Study of Henstock-Kurzweil integrals

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***Abstract*— In this paper a brief introduction to Henstock-Kurzweil integrals is given based on previous studies. This paper explains about definition and some properties of Henstock-Kurzweil integral. Henstock-Kurzweil integral is generalized from Riemann integral. The theory of the Riemann integral was not fully satisfactory. Many important functions do not have Riemann integral. So, Henstock and Kurzweil made the new theory of integral.**

***Keywords*— gauge, δ - fine partition, Henstock-Kurzweil integral.**

I. Introduction

The theory of integration has a long history which dated back to thousand years ago. However the modern theory began with I. Newton (1642 - 1727) and G.W. von Leibniz (1646 - 1716) in the seventeenth century. The idea of fluxions, as Newton called his calculus, was further with application of mechanics, physics and other areas.

The foundation of the modern theory of integration or what we now call classical integration theory, was laid by G.F.B.Riemann (1826 - 1866) in the nineteenth century. This is also the integration theory which is taught in the undergraduate years at the university. However, in 1902, H.Lebesgue (1875 - 1941) following the work of others established what is now known as Lebesgue integral, or in its abstract version measure theory. Of course, many great mathematicians, who came before and after Lebesgue, helped to initiate, to develop, and later perfected the theory. This is the theory that dominates the mathematics arena nowadays. It finds application in virtually every branch of mathematical analysis.

However the Lebesgue integrals has its own defects. For example, it does not integrate the derivatives as the Newton calculus does. An integral that includes Lebesgue and is able to integrate the derivative was first define by A.Denjoy in 1912 and later another version by O.Perron in 1914. It was until 1921 that the two integrals were proved to be equivalent. The fact that it took so many years shows the difficulty of the proof at the time. There has been active research on the Denjoy-Perron integral since then.

Both the Denjoy and the Perron integrals were difficult to handle. The break-through came in 1957-1958 when Henstock and Kurzweil gave independently a Riemann-type definition to the Denjoy-Perron integral. Not only that the definition is now easier, but also the proofs using the Henstock-Kurzweil integral are often simpler.

Here throughout we denote the set of real numbers by R. Now we will see few examples of functions which are neither Riemann integrals nor Lebesgue integrals, but they are Henstock-Kurzweil integral. And so it is important to study Henstock-Kurzweil integrals.

Example 1: Consider the following discontinuous system,
$x' = t^2 x + h(t)$, where $|t| \le 1$, $|x| \le 1$ and $h(t) = \frac{d}{dt}\left(t^2 \sin t^{-2}\right)$.
If $t \neq 0$ and h (0) =0, then $f(t, x) = t^2 x(t) + h(t)$ is a highly oscillating function and is not Lebesgue integrable on $|t| \le 1$. However, with x (0) = 0, the above system has following solution $x(t) = e^{t^3/3} \int_0^t e^{-(s^3/3)} h(s)\,ds$.
The above integral is neither Riemann integral nor Lebesgue integral, it is Henstock-Kurzweil integral.

Example 2: Define a function F on [0, 1] by F (0) = 0, otherwise $F(x) = x^2 \cos(\pi/x^2)$.
The derivative of F is f, given by, f (0) = 0, $f(x) = 2x\cos(\pi/x^2) + (2\pi/x)\sin(\pi/x^2)$, if $x \neq 0$.
The above function f is neither Riemann integrable nor Lebesgue integrable, but f is Henstock-Kurzweil integrable.

II. Basic Definitions and Theory

Definition 1: Let [a, b] be a closed, bounded interval. A tagged partition P of [a, b] consists of partition $\{x_i : 0 \le i \le n\}$ of [a, b] along with a set $\{t_i : 1 \le i \le n\}$ of points that satisfy $x_{i-1} \le t_i \le x_i$ for each i .

Definition 2: Let [a, b] be a closed, bounded interval and Let δ(x): [a, b]→R be a positive function (i.e. δ(x)>0 for all x in [a, b]). A δ- fine tagged partition $P = \{ (t_i, [x_{i-1}, x_i]) : 1 \le i \le n \}$ of [a , b] is a tagged partition of [a , b] that satisfies $[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$ for each $1 \le i \le n$.

Definition 3 : Let f :[a , b]→R and Let $P = \{(t_i, [x_{i-1}, x_i]) : 1 \le i \le n\}$ be a tagged partition of [a , b] . Then the Riemann Sum S(P , f) of f on P is defined by $S(P, f) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$.

Definition 4: A positive function δ: [a, b]→R (i.e. δ (x)>0 for all x in [a, b]) is known as a gauge on [a, b].

Theorem 1:(CousinsLemma) δ - fine partition exists for any gauge δ.

Proof: We prove this theorem by method of contradiction.

Let us assume the contrary. We divide [a, b] in two subintervals [a, c] and [c, b], where c is midpoint of [a, b]. Atleast one of the subintervals does not have δ - fine partition. We denote it by I_1 .

Now, we bisect I_1. Atleast one of the subintervals does not possess a δ - fine partition, and denote it by I_2 .

Continuing we obtain a sequence of nested subintervals $I_n$whose diameter tends to zero.

By Cantor's Intersection Theorem, there exists a unique point x in the intersection. Since diameter tends to zero there exists some n large enough so that $|I_n| \le \delta(x)$. Thus we arrive at a contradiction, and hence the theorem is proved.

Definition 5 : A function f:[a , b]→R is Riemann integrable on [a , b] if there exists a number I such that for each ε>0 there exists δ>0 such that $|S(P, f) - I| < \varepsilon$ for all tagged partitions P of [a , b] with norm of P less than δ .

Now with a small change in the definition of Riemann integrable function we arrive to a more generalized definition and this is the definition of Henstock-Kurzweil integrable function.

Definition 6 :A function f:[a , b]→R is Henstock-Kurzweil integrable on [a , b] if there exists a number I such that for each ε>0 there exists a positive function δ defined on [a , b] such that $|S(P, f) - I| < \varepsilon$ for all δ - fine tagged partition P of [a , b] .

Alternately we can also use the following definition.

Let f be a function that is defined at every point of [a, b]. Then f is said to be Henstock-Kurzweil integrable on [a , b] if it satisfies the following point wise integrability criterion : there is a number I so that for every ε>0 there is a positive function δ:[a , b]→Rwith the property that $| I - \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1}) | < \varepsilon$ whenever points are given $a = x_0 < x_1 < \cdots < x_n = b$ for which $x_i - x_{i-1} < \delta(\alpha_i)$ with associated points α_i in $[x_{i-1}, x_i]$.

Theorem 2: Henstock-Kurzweil integral is well defined i.e. if f is a Henstock-Kurzweil integrable function on an interval [a, b], then the Henstock-Kurzweil integral of f on [a, b] is unique.

Proof: Suppose that I and I' in R are both Henstock-Kurzweil integrals of f on [a, b].

Let ε>0. Then there exists two positive functions δ_1 and δ_2 defined on [a , b] such that $| S(P, f) - I | < \varepsilon/2$ for all δ_1 - fine tagged partition P of [a , b], and such that $| S(P, f) - I' | < \varepsilon/2$ for all δ_2 - fine tagged partition P of [a , b]. Define a function δ:[a, b]→R by $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$. Then, by the triangle inequality, for all δ - fine tagged partition of [a, b],

$$| I - I' | = | I - S(P, f) + S(P, f) - I' | \le | S(P, f) - I | + | S(P, f) - I' | < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

This implies that I = I’.

It follows that Henstock-Kurzweil integral of f on [a, b] is unique. i.e. It is well defined.

Theorem 3: Suppose that f and g are Henstock-Kurzweil integrable functions defined on [a, b] and that k, c, d are constants. Then

(i)kf is Henstock-Kurzweil integrable on [a , b] and $\int_a^b kf = k\int_a^b f$.

(ii)f + g is Henstock-Kurzweil integrable on [a , b] and $\int_a^b f{+}g = \int_a^b f + \int_a^b g$.

(iii)f - g is Henstock-Kurzweil integrable on [a , b] and $\int_a^b f{-}g = \int_a^b f - \int_a^b g$.

(iv) cf + dg is Henstock-Kurzweil integrable on [a, b] and $\int_a^b cf + dg = c\int_a^b f + d\int_a^b g$.

Proof: Let ε>0.

Then there exists a real number $\int_a^b f$ and a positive function δ such that $| S(P, f) - \int_a^b f | < \frac{\varepsilon}{|k|}$ for all δ - fine tagged partition P of [a, b]. Thus for all δ - fine tagged partitions of [a, b],

$$\left|\sum_{i=1}^{n} k\, f(t_i)(x_i - x_{i-1}) - k\int_a^b f\right| = \left|k \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - k\int_a^b f\right|$$

$$= |k|\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \int_a^b f\right|$$

$$< |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon$$

It follows that kf is Henstock-Kurzweil integrable on [a, b] and $\int_a^b kf = k\int_a^b f$.

Now, if g is also Henstock-Kurzweil integrableon [a , b] with ε>0 , then there exists real numbers $\int_a^b f$ and $\int_a^b g$ and positive functions δ_1 and δ_2 such that $\left| S(P_1, f) - \int_a^b f \right| < \varepsilon/2$ for all δ_1 - fine tagged partitions P_1 of [a , b] and $\left|S(P_2, g) - \int_a^b g\right| < \varepsilon/2$ for all δ_2 - fine tagged partitions P_2 of [a , b].

Define a positive function δ: [a, b]→R by $\delta = \min\{\delta_1(x), \delta_2(x)\}$ for each x in [a , b]. Then for each δ - fine tagged partition P, we have

$$\left|\sum_{i=1}^{n}((f+g)(t_i))(x_i - x_{i-1}) - \left(\int_a^b f + \int_a^b g\right)\right| = \left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \int_a^b f + \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) - \int_a^b g\right|$$

$$\leq \left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \int_a^b f\right| + \left|\sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) - \int_a^b g\right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus, f + g is Henstock-Kurzweil integrable on [a, b] and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

Similarly, we can prove Statement (iii) of the theorem.

Statement (iv) of the theorem can be proved by using Statements (i) and (ii).

Thus, cf + dg is Henstock-Kurzweil integrable on [a, b] and $\int_a^b cf + dg = c\int_a^b f + d\int_a^b g$.

Theorem 4: (Cauchy Criterion for Henstock-Kurzweil integrals) Let f be a function defined on the interval [a, b]. Then f is Henstock-Kurzweil integrable on [a , b] if and only if for each ε>0 there exists a positive function δ :[a , b]→Rsuch that $\left| S(P_1, f) - S(P_2, f)\right| < \varepsilon$ for all δ - fine tagged partitions P_1 and $P_2$of [a , b].

Proof: Let us first assume that f is Henstock-Kurzweil integrable on [a, b].

Let ε>0 and choose a positive function δ:[a, b]→R such that $\left| S(P, f) - \int_a^b f\right| < \varepsilon/2$ for all δ - fine tagged partitions of [a, b].

Now, Let P_1 and P_2 be two δ - fine tagged partitions of [a, b]. Then,

$$| S(P_1, f) - S(P_2, f)| \leq \left|S(P_1, f) - \int_a^b f\right| + \left|S(P_2, f) - \int_a^b f\right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

It follows that $\left| S(P_1, f) - S(P_2, f)\right| < \varepsilon$ for all δ - fine tagged partitions P_1 and P_2 of [a, b].

We will now prove the converse of thisstatement.

Suppose that for each ε>0, there exists a positive function defined on [a, b] such that $\left| S(P_1, f) - S(P_2, f)\right| < \varepsilon$ for all δ - fine tagged partitions P_1 and P_2 of [a, b].

For each positive integer n, choose a positive function δ_n such that $\left| S(P_1, f) - S(P_2, f)\right| < 1/n$ for all δ - fine tagged partitions P_1 and P_2 of [a, b].

Without Loss of Generality, we may assume that the sequence $\{\delta_n\}$ is decreasing (i.e. for each n, $\delta_n(x) \geq \delta_{n+1}(x)$ for all x in [a, b]).

Now, for each n, let P_n be aδ - fine tagged partition of [a, b]. If K is a positive integer, and m,n are positive integers greater than or equal to K, then the tagged partitions P_m and P_n are δ_K - fine tagged partitions, since the sequence $\{\delta_n\}$ is decreasing. It follows that $|S(P_n, f) - S(P_m, f)| < 1/K$.

We can conclude that $\{S(P_n, f)\}$ is a Cauchy sequence. Since every Cauchy sequence of real numbers converges, define I in R to be the limit of this sequence.

Let $\varepsilon>0$. Since $\{S(P_n, f)\}$ converges to I, there exists a positive number N such that $1/N < \varepsilon/2$and $|S(P_n, f)-I| < \varepsilon/2$For all$n\geq N$. Define a positive function δ on [a, b] by $\delta(x)= \delta_N(x)$, and suppose that P is a δ - fine tagged partitions of [a, b]. Then,

$$|S(P, f)-I| \leq |S(P, f)-S(P_N, f)|+|S(P_N, f)-I| < 1/N + \varepsilon/2 < \varepsilon$$

It follows that f is Henstock-Kurzweil integrable on [a, b] and $\int_a^b f = I$.

Theorem 5: If f is Henstock-Kurzweil integrable on [a, b], then f is Henstock-Kurzweil integrable on each subinterval of [a, b].

Proof: Supposethat [c, d] is a subinterval of [a, b] and let $\varepsilon>0$.

Since f is Henstock-Kurzweil integrable on [a, b], there exists a positive function δ defined on [a, b] such that $\left| S(P, f) - \int_a^b f \right| < \varepsilon/2$For all δ - fine tagged partitions of [a, b]. Let P_a be a tagged partition of the interval [a , c] such that $[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$and Let $P_b$be a tagged partition of the interval [d , b] such that $[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$.

Assume that for each partition, each tag occurs only once. Let P_1 and P_2 be two δ - fine tagged partitions of [c, d], and suppose thatin each partition, each tag occurs only once. We note that both $P'_1 = P_a \cup P_1 \cup P_b$ and $P'_2 = P_a \cup P_2 \cup P_b$ are δ - fine tagged partitions of [a, b]. Now,

$$|S(P_1,f)-S(P_2,f)| \leq \left|S(P_1,f)+S(P_a,f)+S(P_b,f)-\int_a^b f\right| + \left|S(P_a,f)+S(P_b,f)+S(P_2,f)-\int_a^b f\right|$$
$$= \left|S(P'_1,f)-\int_a^b f\right| + \left|S(P'_2,f)-\int_a^b f\right|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence by Theorem 4, it follows that f is Henstock-Kurzweil integrable on [c, d].

Theorem 6: Let $f:[a, b]\rightarrow R$and let c be in (a, b). If f is Henstock-Kurzweil integrable on the intervals [a, c] and [c, b] then f is Henstock-Kurzweil integrable on [a, b] and $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof: Let $\varepsilon>0$. Since f is Henstock-Kurzweil integrable on [a , c] and [c , b], there exists positive functions $\delta_1 : [a, c] \rightarrow R$ and $\delta_2 : [c, b] \rightarrow R$ such that $\left| S(P_1, f) - \int_a^c f \right| < \varepsilon/2$ for all δ_1 - fine tagged partition P_1 of [a , c] and $\left| S(P_2, f) - \int_c^b f \right| < \varepsilon/2$ for all δ_2 - fine tagged partition P_2 of [c , b].

Define a positive function δ on [a, b] by

$$\delta(x) = \begin{cases} \min\{\delta_1(x), c-x\}, & x \in [a, c) \\ \min\{\delta_1(c), \delta_2(c)\}, & x=c \\ \min\{\delta_2(x), x-c\}, & x \in (c, b] \end{cases}$$

Let $P=\{(t_i,[x_{i-1}, x_i]) : 1\leq i\leq n\}$ be a δ - fine tagged partition of [a, b]. We note that for any tag less c, the right endpoint of the tag's interval is less than c, and for any tag greater than c, the left endpoint of the tag's interval is greater than c. Thus, c must be a tag of P.

Now, Let N be a positive integer such that $t_i<c$ for all indices $i\leq N$ and $t_i>c$for all indices $i>N$. Then $P_1 = \{(t_i,[x_{i-1},x_i]) : 1\leq i\leq N\} \cup (c,[x_N, c])$ is a δ_1 - fine tagged partition of [a, c]

and $P_2 = \{(t_i, [x_{i-1}, x_i]) : N+1 \leq i \leq n\} \cup (c, [c, x_{N+1}])$ is a δ_2 - fine tagged partition of [c , b].

Let σ_1 denote the set of indices of tags of P_1 and let σ_2 denote the set of indices of tags of P_2. Thus,

$$\left|S(P, f) - \left(\int_a^c f + \int_c^b f\right)\right| \leq \left|\sum_{i\in\sigma_1} f(t_i)(x_i - x_{i-1}) - \int_a^c f\right| + \left|\sum_{i\in\sigma_2} f(t_i)(x_i - x_{i-1}) - \int_c^b f\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

It follows that f is Henstock-Kurzweil integrable on [a, b] and $\int_a^b f = \int_a^c f + \int_c^b f$.

Theorem 7: If f is continuous function on [a, b], then f is Henstock-Kurzweil integrable on [a, b].

Proof: Since f is continuous on [a, b]. Hence for any given $\varepsilon>0$, there exists $\delta(x)>0$ with the property that $|f(y)-f(x)|<\frac{\varepsilon}{2(b-a)}$ for all x, y in [a, b] such that $|y-x|<\delta(x)$.

Here the function $\delta(x)$ is a gauge on [a, b].

Let $P_1=\{(t_i, [u_i, v_i]):1\le i\le p\}$ and $P_2=\{(w_j, [x_j, y_j]) : 1\le j\le q\}$ be two δ - fine tagged partitions of [a , b].

If $[u_i, v_i]\cap[x_j, y_j]$ is a non-empty for some $i\in\{1,2,\cdots,p\}$ and $j\in\{1.2.\cdots,q\}$, we select and fix a point $z_{i,j}\in[u_i, v_i]\cap[x_j, y_j]$.

On the other hand, if $[u_r, v_r]\cap[x_s, y_s]$ is empty for some $r\in\{1, 2, \cdots, p\}$ and $s\in\{1.2.\cdots, q\}$ we set $z_{r,s}=a$.

Let $d(\emptyset)=0$ and Let $d([\alpha, \beta])=\beta-\alpha$ for each pair of real numbers α and β satisfying $\alpha\le\beta$.

$$|S(P_1,f)-S(P_2,f)|=\left|\sum_{i=1}^{p} f(t_i)(v_i-u_i)-\sum_{j=1}^{q} f(w_j)\left(y_j-x_j\right)\right|$$

$$=\left|\sum_{i=1}^{p}\sum_{j=1}^{q} f(t_i)d\left([u_i, v_i]\cap[x_j, y_j]\right)-\sum_{i=1}^{p}\sum_{j=1}^{q} f(w_i)d\left([u_i, v_i]\cap[x_j, y_j]\right)\right|$$

$$\le\left|\sum_{i=1}^{p}\sum_{j=1}^{q}\left(f(t_i)-f(z_{i,j})\right)d\left([u_i, v_i]\cap[x_j, y_j]\right)\right|+\left|\sum_{i=1}^{p}\sum_{j=1}^{q}\left(f(w_j)-f(z_{i,j})\right)d\left([u_i, v_i]\cap[x_j, y_j]\right)\right|$$

$$<\frac{\varepsilon}{2(b-a)}(b-a)+\frac{\varepsilon}{2(b-a)}(b-a)=\varepsilon$$

Hence by Theorem 4, f is Henstock-Kurzweil integrable on [a, b].

Theorem 8: If f is Henstock-Kurzweil integrable on [a, b], then f^2 is also Henstock-Kurzweil integrable on [a, b].

Proof: Since f is bounded, there exists a positive real number M such that for all x in [a, b], we have $|f(x)|<M$.

Since f is Henstock-Kurzweil integrable. Let $\varepsilon>0$ and choose a positive function $\delta:[a, b]\to R$.

Let P_1 and P_2 be two δ - fine tagged partition of [a, b]. Therefore by Theorem 4, $|S(P_1, f)-S(P_2, f)|<\varepsilon/_{nM}$

Now, $S(P_1, f^2)=\sum_{i=1}^{n} f^2(t_i)(x_i-x_{i-1})=\sum_{i=1}^{n} f(t_i)f(t_i)(x_i-x_{i-1})\le\sum_{i=1}^{n} f(t_i)\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})=\sum_{i=1}^{n} f(t_i)\, S(P_1, f)$

Therefore, $S(P_1, f^2)\le\sum_{i=1}^{n} f(t_i)\ S(P_1, f)$

Thus, $|S(P_1, f^2)-S(P_2, f^2)|\le|\sum_{i=1}^{n} f(t_i)\ S(P_1, f)-\sum_{i=1}^{n} f(t_i)\ S(P_2, f)|$

$$\le\left|\sum_{i=1}^{n} f(t_i)\right||S(P_1, f)-S(P_2, f)|$$

$$<nM\ \varepsilon/_{nM}=\varepsilon$$

Hence by Theorem 4, f^2 is Henstock-Kurzweil integrable on [a, b].

Theorem 9: If f and g are Henstock-Kurzweil integrable on [a, b], then fg is Henstock-Kurzweil integrable on [a, b].

Proof: Given that f and g are Henstock-Kurzweil integrable on [a, b].

We know that, $fg=\frac{1}{4}\left[(f+g)^2+(f-g)^2\right]$

Hence by Theorem 3 and Theorem 8, fg is Henstock-Kurzweil integrable on [a, b].

Theorem 10: (Fundamental Theorem of Calculus) Let $f:[a, b]\to R$ and Let f be a continuous function on [a, b]. If F is differentiable on (a, b) and $F'(x)=f(x)$ for all x in (a, b), then f is Henstock-Kurzweil integrable on [a, b] and $\int_a^b f=F(b)-F(a)$.

Proof of the theorem can be seen in [3].

III. Conclusion

In this paper, we have seen how a small change in to the definition of the Riemann integral and the introduction of the δ - fine tagged partition has provided us with more generalized and much more stronger form of integral. This allows us to handle some improper Riemann integrals. We have seen a more generalized form of Fundamental Theorem of Calculus and some properties of integrals. Still there are many more results of Henstock-Kurzweil Integrals which are not proved and lot of work can be done in this area.

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