Face and Total face edge product cordial graphs

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Abstract - In this paper, the face edge product cordial labeling of planar graphs T_n for even n, $M(P_n)$ for odd n, the star of cycle C_n for odd n, the graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length and the path union of k copies of cycle C_n except for odd k and even n are presented. We also discussed the total face edge product cordial labeling of f_n, W_n and the star of cycle C_n and face product cordial labeling of the graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length and the path union of k copies of cycle C_n except for odd k and even n.

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I. Introduction

By a graph, we mean a simple, finite, planar and undirected unless otherwise specified. A (p,q) planar graph G means a graph G = (V,E), where V is the set of vertices with |V| = p, E is the set of edges with |E| = q and F is the set of interior faces of G with |F| = number of interior faces of G, for terms not defined here, we refer to Harary [4]. For standard terminology and notations related to graph labeling, we refer to Gallian [3]. In [2], Cahit introduce the concept of cordial labeling of graph. The concept of product cordial labeling of a graph was introduced by Sundaram et.al., [9]. In [10], Sundaram et al. also have introduced total product cordial labeling of graph. The concept of signed product cordial labeling was introduced by Baskar Babujee et al. [1]. In [11], Vaidya et al. introduced the concept of edge product cordial labeling of graph. The edge product cordial labeling of various types of graph are presented in [12]. The concept of total edge product cordial labeling is introduced by Vaidya et al. [13]. Sedlacek [8] defined a graph to be magic if it had an edge-labeling, with range the real numbers, such that the sum of the labels around any vertex equals some constant, independent of the choice of vertex. In 1983, Lih [7] introduced magic labelings of planar graphs where labels extended to faces as well as edges and vertices, an idea which he traced back to 13th century Chinese roots. Motivated by the concept of various types of product cordial labeling and magic labeling, we introduce face product cordial labeling, total face product cordial labeling, face edge product cordial labeling, total face edge product cordial labeling, face signed product cordial labeling and total face signed product cordial labeling of graph. In [5], Lawrence et al. proved the face signed product cordial labeling of the Pl_n, $n \geq 5$ except $n \not\equiv 0 \pmod 4$ and the graph $Pl_{m,n}$, $m,n \geq 3$. Here we also prove the total face signed product cordial labeling of the Pl_n, $n \geq 4$ and the graph $Pl_{m,n}$, $m, n \geq 3$. In [6], Lawrence et al. proved the face product cordial labeling of fan, $M(P_n)$, $S(P_n)$ except for odd n, $T(P_n)$, T_n, H_n, S_n except for even n and one vertex union of mC_n and C_{mn}. Here we also proved the total face product cordial labeling of H_n, S_n and W_n. As every edge product cordial graph does not admit face edge product cordial labeling it is very interesting to find out graphs or graph families which admit face edge product cordial labeling. The brief summaries of definition which are necessary for the present investigation are provided below.

Definition : 1.1

A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph.

Definition : 1.2

A mapping $f : V(G) \rightarrow \{0,1\}$ is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f. If for an edge e = uv, the induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$, where $v_f(i)$ = number of vertices of having label i under f and $e_f(i)$ = number of edges of having label i under f*.

Definition : 1.3

A binary vertex labeling f of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.

Remarks : 1.1

G is an edge product cordial planar graph, then $G \cup G$ is also a face edge product cordial graph and any unicyclic edge product cordial graph is also a face edge product cordial graph.

Definition : 1.4

A wheel W_n is a graph with n+1 vertices, formed by connecting a single vertex to all the vertices of cycle C_n. It is denoted by $W_n = C_n + K_1$.

Definition : 1.5

The path union of n copies of G is obtained by adding an edge between G_i to G_{i+1} for i = 1,2,...., n–1, where $G_1, G_2, \ldots, G_n$, $n \geq 2$ be n copies of a fixed graph G.

Definition : 1.6

The shell S_n is the graph obtained by taking n−3 concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan f_{n-1}. Thus $S_n = f_{n-1} = P_{n-1} + K_1$.

Definition : 1.7

The middle graph M(G) of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.

Definition : 1.8

A product cordial labeling of a graph G with vertex set V is a function f from V to {0,1} such that if each edge uv is assigned a label f(u) f(v) then (i) the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and (ii) the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph.

Definition : 1.9

A total product cordial labeling of a graph G with vertex set V is a function f from V to {0, 1} such that if each edge uv is assigned a label f(u) f(v), the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a total product cordial graph.

Definition : 1.10

A vertex labeling of graph G, $f : V(G) \to \{-1,1\}$ with induced edge labeling $f^* : E(G) \to \{-1,1\}$ defined by $f^*(uv) = f(u)\, f(v)$ is called a signed product cordial labeling if $|v_f(-1) - v_f(1)| \leq 1$ and $|e_f(-1) - e_f(1)| \leq 1$, where $v_f(-1)$ is the number of vertices labeled with –1, $v_f(1)$ is the number of vertices labeled with 1, $e_f(-1)$ is the number of edges labeled with –1 and $e_f(1)$ is the number of edges labeled with 1. A graph G is signed product cordial if it admits signed product cordial labeling.

Definition : 1.11

For graph G, the edge labeling function is defined as $f : E(G) \to \{0,1\}$ and induced vertex labeling function $f^* : V(G) \to \{0,1\}$ is given as if $e_1, e_2, \ldots, e_n$ are the edges incident to vertex v then $f^*(v) = f(e_1)\, f(e_2) \ldots f(e_n)$. Let us denote $v_f(i)$ is the number of vertices of G having label i under f^* and $e_f(i)$ is the number of edges of G having label i under f for i = 1,2. f is called edge product cordial labeling of graph G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called edge product cordial if it admits edge product cordial labeling.

Definition : 1.12

For a graph G, an edge labeling function $f^* : E(G) \to \{0, 1\}$ induces a vertex labeling function $f : V(G) \to \{0, 1\}$ defined as $f(v) = \Pi\{f^*(uv)/uv \in E(G)\}$. The function f^* is called a total edge product cordial labeling of G if $|(v_f(0)+e_f(0)) - (v_f(1)+e_f(1))| \leq 1$. A graph is called total edge product cordial if it admits total edge product cordial labeling in G.

Definition : 1.13 [6]

For a planar graph G, the vertex labeling function is defined as $g : V(G) \to \{0,1\}$ and g(v) is called the label of the vertex v of G under g, induced edge labeling function $g^* : E(G) \to \{0,1\}$ is given as if e = uv then $g^*(e) = g(u)\, g(v)$ and induced face labeling function $g^{**} : F(G) \to \{0,1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face f, then $g^{**}(f) = g(v_1)\, g(v_2) \ldots g(v_n)\, g^*(e_1)\, g^*(e_2) \ldots g^*(e_m)$. Let us denote $v_g(i)$ is the number of vertices of G having label i under g, $e_g(i)$ is the number of edges of G having label i under g^* and $f_g(i)$ is the number of interior faces of G having label i under g^{**} for i = 1,2. g is called face product cordial labeling of graph G if $|v_g(0) - v_g(1)| \leq 1$, $|e_g(0) - e_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$. A graph G is face product cordial if it admits face product cordial labeling.

Definition : 1.14 [6]

For a planar graph G, the vertex labeling function is defined as $g : V(G) \to \{0,1\}$ and g(v) is called the label of the vertex v of G under g, induced edge labeling function $g^* : E(G) \to \{0,1\}$ is given as if e = uv then $g^*(e) = g(u)\, g(v)$ and induced face labeling function $g^{**} : F(G) \to \{0,1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face f, then $g^{**}(f) = g(v_1)\, g(v_2) \ldots g(v_n)\, g^*(e_1)\, g^*(e_2) \ldots g^*(e_m)$. Let g(0), g(1) be the sum of the number of vertices, edges and interior faces having labels 0 and 1 respectively. g is called total face product cordial labeling of graph G if $|g(0)-g(1)|\le 1$. A graph G is total face product cordial if it admits total face product cordial labeling.

Definition : 1.15 [5]

For a planar graph G, the vertex labeling function is defined as $g : V(G) \to \{-1,1\}$ and g(v) is called the label of the vertex v of G under g, induced edge labeling function $g^* : E(G) \to \{-1,1\}$ is given as if e = uv then $g^*(e) = g(u)\, g(v)$ and induced face labeling function $g^{**} : F(G) \to \{-1,1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face f , then $g^{**}(f) = g(v_1)\, g(v_2) \ldots g(v_n)\, g^*(e_1)\, g^*(e_2) \ldots g^*(e_m)$. Let us denote $v_g(i)$ is the number of vertices of G having label i under g, $e_g(i)$ is the number of edges of G having label i under g^* and $f_g(i)$ is the number of interior faces of G having label i under g^{**} for $i = -1,1$. g is called face signed product cordial labeling of graph G if $|v_g(-1)-v_g(1)| \le 1$, $|e_g(-1)-e_g(1)| \le 1$ and $|f_g(-1)-f_g(1)| \le 1$. A graph G is face signed product cordial if it admits face signed product cordial labeling.

Definition : 1.16 [5]

For a planar graph G, the vertex labeling function is defined as $g : V(G) \to \{-1,1\}$ and g(v) is called the label of the vertex v of G under g, induced edge labeling function $g^* : E(G) \to \{-1,1\}$ is given as if e = uv then $g^*(e) = g(u)\, g(v)$ and induced face labeling function $g^{**} : F(G) \to \{-1,1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face f, then $g^{**}(f) = g(v_1)\, g(v_2) \ldots g(v_n)\, g^*(e_1)\, g^*(e_2) \ldots g^*(e_m)$. Let g(–1), g(1) be the sum of the number of vertices, edges and interior faces having labels –1 and 1 respectively. g is called total face signed product cordial labeling of graph G if $|g(-1)-g(1)|\le 1$. A graph G is total face signed product cordial if it admits total face signed product cordial labeling.

Definition : 1.17

For a planar graph G, the edge labeling function is defined as $g : E(G) \to \{0,1\}$ and g(e) is called the label of the edge e of G under g, induced vertex labeling function $g^* : V(G) \to \{0,1\}$ is given as if $e_1,e_2,\ldots,e_m$ are the edges incident to vertex v, then $g^*(v) = g(e_1)g(e_2)\ldots g(e_m)$ and induced face labeling function $g^{**} : F(G) \to \{0,1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face f then $g^{**}(f) = g^*(v_1)g^*(v_2)\ldots g^*(v_n)\, g(e_1)g(e_2)\ldots g(e_m)$. Let us denote $v_g(i)$ is the number of vertices of G having label i under g^*, $e_g(i)$ is the number of edges of G having label i under g and $f_g(i)$ is the number of interior faces of G having label i under g^{**} for i = 1,2. g is called face edge product cordial labeling of graph G if $|v_g(0)-v_g(1)|\le 1$, $|e_g(0)-e_g(1)|\le 1$ and $|f_g(0) - f_g(1)| \le 1$. A graph G is face edge product cordial if it admits face edge product cordial labeling.

Definition : 1.18

For a planar graph G, the edge labeling function is defined as $g : E(G) \to \{0,1\}$ and g(e) is called the label of the edge e of G under g, induced vertex labeling function $g^* : V(G) \to \{0,1\}$ is given as if $e_1,e_2,\ldots,e_m$ are the edges incident to vertex v, then $g^*(v) = g(e_1)g(e_2)\ldots g(e_m)$ and induced face labeling function $g^{**} : F(G) \to \{0,1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face f then $g^{**}(f) = g^*(v_1)g^*(v_2)\ldots g^*(v_n)\, g(e_1)g(e_2)\ldots g(e_m)$. Let g(0), g(1) be the sum of the number of vertices, edges and interior faces having labels 0 and 1 respectively. g is called total face edge product cordial labeling of graph G if $|g(0)-g(1)|\le 1$. A graph G is total face edge product cordial if it admits total face edge product cordial labeling.

II. Main Theorems

Theorem 2.1

The graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length is face edge product cordial graph.

Proof.

Let G be the graph obtained by joining two copies of planar graph G by a path P_k.

Let $u_1, u_2, \ldots, u_n$ be the vertices, $e_1, e_2, \ldots, e_m$ be the edges and $f_1, f_2, \ldots, f_s$ be the interior faces of first copy of planar graph G′, say G'_1.

Let $u'_1, u'_2, \ldots, u'_n$ be the vertices, $e'_1, e'_2, \ldots, e'_m$ be the edges and $f'_1, f'_2, \ldots, f'_s$ be the interior faces of second copy of planar graph G′, say G'_2.

Let $w_1,w_2,\ldots,w_k$ be the vertices of path P_k with $u_1 = w_1$ and $u'_1 = w_k$, $e''_1, e''_2, \ldots, e''_{k-1}$ be the edges of path P_k.

Then $|V(G)| = 2n + k - 2$, $|E(G)| = 2m + k - 1$ and $|F(G)| = 2s$.

Define edge labeling $g : E(G) \rightarrow \{0,1\}$ as follows

$g(e_i) = 1$, for $1 \leq i \leq m$

$g(e'_i) = 0$, for $1 \leq i \leq m$

Case (i) : k is odd

$g(e''_i) = 1$, for $1 \leq i \leq \frac{k-1}{2}$

$g(e''_i) = 0$, for $\frac{k-1}{2} + 1 \leq i \leq k - 1$

In view of the above defined labeling pattern we have

$$e_g(0) = e_g(1) = m + \frac{k-1}{2},\ v_g(0) = v_g(1) + 1 = n + \frac{k-1}{2} \text{ and } f_g(0) = f_g(1) = s.$$

Then $|v_g(0) - v_g(1)| \leq 1$, $|e_g(0) - e_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$

Case (ii) : k is even

$g(e''_i) = 1$, for $1 \leq i \leq \frac{k}{2}$

$g(e''_i) = 0$, for $\frac{k}{2} + 1 \leq i \leq k - 1$

In view of the above defined labeling pattern we have

$$v_g(0) = v_g(1) = n + \frac{k-2}{2},\ e_g(1) = e_g(0) + 1 = m + \frac{k}{2} \text{ and } f_g(0) = f_g(1) = s.$$

Then $|v_g(0) - v_g(1)| \leq 1$, $|e_g(0) - e_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$

Therefore, the graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length is face edge product cordial graph.

Example : 2.1

The graph G obtained by joining two copies of planar graph C_5 by a path P_5 and its face edge product cordial labeling is given in figure 2.1.

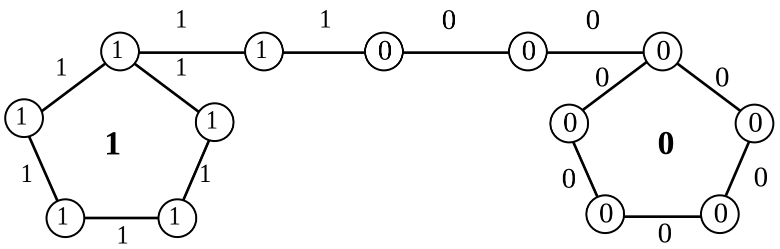

Figure 2.1

Theorem : 2.2

The path union of k copies of cycle C_n is a face edge product cordial graph except for odd k and even n.

Proof.

Let $G_1, G_2, \ldots, G_k$ be k copies of the cycle C_n and G be the path union of cycle C_n.

Let us denote the successive vertices and edges of the i^{th} copy G_i by $u_{i1}, u_{i2}, \ldots, u_{in}$ and $e_{i1}, e_{i2}, \ldots, e_{in}$.

Let $e_i = u_{i1}u_{(i+1)1}$ be the edge joining G_i and G_{i+1} for $i = 1,2,\ldots, k–1$.

Let $f_1, f_2, \ldots, f_n$ be the interior faces of G

Then $|V(G)| = nk$, $|E(G)| = nk + (k–1)$ and $|F(G)| = k$.

To define binary edge labeling $g : E(G) \rightarrow \{0,1\}$ we consider the following cases.

Case (i) : k is even and n is odd or even.

$g(e_{ij}) = 1$, for $1 \leq j \leq n$, $1 \leq i \leq \frac{k}{2}$

$g(e_{ij}) = 0$, for $1 \leq j \leq n$, $\frac{k}{2} + 1 \leq i \leq k$

$g(e_i) = 1,$ for $1 \le i \le \frac{k}{2}$

$g(e_i) = 0,$ for $\frac{k}{2} + 1 \le i \le k-1$

In view of the above defined labeling pattern

$v_f(0) = v_f(1) = \frac{nk}{2}$, $e_f(0) + 1 = e_f(1) = \frac{k(n+1)}{2}$ and $f_g(0) = f_g(1) = \frac{k}{2}$.

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Thus the graph G satisfies the condition for face edge product cordial.

Case (ii) : k is odd and n is odd

$g(e_{ij}) = 1,$ for $1 \le j \le n,\ 1 \le i \le \frac{k-1}{2}$

$$\left.\begin{aligned} g(e_{ij}) &= 1, \quad 1 \le j \le \frac{n+1}{2}, \\ &= 0, \quad \frac{n+1}{2} + 1 \le j \le n, \end{aligned}\right\} \text{for } i = \frac{k+1}{2}$$

$g(e_{ij}) = 0,$ for $1 \le j \le n,\ \frac{k+1}{2} + 1 \le i \le k$

$g(e_i) = 1,$ for $1 \le i \le \frac{k-1}{2}$

$g(e_i) = 0,$ for $\frac{k-1}{2} + 1 \le i \le k-1$

In view of the above defined labeling pattern

$v_f(1) + 1 = v_f(0) = \frac{nk+1}{2}$, $e_f(0) + 1 = e_f(1) = \frac{k(n+1)}{2}$ and $f_g(0) = f_g(1) + 1 = \frac{k+1}{2}$.

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Thus the graph G satisfies the condition for face edge product cordial.

Case (iii) : k is odd and n is even

In order to satisfy the edge condition for G, it is essential to assign label 1 and 0 to exactly $\frac{nk+k-2}{2}$ edges.

Any pattern assigning edge labels satisfying edge condition will induce vertex labels for nk number of vertices in such a way that $|v_g(0) - v_g(1)| \ge 2$, that is vertex condition for G is violated.

Thus the graph G under consideration is not a face edge product cordial graph when n is even and k is odd.

The path union of k copies of cycle C_n is a face edge product cordial graph except for odd k and even n.

Example : 2.2

The path union of 3 copies of cycle C_5 and its face edge product cordial labeling is given in figure 2.2.

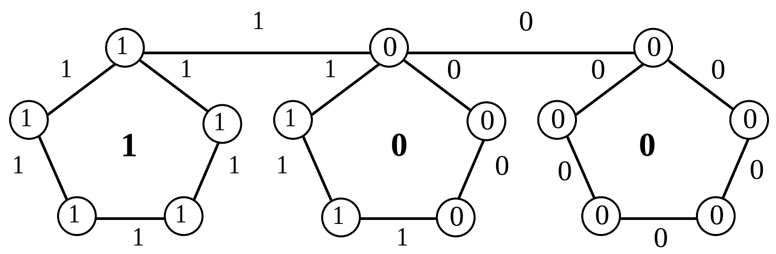

Figure 2.2

Theorem : 2.3

The graph T_n is face edge product cordial graph for even n and not face edge product cordial for odd n.

Proof.

Let path P_n having vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-1}$.

To construct triangular snake T_n from path P_n join v_i and v_{i+1} to new vertex w_i by edges $e'_{2i-1} = v_i w_i$ and $e'_{2i} = v_{i+1} w_i$ for $i = 1,2,\ldots, n-1$ and interior faces $f_i = v_i w_i v_{i+1}$ for $i = 1,2,\ldots, n-1$

Then $|V(T_n)| = 2n - 1$, $|E(T_n)| = 3n - 3$ and $|F(T_n)| = n - 1$.

Define edge labeling $g : E(T_n) \rightarrow \{0,1\}$ as follows

We consider following two cases.

Case (i): When n is even.

$g(e_i) = 1$, for $1 \le i \le \frac{n}{2}$

$g(e_i) = 0$, for $\frac{n}{2} + 1 \le i \le n-1$

$g(e'_i) = 1$, for $1 \le i \le n-1$

$g(e'_i) = 0$, for $n \le i \le 2n-2$

In view of the above defined labeling pattern we have

$$v_g(0) = v_g(1) + 1 = n-1,\ e_g(1) = e_g(0) + 1 = \frac{3n}{2} - 1 \text{ and } f_g(0) = f_g(1) + 1 = \frac{n}{2}.$$

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

The graph T_n is face edge product cordial graph for even n.

Case 2: When n is odd.

In order to satisfy the face condition for graph T_n, it is essential to assign label 1 to at least $\frac{3n+1}{2}$ edges out of $3n - 3$ edges. Then $|e_g(1) - e_g(0)| = 4$.

Thus the edge condition for T_n is violated.

Therefore, the graph T_n is not face edge product cordial for odd n.

Hence, the graph T_n is face edge product cordial graph for even n and not face edge product cordial for odd n.

Example : 2.3

The graph T_6 and its face edge product cordial labeling is shown in figure 2.3.

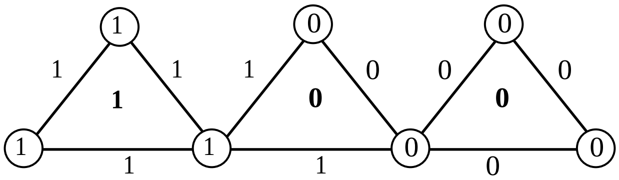

Figure 2.3

Theorem : 2.4

The graph $M(P_n)$ is face edge product cordial graph for odd n and not face edge product cordial for even n.

Proof :

Let path P_n having vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-1}$.

$v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$ are vertices, $e'_{2i-1} = v_i e_i$, $e'_{2i} = e_i v_{i+1}$ for $i = 1,2,\ldots, n-1$ and $e''_i = e_i e_{i+1}$ for $i = 1,2,\ldots, n-2$ are edges and $f_i = e_i v_{i+1} e_{i+1}$ for $i = 1,2,\ldots, n-2$ are interior faces of $M(P_n)$

Then $|V(M(P_n))| = 2n-1$, $|E(M(P_n))| = 3n-4$ and $|F(M(P_n))| = n-2$.

Define edge labeling $g : E(M(P_n)) \rightarrow \{0,1\}$ as follows

We consider following two cases.

Case (i): When n is odd.

$g(e''_i) = 1$, for $1 \le i \le \frac{n-1}{2}$

$g(e''_i) = 0$, for $\frac{n-1}{2} + 1 \le i \le n-2$

$g(e'_i) = 1$, for $1 \le i \le n-1$

$g(e'_i) = 0$, for $n \le i \le 2n-2$

In view of the above defined labeling pattern we have

$$v_g(0) = v_g(1) + 1 = n - 1,\ e_g(1) = e_g(0) + 1 = \frac{3n}{2} - 1 \text{ and } f_g(0) = f_g(1) + 1 = \frac{n-1}{2}.$$

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

The graph T_n is face edge product cordial graph for even n.

Case 2: When n is even.

In order to satisfy the face condition for graph $M(P_n)$, it is essential to assign label 1 to at least $\frac{3n}{2}$ edges out of $3n - 4$ edges. Then $|e_g(1) - e_g(0)| = 4$.

Thus the edge condition for $M(P_n)$ is violated.

Therefore, the graph $M(P_n)$ is not face edge product cordial for even n.

Hence, the graph $M(P_n)$ is face edge product cordial graph for odd n and not face edge product cordial for even n.

Example : 2.4

The graph $M(P_5)$ and its face edge product cordial labeling is shown in figure 2.4.

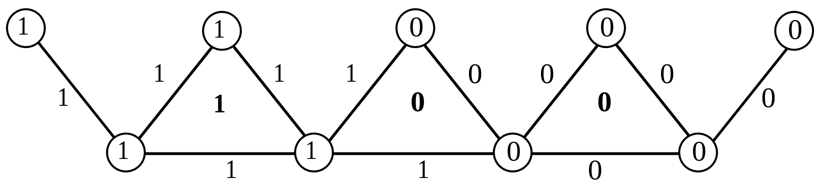

Figure 2.4

Theorem : 2.5

The star of cycle C_n is face edge product cordial graph for odd n and not face edge product cordial for even n.

Proof.

Let $v_1, v_2, ..., v_n$ and $e_1, e_2, ..., e_n$ be the vertices and edges of central cycle C_n and v_{ij} and e_{ij} be the vertices and edges of the cycle C_n^i, where $1 \le i \le n$ and $1 \le j \le n$, v_{i1} be adjacent to the i^{th} vertex of central cycle C_n and $e'_i = v_i v_{i1}$, where $1 \le i \le n$ and $f_1, f_2, ..., f_{n+1}$ are interior faces of star of cycle C_n. Let G be a star of cycle C_n.

Then $|V(G)| = n(n+1)$, $|E(G)| = n(n+2)$ and $|F(G)| = n + 1$.

Define edge labeling $g : E(G) \to \{0,1\}$ as follows

We consider following two cases.

Case (i) : n is odd.

$g(e_{ij}) = 0$, for $1 \le j \le n$, $1 \le i \le \frac{n-1}{2}$

$g(e_{ij}) = 1$, for $1 \le j \le n$, $\frac{n-1}{2} + 1 \le i \le n$

$g(e_i) = 1$, for $1 \le i \le n$

$g(e'_i) = 0$, for $1 \le i \le \frac{n-1}{2}$

$g(e'_i) = 1$, for $\frac{n-1}{2} + 1 \le i \le n$

In view of the above defined labeling pattern

$$v_f(0) = v_f(1) = \frac{n(n+1)}{2},\ e_f(0) + 1 = e_f(1) = n(n+1) \text{ and } f_g(0) = f_g(1) = \frac{n+1}{2}.$$

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Thus the graph G satisfies the condition for face edge product cordial for n is odd.

Case (ii) : n is even

In order to satisfy the edge condition for G, it is essential to assign label 1 and 0 to exactly $\frac{n(n+2)}{2}$ edges.

Any pattern assigning edge labels satisfying edge condition will induce vertex labels for n(n+1) number of vertices in such a way that $| v_g(0) – v_g(1) | \geq 2$, that is vertex condition for G is violated.

Thus the graph G under consideration is not a face edge product cordial graph when n is even.

Therefore, the star of cycle C_n is face edge product cordial graph for odd n and not face edge product cordial for even n.

Example : 2.5

The star of cycle C_5 and its face edge product cordial labeling of graph is shown in figure 2.5.

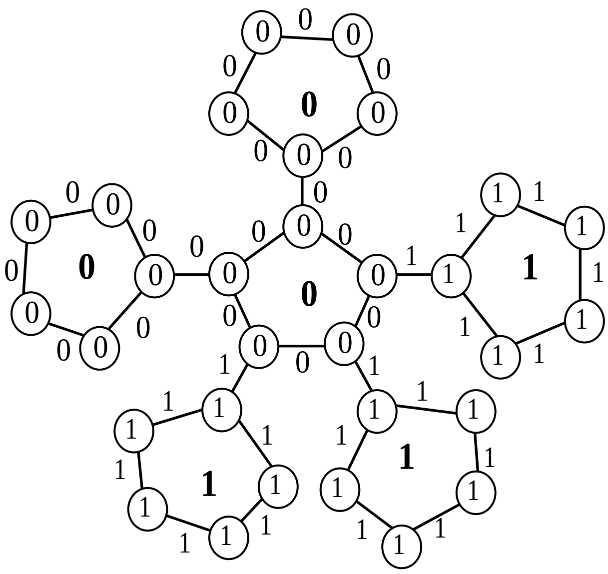

Figure 2.5

Theorem : 2.6

The graph DT_n is not a face edge product cordial graph.

Proof.

Let $v_1, v_2,\ldots,v_n$ be the vertices and $e_1,e_2,\ldots,e_{n-1}$ be the edges of path P_n .

To construct double triangular snake DT_n from path P_n join v_i and v_{i+1} to two new vertices w_i and w'_i by edges $e'_{2i-1} = v_i w_i$, $e'_{2i} = v_{i+1} w_i$, $e''_{2i-1} = v_i w'_i$ and $e''_{2i} = v_{i+1} w'_i$ for i = 1,2,…, n−1 and interior faces $f_i = v_i\, w_i\, v_{i+1}$ and $f'_i = v_i\, w'_i\, v_{i+1}$ for i = 1,2,…, n−1.

Then $|V(DT_n)| = 3n – 2$, $|E(DT_n)| = 5n – 5$ and $|F(T_n)| = 2n – 2$.

Define edge labeling $g : E(DT_n) \to \{0,1\}$ as follows

Case 1: When n is even.

In order to satisfy the face condition for graph DT_n, it is essential to assign label 1 to at least $\frac{5n+6}{2}$ edges out of $5n - 5$ edges. Then $|e_g(1) - e_g(0)| = 11$.

Thus the edge condition for DT_n is violated.

Therefore, the graph DT_n is not face edge product cordial for even n.

Case (ii): When n is odd.

In order to satisfy the face condition for graph DT_n, it is essential to assign label 1 to at least $\frac{5n+1}{2}$ edges out of $5n - 5$ edges. Then $|e_g(1) - e_g(0)| = 6$.

Thus the edge condition for DT_n is violated.

Therefore, the graph DT_n is not face edge product cordial for odd n.

Hence, the graph DT_n is not face edge product cordial graph.

Theorem : 2.7

The shell S_n is not face edge product cordial graph.

Proof.

Let $v_1,v_2,\ldots,v_n$ be the vertices, $e_1,e_2,\ldots,e_{2n-3}$ be the edges and $f_1,f_2,\ldots,f_{n-2}$ be interior faces of shell S_n.
Then $|V(S_n)| = n$, $|E(S_n)| = 2n - 3$ and $|F(S_n)| = n - 2$.
Define edge labeling $g : E(S_n) \to \{0,1\}$ as follows

Case (i): When n is even.

In order to satisfy the face condition for shell S_n, it is essential to assign label 1 to at least $\frac{3n-2}{2}$ edges out of $2n - 3$ edges. Then $|e_g(1) - e_g(0)| \geq 2$.
Thus, the edge condition for S_n is violated.
Therefore, the graph S_n is not face edge product cordial for even n.

Case (ii): When n is odd.

In order to satisfy the face condition for shell S_n, it is essential to assign label 1 to at least $\frac{3n-3}{2}$ edges out of $2n - 3$ edges. Then $|e_g(1) - e_g(0)| \geq 2$.
Thus, the edge condition for S_n is violated.
Therefore, the graph S_n is not face edge product cordial for odd n.
Hence, the graph S_n is not face edge product cordial graph.

Theorem : 2.8

The wheel W_n is a total edge product cordial graph.

Proof.

Let $v_1, v_2, \ldots, v_n$ be the rim vertices and v be an apex vertex of wheel W_n and $e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_n$ be the edges of wheel W_n, where $e_i = vv_i$, for $i = 1,2,\ldots,n$, $e'_i = v_iv_{i+1}$ for $i = 1,2,\ldots,n-1$ and $e_n = v_nv_1$. $f_1,f_2,\ldots,f_n$ be interior faces of wheel W_n.
To define edge labeling $g : E(W_n) \to \{0, 1\}$ as follows.
Then $|V(W_n)| = n + 1$, $|E(W_n)| = 2n$ and $|F(W_n)| = n$.

Case 1: When n is odd.

$g(e'_i) = 0$; for $1 \leq i \leq \frac{n-3}{2}$

$g(e'_i) = 1$; for $\frac{n-3}{2} + 1 \leq i \leq n$

$g(e_1) = 1$,
$g(e_2) = 0$,
$g(e_i) = 1$, for $3 \leq i \leq n$.

In view of the above defined labeling pattern we have
$g(0) + 1 = g(1) = 2n+1$
Then, $|g(0) - g(1)| \leq 1$.
Then, the wheel W_n is a total face edge product cordial graph for n is odd.

Case 2: When n is even.

$g(e'_i) = 0$; for $1 \leq i \leq \frac{n-2}{2}$

$g(e'_i) = 1$; for $\frac{n-2}{2} + 1 \leq i \leq n$

$g(e_1) = 1$,
$g(e_2) = 0$,
$g(e_i) = 1$, for $3 \leq i \leq n$.

In view of the above defined labeling pattern we have
$g(0) = g(1) + 1 = 2n+1$
Then, $|g(0) - g(1)| \leq 1$.
Then, the wheel W_n is a total face edge product cordial graph for n is even.
Thus in both the cases we have $|g(0) - g(1)| \leq 1$.
Hence, the wheel W_n is a total face edge product cordial graph.

Example : 2.6

The wheel W_5 and its total face edge product cordial labeling is shown in figure 2.6.

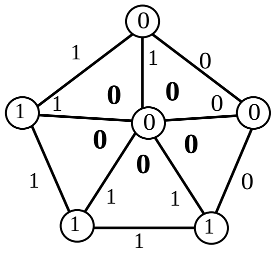

Figure 2.6

Theorem : 2.9

The fan f_n is total face edge product cordial graph.

Proof.

Let $v, v_1, v_2, \ldots, v_n$ be the vertices, $e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_{n-1}$ be the edges and $f_1, f_2, \ldots, f_{n-1}$ be interior faces of f_n, where $e_i = vv_i$, for $i = 1,2,\ldots,n$, $e'_i = v_i v_{i+1}$ for $i = 1,2,\ldots,n-1$.

To define edge labeling $g : E(W_n) \rightarrow \{0, 1\}$ as follows.

Then $|V(f_n)| = n+1$, $|E(f_n)| = 2n-1$ and $|F(f_n)| = n-1$.

Define edge labeling $g : E(f_n) \rightarrow \{0,1\}$ as follows

Case (i): When n is even.

$g(e'_i) = 1;$ for $1 \le i \le n-1$

$g(e_{2i-1}) = 1,$ for $1 \le i \le \frac{n}{2}$

$g(e_{2i}) = 0,$ for $1 \le i \le \frac{n}{2}$

In view of the above defined labeling pattern we have

$g(0) = g(1) + 1 = 2n$

Then, $|g(0) - g(1)| \le 1$.

Then, the fan f_n is a total face edge product cordial graph for n is even.

Case (ii): When n is odd.

$g(e'_i) = 1;$ for $1 \le i \le n-1$

$g(e_{2i-1}) = 1,$ for $1 \le i \le \frac{n+1}{2}$

$g(e_{2i}) = 0,$ for $1 \le i \le \frac{n-1}{2}$

In view of the above defined labeling pattern we have

$g(0) + 1 = g(1) = 2n$

Then, $|g(0) - g(1)| \le 1$.

Then, the fan f_n is a total face edge product cordial graph for n is odd.

Thus in both the cases we have $|g(0) - g(1)| \le 1$.

Therefore, the fan f_n is a total face edge product cordial graph.

Example : 2.7

The fan f_5 and its total face edge product cordial labeling is shown in figure 2.7.

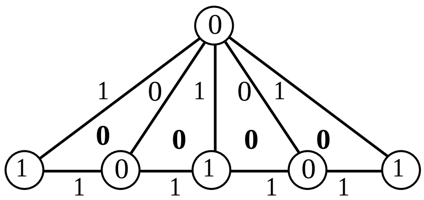

Figure 2.7

Theorem : 2.10

The star of cycle C_n is total face edge product cordial graph.

Proof.

Let $v_1, v_2, ..., v_n$ and $e_1, e_2, ..., e_n$ be the vertices and edges of central cycle C_n and v_{ij} and e_{ij} be the vertices and edges of the cycle C_n^i, where $1 \le i \le n$ and $1 \le j \le n$, v_{i1} be adjacent to the i^{th} vertex of central cycle C_n and $e'_i = v_i v_{i1}$, where $1 \le i \le n$ and $f_1, f_2, ..., f_{n+1}$ are interior faces of star of cycle C_n. Let G be a star of cycle C_n.

Then $|V(G)| = n(n+1)$, $|E(G)| = n(n+2)$ and $|F(G)| = n+1$.

Define edge labeling $g : E(G) \to \{0,1\}$ as follows

We consider following two cases.

Case (i) : n is odd.

$g(e_{ij}) = 0$, for $1 \le j \le n$, $1 \le i \le \frac{n-1}{2}$

$g(e_{ij}) = 1$, for $1 \le j \le n$, $\frac{n-1}{2}+1 \le i \le n$

$g(e_i) = 1$, for $1 \le i \le n$

$g(e'_i) = 0$, for $1 \le i \le \frac{n-1}{2}$

$g(e'_i) = 1$, for $\frac{n-1}{2}+1 \le i \le n$

In view of the above defined labeling pattern

$v_f(0) = v_f(1) = \frac{n(n+1)}{2}$, $e_f(0) + 1 = e_f(1) = \frac{n(n+2)+1}{2}$ and

$f_g(0) = f_g(1) = \frac{n+1}{2}$.

Thus $g(0) + 1 = g(1) = (n+1)(n+1)$

Then, $|g(0) - g(1)| \le 1$.

Thus the graph G satisfies the condition for total face edge product cordial for n is odd.

Case (ii) : n is even

$g(e_{ij}) = 1$, for $1 \le j \le n$, $1 \le i \le \frac{n}{2}$

$g(e_{ij}) = 0$, for $1 \le j \le n$, $\frac{n}{2}+1 \le i \le n$

$g(e_i) = 1$, for $1 \le i \le \frac{n}{2}$

$g(e_i) = 0$, for $\frac{n}{2}+1 \le i \le n-1$

$g(e_n) = 1$.

$g(e'_i) = 0$, for $1 \le i \le \frac{n}{2}$

$g(e'_i) = 1$, for $\frac{n}{2}+1 \le i \le n$

In view of the above defined labeling pattern

$v_f(0) = v_f(1) = \frac{n(n+1)}{2}$, $e_f(0) + 2 = e_f(1) = \frac{n(n+2)}{2}+1$ and

$f_g(0) = f_g(1) + 1 = \frac{n+2}{2}$.

Thus $g(0) + 1 = g(1) = (n+1)(n+1)$

Then, $|g(0) - g(1)| \le 1$.

Thus the graph G satisfies the condition for total face edge product cordial for n is even.

Therefore, the star of cycle C_n is total face edge product cordial graph.

Example : 2.8

The star of cycle C_5 and its face edge product cordial labeling of graph is shown in figure 2.8.

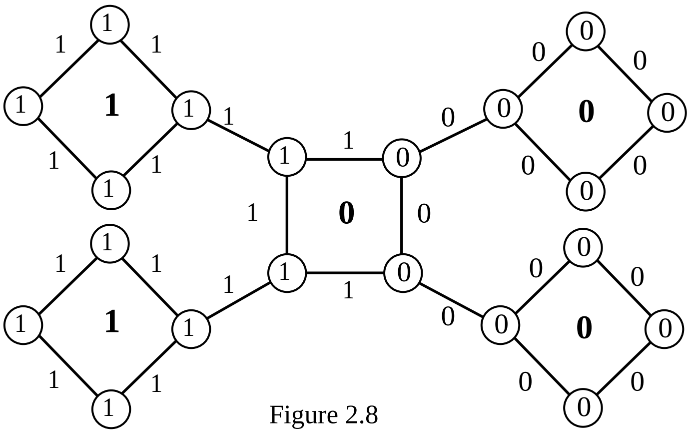

Figure 2.8

Theorem : 2.11

The graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length is face product cordial graph.

Proof.

Let G be the graph obtained by joining two copies of planar graph G by a path P_k.

Let $u_1, u_2, \dots, u_n$ be the vertices, $e_1, e_2, \dots, e_m$ be the edges and $f_1, f_2, \dots, f_s$ be the interior faces of first copy of planar graph G′, say G'_1.

Let $u'_1, u'_2, \dots, u'_n$ be the vertices, $e'_1, e'_2, \dots, e'_m$ be the edges and $f'_1, f'_2, \dots, f'_s$ be the interior faces of second copy of planar graph G′, say G'_2.

Let $w_1, w_2, \dots, w_k$ be the vertices of path P_k with $u_1 = w_1$ and $u'_1 = w_k$, $e''_1, e''_2, \dots, e''_{k-1}$ be the edges of path P_k.

Then $|V(G)| = 2n + k - 2$, $|E(G)| = 2m + k - 1$ and $|F(G)| = 2s$.

Define vertex labeling $g : V(G) \rightarrow \{0,1\}$ as follows

$g(v_i) = 1,\quad$ for $1 \le i \le n$

$g(v'_i) = 0,\quad$ for $1 \le i \le n$

Case (i) : k is odd

$g(v''_{i+1}) = 1,\quad$ for $1 \le i \le \dfrac{k-1}{2}$

$g(v''_{i+1}) = 0,\quad$ for $\dfrac{k-1}{2} + 1 \le i \le k-2$

In view of the above defined labeling pattern we have

$$v_g(1) = v_g(0) + 1 = n + \frac{k-1}{2},\ f_g(0) = f_g(1) = s \text{ and } e_g(0) = e_g(1) = m + \frac{k-1}{2}.$$

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Case (ii) : k is even

$g(e''_{i+1}) = 1,\quad$ for $1 \le i \le \dfrac{k-2}{2}$

$g(e''_{i+1}) = 0,\quad$ for $\dfrac{k-2}{2} + 1 \le i \le k-2$

In view of the above defined labeling pattern we have

$$v_g(0) = v_g(1) = n + \frac{k-2}{2},\ f_g(0) = f_g(1) = s \text{ and } e_g(0) = e_g(1) + 1 = m + \frac{k}{2}.$$

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Therefore, the graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length is face product cordial graph.

Example : 2.9

The graph G obtained by joining two copies of planar graph C_5 by a path P_5 and its face product cordial labeling is given in figure 2.9.

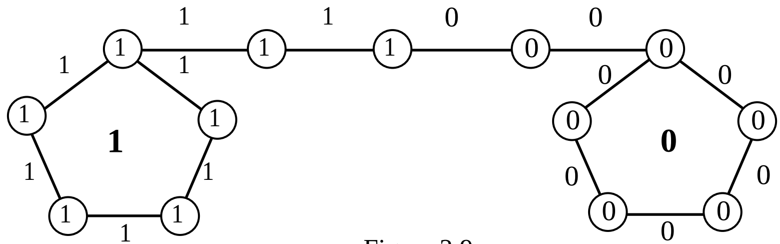

Figure 2.9

Theorem : 2.12

The path union of k copies of cycle C_n is a face product cordial graph except for odd k and even n.

Proof.

Let $G_1, G_2, \ldots, G_k$ be k copies of the cycle C_n and G be the path union of cycle C_n.

Let us denote the successive vertices and edges of the i^{th} copy G_i by $u_{i1}, u_{i2}, \ldots, u_{in}$ and $e_{i1}, e_{i2}, \ldots, e_{in}$.

Let $e_i = u_{i1}u_{(i+1)1}$ be the edge joining G_i and G_{i+1} for $i = 1,2,\ldots, k-1$.

Let $f_1, f_2, \ldots, f_n$ be the interior faces of G

Then $|V(G)| = nk$, $|E(G)| = nk + (k-1)$ and $|F(G)| = k$.

To define binary vertex labeling $g : V(G) \to \{0,1\}$ we consider the following cases.

Case (i) : k is even and n is odd or even.

$$g(u_{ij}) = 1, \qquad \text{for } 1 \le j \le n,\ 1 \le i \le \frac{k}{2}$$

$$g(u_{ij}) = 0, \qquad \text{for } 1 \le j \le n,\ \frac{k}{2} + 1 \le i \le k$$

In view of the above defined labeling pattern

$$v_f(0) = v_f(1) = \frac{nk}{2},\ e_f(0) = e_f(1) + 1 = \frac{k(n+1)}{2} \text{ and } f_g(0) = f_g(1) = \frac{k}{2}.$$

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Thus the graph G satisfies the condition for face product cordial.

Case (ii) : k is odd and n is odd

$$g(u_{ij}) = 0, \qquad \text{for } 1 \le j \le n,\ 1 \le i \le \frac{k-1}{2}$$

$$\left.\begin{array}{ll} g(u_{ij}) = 1, & 1 \le j \le \frac{n+1}{2}, \\ \quad = 0, & \frac{n+1}{2} + 1 \le j \le n, \end{array}\right\} \text{ for } i = \frac{k+1}{2}$$

$$g(u_{ij}) = 1, \qquad \text{for } 1 \le j \le n,\ \frac{k+1}{2} + 1 \le i \le k$$

In view of the above defined labeling pattern

$$v_f(1) = v_f(0) + 1 = \frac{nk+1}{2},\ e_f(0) = e_f(1) + 1 = \frac{k(n+1)}{2} \text{ and } f_g(0) = f_g(1) + 1 = \frac{k+1}{2}.$$

Then $|v_g(0) - v_g(1)| \le 1$, $|e_g(0) - e_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$

Thus the graph G satisfies the condition for face product cordial.

Case (iii) : k is odd and n is even

In order to satisfy the vertex condition for G, it is essential to assign label 1 and 0 to exactly $\frac{nk}{2}$ vertices.

Any pattern assigning vertex labels satisfying vertex condition will induce edge labels for $nk + k - 1$ number of edges in such a way that $|e_f(0) - e_f(1)| \ge 2$, that is edge condition for G is violated.

Thus the graph G under consideration is not a face product cordial graph when n is even and k is odd.

The path union of k copies of cycle C_n is a face product cordial graph except for odd k and even n.

Example : 2.10

The path union of 3 copies of cycle C_5 and its face product cordial labeling is given in figure 2.10.

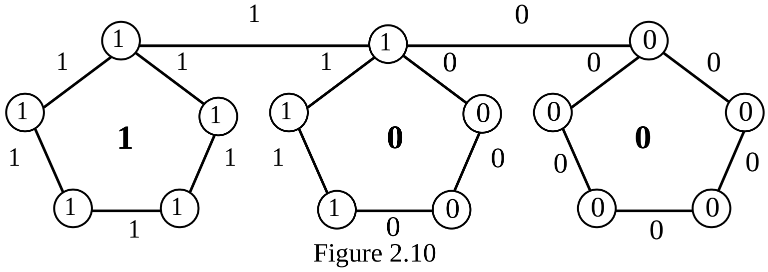

Figure 2.10

III. Conclusions

In this paper, we prove the face edge product cordial labeling of planar graphs T_n for even n, $M(P_n)$ for odd n, the star of cycle C_n for odd n, the graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length and the path union of k copies of cycle C_n except for odd k and even n are presented. We also discussed the total face edge product cordial labeling of f_n, W_n and the star of cycle C_n and face product cordial labeling of the graph G obtained by joining two copies of planar graph G′ by a path of arbitrary length and the path union of k copies of cycle C_n except for odd k and even n.

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