of Consecutive Primes using Maximal Gap

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Abstract

In this paper an attempt has been made to find the sum of squares of consecutive primes using $G(x)$, familiarly known as the maximal gap between consecutive primes. Here conjectures of [1,2] namely

$$G(x) \sim logx(logx - 2loglogx + C),$$

$$G(x) \sim (logx)^2,$$

$$G(x) \sim logx(logx + logloglogx)$$

have been considered. Relations among values of x and the gaps between consecutive primes are presented here. The results are analyzed for the primes $\leq 10^6$ and to a $gap\ of\ 72$.

Keywords- Maximal gaps between primes, Polygonal numbers.

Notations-$t_{m,n}$- polygonal number of rank n and side m.$[x]$-integral part of x.

Introduction

In [3] Guangshi Lii showed that each sufficiently large integer $N \not\equiv 1(3)$ can be written as $p_1 + p_2^2 + p_3^2 + p_4^2$ with $\left|p - \frac{N}{5}\right| \leq \sqrt{\frac{N}{5}}U$ where $U = N^{\frac{41}{100}+\epsilon}$ and p, p_j are primes. This result was an improvement on a previous result with $U = N^{\frac{41}{100}+\epsilon}$ replaced by $U = N^{\frac{5}{11}+\epsilon}$ .Continued with the study of [3] the exceptional set of integers not restricted by elementary congruence conditions which cannot be represented as sums of 3 or 4 squares of primes was improved in [4,5,6,7]. Motivated by the above results, we propose to find the sum of squares of consecutive primes.

Here, our study is based on $G(x)$, the maximal gap between consecutive primes. Though there are many papers providing bounds for $G(x)$, for analytic calculation we use only three conjectures which provide approximate values for sum of squares of consecutive primes.

Analysis

Denoting the n^{th} prime as p_n, p_{n+1} is taken as $p_n + G(x)$. Here x is taken as an integer which provides the gap.

The 3 conjectures used are

$$G_1(x) \sim logx(logx - 2loglogx + C),$$

$$G_2(x) \sim (logx)^2,$$

$$G_3(x) \sim logx(logx + logloglogx)$$

In the first conjecture given here the value of x which gives the gap is nearly 2.5 times the gap. In symbols, $x \sim gap \times 2.5$. In the second and third conjectures $x \sim gap \times .25$

From the illustration exhibited below in table I it is observed that the sum of squares using the first conjecture gives the value nearest to the actual sum of squares.

Table I: Numerical Illustration

P_n	P_{n+1}	gap	$P_n^2 + P_{n+1}^2$	$P_n^2 + (P_n + G_1(x))^2$	$P_n^2 + (P_n + G_2(x))^2$	$P_n^2 + (P_n + G_3(x))^2$
99989	99991	2	19996000202	19996020371	19996118250	19996049045
99877	99881	4	19951629290	19951630887	19951694028	19951689627
99923	99929	6	19970410970	19970420337	19970445897	19970496018
99809	99817	8	19925269970	19925272002	19925275380	19925298330
99377	99391	14	19754359010	19754359834	19754388157	19754391220
99971	99989	18	19992000962	19992001152	19992010903	19992003813
99881	99901	20	19956423962	19956424024	19956433262	19956428177
99529	99551	22	19816423442	19816424805	19816425183	19816440344
88609	88643	34	15709136330	15709136737	15709138254	15709139327
98737	98773	36	19505100698	19505100798	19505104052	19505105974
98057	98101	44	19238981450	19238981536	19238981629	19238984741
95651	95707	56	18308943650	18308943654	18308945059	18308945031

Definition:

We choose x in such a way that $G(x)$ gives the gap between consecutive primes. Index of the gap is defined as $\left[\frac{x}{gap}\right]$ .We present below some of the indices of the $gaps$.

TableII:Some of the indices of the gaps

Gap	Index [1]	Index [2]	Index[3]
2	$\left[\frac{10}{2}\right]$	$\left[\frac{5}{2}\right]$	$\left[\frac{6}{2}\right]$
10	$\left[\frac{140}{10}\right]$	$\left[\frac{24}{10}\right]$	$\left[\frac{23}{10}\right]$
26	$\left[\frac{1432}{26}\right]$	$\left[\frac{164}{26}\right]$	$\left[\frac{131}{26}\right]$
34	$\left[\frac{3304}{34}\right]$	$\left[\frac{341}{34}\right]$	$\left[\frac{262}{34}\right]$
46	$\left[\frac{9608}{46}\right]$	$\left[\frac{883}{46}\right]$	$\left[\frac{651}{46}\right]$
58	$\left[\frac{24163}{58}\right]$	$\left[\frac{2030}{58}\right]$	$\left[\frac{1452}{58}\right]$
72	$\left[\frac{62625}{72}\right]$	$\left[\frac{4843}{72}\right]$	$\left[\frac{3374}{72}\right]$

In the table II above Index [j] represents index of $gap\ G_j, j = 1,2,3$ respectively

Observation:

If the index $[j], j = 1,2,3$ is represented as $\left[\frac{p_i}{q_i}\right]$, $i = 1\ to\ 36$, it satisfies the relations

$$\left[\frac{p_{i+\alpha}+p_{i+\gamma}}{p_{i+\beta}}\right] = \left[\frac{q_{i+\alpha}+q_{i+\gamma}}{q_{i+\beta}}\right]\ and\ \left[\frac{p_{i+\alpha}+p_{i+\gamma}}{p_{i+\beta}}\right] = 1$$

where $0 \leq \alpha < \beta < \gamma, 0 \leq i+\alpha \leq 33, 3 \leq i+\gamma \leq 36$

In Particular, $$\left[\frac{p_{i-1}+p_{i+1}}{p_i}\right] = \frac{q_{i-1}+q_{i+1}}{q_i}$$

Also it is noted that $$\left[\frac{p_{i-1}+p_{i+1}}{p_i}\right] = 2.$$

For simplicity and clear understanding a few example are given below for $G_1(x)$

For $i = 1$ $\left[\frac{p_0+p_2}{p_1}\right] = \left[\frac{10+55}{28}\right]$; $\frac{q_0+q_2}{q_1} = \frac{2+6}{4}$

For $i = 18$ $\left[\frac{p_{17}+p_{19}}{p_{18}}\right] = \left[\frac{3999+5761}{4813}\right]$; $\frac{q_{17}+q_{19}}{q_{18}} = \frac{36+40}{38}$

For $i = 35$ $\left[\frac{p_{34}+p_{36}}{p_{35}}\right] = \left[\frac{48239+62625}{55019}\right]$; $\frac{q_{34}+q_{36}}{q_{35}} = \frac{68+72}{70}$

Similar results are true for $G_2(x)$ and $G_3(x)$.

Representation of x in terms of polygonal numbers:

We represent x as a sum of a polygonal number and sum of squares of integers.

Proof:

Let the gap between primes be m.

We Know that $t_{m,n} = \frac{(m-2)n(n-1)}{2} + n$ denotes a polygonal number of rank n and side m.

Taking $m = gap$, we have $\frac{(m-2)n(n-1)}{2} + n = x$

$$(m-2)n^2 - (m-4)n - 2x = 0$$

Treating the above equation as a quadratic in n and solving for n, we have

$$n = \frac{(m-4) \pm \sqrt{(m-4)^2 + 4(m-2)(2x)}}{2(m-2)} = \mathcal{N}, say$$

The negative sign before the square root on the right hand side of the above equation is neglected since n is to be considered as the rank of polygonal number. For given $x \; and \; m$,consider n as $n = [\mathcal{N}]$. It is observed that the value of x is represented as the sum of $t_{m,[\mathcal{N}]}$ and squares of integers. A few illustrations are exhibited in Table III below:

Table III: Illustrations.

x in				Representation of x in		
m	$G_1(x)$	$G_2(x)$	$G_3(x)$	$G_1(x)$	$G_2(x)$	$G_3(x)$
6	55	12	13	$t_{6,5} + 3^2 + 1^2$	$t_{6,2} + 2^2 + 1^2 + 1^2$	$t_{6,3} + 2^2 + 1^2 + 1^2 + 1^2$
12	205	32	30	$t_{12,6} + 7^2$	$t_{12,2} + 4^2 + 2^2$	$t_{12,2} + 4^2 + 1^2 + 1^2$
24	1137	135	109	$t_{24,10} + 11^2 + 4^2$	$t_{24,3} + 8^2 + 1^2 + 1^2$	$t_{24,3} + 6^2 + 2^2$
30	2209	240	188	$t_{30,13} + 3^2 + 1^2 + 1^2 + 1^2$	$t_{30,4} + 8^2 + 2^2$	$t_{30,4} + 4^2$
40	5761	559	420	$t_{40,17} + 24^2$	$t_{40,5} + 13^2 + 2^2 + 1^2$	$t_{40,5} + 5^2 + 3^2 + 1^2$
48	11299	1021	748	$t_{48,22} + 25^2 + 5^2 + 1^2$	$t_{48,7} + 6^2 + 3^2 + 1^2 + 1^2 + 1^2$	$t_{48,6} + 7^2 + 1^2 + 1^2 + 1^2$
56	20882	1779	1278	$t_{56,28} + 21^2 + 1^2$	$t_{56,8} + 16^2 + 1^2 + 1^2 + 1^2$	$t_{56,7} + 11^2 + 4^2$
60	27882	2313	1647	$t_{60,31} + 29^2 + 6^2 + 2^2$	$t_{60,9} + 14^2 + 4^2 + 2^2$	$t_{60,8} + 3^2 + 2^2 + 1^2 + 1^2$
64	36846	2981	2107	$t_{64,34} + 45^2 + 2^2 + 1^2$	$t_{64,10} + 13^2 + 3^2 + 1^2 + 1^2 + 1^2$	$t_{64,8} + 19^2 + 1^2 + 1^2$
70	55019	4301	3006	$t_{70,40} + 44^2 + 1^2 + 1^2 + 1^2$	$t_{70,11} + 23^2 + 4^2 + 2^2 + 1^2$	$t_{70,9} + 23^2 + 4^2 + 2^2$

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