International Journal of Mathematics Trends and Technology- Volume 19 Number 2 Mar 2015

2(1)-SEMIPRIME PARTIAL IDEALS OF PARTIAL SEMIRINGS

P. V. SRINIVASA RAO

Abstract. A partial semiring is a structure possessing an infinitary partial addition and a binary multiplication, subject to a set of axioms. The partial functions under disjoint-domain sums and functional composition is a partial semiring. In this paper we introduce the notions of 2(1)-semiprime partial ideal and $p_2(p_1)$-system in partial semirings and we obtain some characteristics of 2(1)-semiprime partial ideals.

Index Terms. Subtractive partial ideal, prime partial ideal, 2(1)-prime partial ideal, semiprime partial ideal, 2(1)-semiprime partial ideal, $p_2(p_1)$-system of a partial semiring.

Introduction

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming language semantics. The general cardinal algebras studied by Tarski in 1949, Housdorff topological commutative groups studied by Bourbaki in 1966, Σ-structures studied by Higgs in 1980, sum ordered partial monoids and sum ordered partial semirings studied by Arbib, Manes, Benson[2], [4] and Streenstrup[6] are some of the algebraic structures of the above type.

In [7] and [9] we developed the ideal theory in partial semirings. In [8] the notion of 1-(2-)prime partial ideals in partial semirings is introduced and obtained their characteristics. In this paper we introduce the notions of 2(1)-semiprime partial ideals in partial semirings and obtain the equivalent conditions of 2(1)-semiprime partial ideals. We also introduce the notions of $p_2(p_1)$-systems in partial semirings and obtain the characteristics of 2(1)-semiprime partial ideals.

1. Preliminaries

In this section we collect some important definitions and results for our use in this paper.

Definition 1.1. [4] *A* partial monoid *is a pair* (M, Σ) *where* M *is a nonempty set and* Σ *is a partial addition defined on some, but not necessarily all families* $(x_i : i \in I)$ *in* M *subject to the following axioms:*

(1) Unary Sum Axiom. If $(x_i : i \in I)$ *is a one element family in* M *and* $I = \{j\}$*, then*

2010 *Mathematics Subject Classification.* 16Y60.

Department of Science & Humanities, DVR & Dr. HS MIC College of Technology, Kanchikacherla-521180, Andhra Pradesh, INDIA, Email: $srinu\_fu2004@yahoo.co.in$

$\Sigma(x_i : i \in I)$ is defined and equals x_j.

(2) Partition-Associativity Axiom. If $(x_i : i \in I)$ is a family in M and $(I_j : j \in J)$ is a partition of I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J and $(\Sigma(x_i : i \in I_j) : j \in J)$ is summable. we write $\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j) : j \in J)$.

Definition 1.2. [6] *A* partial semiring *is a quadruple $(R, \Sigma, \cdot, 1)$, where (R, Σ) is a partial monoid, $(R, \cdot, 1)$ is a monoid with multiplicative operation $\cdot$ and unit 1, and the additive and multiplicative structures obey the following distributive laws. If $\Sigma(x_i : i \in I)$ is defined in R, then for all y in R, $\Sigma(y \cdot x_i : i \in I)$ and $\Sigma(x_i \cdot y : i \in I)$ are defined and $y \cdot [\Sigma_i x_i] = \Sigma_i(y \cdot x_i)$; $[\Sigma_i x_i] \cdot y = \Sigma_i(x_i \cdot y)$.*

Definition 1.3. [1] *Let R be a partial semiring. A subset N of R is said to be a* partial ideal *of R if the following are satisfied*

(i) if $(x_i : i \in I)$ is summable family in R and $x_i \in N$ for every $i \in I$ then $\Sigma x_i \in N$,

(ii) if $x \in N$ and $r \in R$ then $xr, rx \in N$.

Definition 1.4. [7] *Let R be a partial semiring and let S be a subset of R. Then the intersection of all partial ideals of R containing S is called the* partial ideal generated by *S and is denoted by $< S >$.*

Definition 1.5. [7] *Let N and P be partial ideals of a partial semiring R. Then we define $NP = \{x \in R \mid x = \Sigma a_i b_i$ for some $a_i \in N, b_i \in P\}$.*

Theorem 1.6. [7] *Let R be a partial semiring, then for any S contained in R, $< S > = \{x \in R \mid x = \Sigma r_i x_i r_i', x_i \in S, r_i, r_i' \in R\}$.*

Definition 1.7. [7] *A nonempty subset A of a partial semiring R is said to be* subtractive *if for any $a, b \in R$, $a + b \in A$ and $a \in A$ implies that $b \in A$.*

Definition 1.8. [7] *Let A be a nonempty subset of a partial semiring R. Then the intersection of all subtractive partial ideals of R containing A is called* subtractive closure of *A, denoted by $\overline{A}$.*

Definition 1.9. [7] *A proper partial ideal P of a partial semiring R is said to be* prime *if and only if for any partial ideals A, B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

Definition 1.10. [8] *A proper partial ideal P of a partial semiring R is said to be* 2-prime *if and only if for any subtractive partial ideals A, B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

Definition 1.11. [8] *A proper partial ideal P of a partial semiring R is said to be* 1-prime *if and only if for any partial ideal A and subtractive partial ideal B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

A partial semiring is said to be complete if and only if every family in it is summable.

Lemma 1.12. [8] *Let R be a complete partial semiring and A be a partial ideal of R. Then the subtractive closure of A is* $\overline{A} = \{a \in R \mid a + x \in A, \text{ for some } x \in A\}$.

Definition 1.13. [8] *Let R be a partial semiring. Then we define*

(i) $(A : B)_l = \{r \in R \mid rB \subseteq A\}$,

(ii) $(A : B)_r = \{r \in R \mid Br \subseteq A\}$,

(iii) $(A : B) = \{r \in R \mid rB \subseteq A \text{ and } Br \subseteq A\}$.

Lemma 1.14. [8] *Let R be a partial semiring. Then we have the following:*

(i) If A and B are left (right) partial ideals of R then $(A : B)_l$ ($(A : B)_r$) is a partial ideal of R.

(ii) If A is a subtractive left (right) partial ideal and B is a left (right) partial ideal of R then $(A : B)_l$ ($(A : B)_r$) is a subtractive partial ideal of R.

Definition 1.15. [8] *Let R be a partial semiring and M be a subset of R. Then M is called an* $m_2(m_1)$-system *of R if for any $a, b \in R$, there exists $a_1 \in \overline{< a >}$ and $b_1 \in \overline{< b >}$ ($b1 \in < b >$) such that $a_1 b_1 \in M$.*

Theorem 1.16. [8] *Let R be a complete partial semiring, Q be a partial ideal of R, and M be an $m_2(m_1)$-system of R such that $\overline{Q} \bigcap M = \emptyset$. Then there exists a 2(1)-prime partial ideal P of R such that $Q \subseteq P$ with $\overline{Q} \bigcap M = \emptyset$ and P is maximal with respect to this property.*

Definition 1.17. [7] *A proper partial ideal I of a partial semiring R is said to be* semiprime *if and only if for any partial ideal H of R, $H^2 \subseteq I$ implies $H \subseteq I$.*

Definition 1.18. [7] *A nonempty subset A of a partial semiring R is called a* p-system *if and only if for any $a \in A$, $\exists r \in R \ni ara \in A$.*

2. 2(1)-Semiprime Partial Ideals

Following the notion of 2-semiprime ideal in semirings by Nanda Kumar[5], we introduce the notion in partial semirings as follows:

Definition 2.1. *A proper partial ideal P of a partial semiring R is said to be* 2-semiprime *if and only if for any subtractive partial ideal A of R, $A^2 \subseteq P$ implies $A \subseteq P$.*

Definition 2.2. *A proper partial ideal P of a partial semiring R is said to be* 1-semiprime *if and only if for any partial ideal A of R, $\overline{A}A \subseteq P$ implies $A \subseteq P$.*

Note that every semiprime partial ideal of a partial semiring R is a 2(1)-semiprime partial ideal of R. Following is an example of a partial semiring R in which a 2(1)-semiprime partial ideal is not semiprime.

Example 2.3. *Let $R = \{0, 1, 2, 3\}$. Define Σ on R as*

$$\Sigma x_i = \begin{cases} x_j, \ if \ x_i = 0 \ \forall i \neq j, \ for \ some \ j, \\ 2, \ if \ x_j = x_k = 1 \ for \ some \ j, k \ and \ x_i = 0 \ \forall i \neq j, k, \\ 3, \ if \ J = \{i \mid x_i \neq 0\} \ is \ finite \ and \ \Sigma(x_i : i \in J) \geq 3, \\ undefined, \ otherwise. \end{cases}$$

and $\cdot$ defined by the following table:

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

Then $(R, \Sigma, \cdot)$ is a partial semiring. Clearly the partial ideal $P = \{0, 3\}$ is a 2(1)-semiprime partial ideal of R. For the partial ideal $A = \{0, 2, 3\}$ of R, $A^2 = \{0, 3\} \subseteq P$ but $A \nsubseteq P$. Hence P is not a semiprime partial ideal of R.

Clearly every 2(1)-prime partial ideal of a partial semiring R is a 2(1)-semiprime partial ideal of R. Following is an example of a partial semiring R in which a 2(1)-semiprime partial ideal is not 2(1)-prime.

Example 2.4. *Consider the partial semiring $R = \{0, u, v, x, y, 1\}$ with Σ defined on R by*

$$\Sigma x_i = \begin{cases} x_j, \ if \ x_i = 0 \ \forall i \neq j, \ for \ some \ j, \\ undefined, \ otherwise. \end{cases}$$

and $\cdot$ defined by the following table:

.	0	u	v	x	y	1
0	0	0	0	0	0	0
u	0	u	0	0	0	u
v	0	0	v	0	0	v
x	0	0	0	0	0	x
y	0	0	0	0	0	y
1	0	u	v	x	y	1

Then the partial ideal $P = \{0, x, y\}$ is 2(1)-semiprime. For the subtractive partial ideals $A = \{0, u\}$, $\{0, v\}$ of R, $AB = \{0\} \subset P$ but $A \nsubseteq P$ and $B \nsubseteq P$. Hence P is not a 2(1)-prime partial ideal of R.

Theorem 2.5. *If P is a proper partial ideal of a complete partial semiring R then the following conditions are equivalent*

(i) P is a 2(1)-semiprime partial ideal of R

(ii) for any $a \in R$, $(\overline{<a>})^2 \subseteq P$ $(\overline{<a>} <a> \subseteq P)$ implies $a \in P$.

Proof. (i)⇒(ii): Suppose P is a 2(1)-semiprime partial ideal of R. Let $a \in R$ such that $(\overline{<a>})^2 \subseteq P$. Then $\overline{<a>} \subseteq P$. Since $a \in <a> \subseteq \overline{<a>}$, $a \in P$.

(ii)⇒(i): Suppose for any $a \in R$, $(\overline{<a>})^2 \subseteq P$ implies $a \in P$. Let A be a subtractive partial ideal of R such that $A^2 \subseteq P$. Let $a \in A$. First we prove that $(\overline{<a>})^2 \subseteq A^2$: Let $x \in (\overline{<a>})^2$. Then $x = \Sigma_i x_i y_i$ for some $x_i, y_i \in \overline{<a>}$ $\forall\, i \in I$. $\Rightarrow x = \Sigma_i x_i y_i$, where $x_i + a_i, y_i + a_i' \in <a> \subseteq A$ for some $a_i, a_i' \in <a> \subseteq A$ $\forall\, i \in I$. Since A is subtractive, $x_i, y_i \in A$ $\forall\, i \in I$. $\Rightarrow x = \Sigma_i x_i y_i \in A^2$. Therefore $(\overline{<a>})^2 \subseteq A^2$. $\Rightarrow (\overline{<a>})^2 \subseteq P$. By assumption, $a \in P$ and hence $A \subseteq P$. Hence P is a 2(1)-semiprime partial ideal of R. □

Theorem 2.6. *Let P be a subtractive partial ideal of a partial semiring R. Then P is semiprime if and only if it is 2(1)-semiprime.*

Proof. If P is a semiprime partial ideal of R then clearly P is a 2-semiprime partial ideal of R. Conversely suppose that P is a 2-semiprime partial ideal of R. Let A be a partial ideal of R such that $A^2 \subseteq P$. Then $A \subseteq (P : A)_l$. Since P is subtractive, by the Lemma 1.14, $(P : A)_l$ is a subtractive partial ideal of R containing A. $\Rightarrow \overline{A} \subseteq (P : A)_l$. $\Rightarrow \overline{A}A \subseteq P$. $\Rightarrow A \subseteq (P : \overline{A})_r$. Since P is subtractive, by the Lemma 1.14, $(P : \overline{A})_r$ is a subtractive partial ideal of R containing A. $\Rightarrow \overline{A} \subseteq (P : \overline{A})_r$. $\Rightarrow (\overline{A})^2 \subseteq P$. Since P is 2-semiprime and $\overline{A}$ is a subtractive partial ideal of R such that $(\overline{A})^2 \subseteq P$, we have $\overline{A} \subseteq P$. $\Rightarrow A \subseteq P$. Hence P is a semiprime partial ideal of R. □

3. $p_2(p_1)$-System of R

We introduce the notion of $p_2(p_1)$-system in partial semirings as follows:

Definition 3.1. *Let R be a partial semiring and N be a subset of R. Then N is called a $p_2(p_1)$-*system *of R if for any $a \in R$, there exists $a_1 \in \overline{<a>}$ and $a_2 \in \overline{<a>}$ ($a_2 \in <a>$) such that $a_1a_2 \in N$.*

Note that every p-system of a partial semiring R is a $p_2(p_1)$-system of R. Following is an example of a partial semiring R in which a $p_2(p_1)$-system is not a p-system.

Example 3.2. *Consider the partial semiring $R = \{0, 1, 2, 3\}$ as in the Example 2.3. Then a subset $N = \{1, 2\}$ of R is a $p_2(p_1)$-system of R. For $2 \in N$, $\exists$ no $r \in R \ni 2r2 = 3 \notin N$. Hence $N = \{1, 2\}$ is not a p-system of R.*

Clearly every $m_2(m_1)$-system of a partial semiring R is a $p_2(p_1)$-system of R. Following is an example of a partial semiring R in which a $p_2(p_1)$-system is not an $m_2(m_1)$-system.

Example 3.3. *Consider the partial semiring $R = \{0, u, v, x, y, 1\}$ as in the Example 2.4. Then a subset $N = \{u, v\}$ of R is a $p_2(p_1)$-system of R. For $u, v \in N$, $\exists$ no $x \in \overline{<u>} = \{0, u\}$, $y \in \overline{<v>} = \{0, v\} \ni xy \in N$. Hence $N = \{u, v\}$ is not an $m_2(m_1)$-system of R.*

Lemma 3.4. *Let R be a complete partial semiring and P be a proper partial ideal of R. Then P is 2(1)-semiprime if and only if $R \setminus P$ is a $p_2(p_1)$-system of R.*

Proof. Suppose P is a 2-semiprime partial ideal of R. Let $a \in R \setminus P$. Then $a \notin P$. By the Theorem 2.5, $(\overline{<a>})^2 \nsubseteq P$. $\Rightarrow \exists\, x \in (\overline{<a>})^2 \ni x \notin P$. $\Rightarrow x = \Sigma_i a_i a_i'$ for some $a_i, a_i' \in \overline{<a>}$. Since $x \notin P$, $\Sigma_i a_i a_i' \notin P$. $\Rightarrow \exists$ some $a_j, a_j' \in \overline{<a>} \ni a_j a_j' \in R \setminus P$. Hence $R \setminus P$ is a p_2-system of R.

Conversely suppose $R \setminus P$ is a p_2-system of R. Let $a \in R$ such that $(\overline{<a>})^2 \subseteq P$. Suppose if $a \notin P$. Then $a \in R \setminus P$. $\Rightarrow \exists\, a_1, a_1' \in \overline{<a>} \ni a_1 a_1' \in R \setminus P$. $\Rightarrow a_1 a_1' \in (\overline{<a>})^2 \ni a_1 a_1' \notin P$. $\Rightarrow (\overline{<a>})^2 \nsubseteq P$, a contradiction. Hence P is a 2-semiprime partial ideal of R. □

Lemma 3.5. *Let R be a partial semiring, A be a $p_2(p_1)$-system of R and $a \in A$. Then there exists an $m_2(m_1)$-system M of R such that $a \in M \subseteq A$.*

Proof. Let $a \in A$. Then $\exists\, a_1, a_2 \in \overline{<a>} \ni a_1 a_2 \in A$. $\Rightarrow \exists\, a_1', a_2' \in \overline{<a_1a_2>} \ni a_1' a_2' \in A$ and $\overline{<a>} \supseteq \overline{<a_1a_2>}$. Continuing this process, we get a sequence $\{a, a_1a_2, a_1'a_2', a_1''a_2'', ...\}$ elements of A such that for every positive integer k, $a_1^k a_2^k \in A$ with $\overline{<a>} \supseteq \overline{<a_1a_2>} \supseteq \overline{<a_1'a_2'>} \supseteq ....$ Take $M = \{a, a_1a_2, a_1'a_2', a_1''a_2'', ...\}$. Then clearly $M \subseteq A$ and $a \in M$. Now it is enough to prove that M is an m_2-system of R. Let $a_1^l a_2^l, a_1^k a_2^k \in M \subseteq A$. Without loss of generality we may assume

$k \leq l$. $\Rightarrow \overline{< a_1^k a_2^k >} \supseteq \overline{< a_1^l a_2^l >}$. Now $\exists\ a_1^{l+1} \in \overline{< a_1^l a_2^l >}$ and $a_2^{l+1} \in \overline{< a_1^l a_2^l >} \subseteq \overline{< a_k^l a_2^k >} \ni a_1^{l+1} a_2^{l+1} \in M$. Hence M is an m_2-system of R. Hence the lemma. □

Definition 3.6. *Let A be a partial ideal of a partial semiring R. Then we define $\mathbb{B}_2(A)(\mathbb{B}_1(A)) = \bigcap\{P \mid P$ is a 2(1)-prime partial ideal of R and $A \subseteq P\}$.*

Theorem 3.7. *Let R be a partial semiring and A be a partial ideal of R. Then $\overline{A}$ is 2(1)-semiprime partial ideal of R if and only if $\mathbb{B}_2(\overline{A}) = \overline{A}$ ($\mathbb{B}_1(\overline{A}) = \overline{A}$).*

Proof. Suppose $\mathbb{B}_2(\overline{A}) = \overline{A}$. Then $\overline{A}$ is the intersection of all 2-prime partial ideals of R containing $\overline{A}$. Since intersection of 2-prime partial ideals of R is a 2-semiprime partial ideal of R, $\overline{A}$ is a 2-semiprime partial ideal of R.

Conversely suppose $\overline{A}$ is a 2-semiprime partial ideal of R. Clearly $\overline{A} \subseteq \mathbb{B}_2(\overline{A})$. Let $a \in R \setminus \overline{A}$. Then by the Lemma 3.4, $R \setminus \overline{A}$ is a p_2-system of R. By the Lemma 3.5, $\exists$ an m_2-system M of $R \ni a \in M \subseteq R \setminus \overline{A}$. By the Theorem 1.16, $\exists$ a 2-prime partial ideal P of $R \ni A \subseteq P$ and $\overline{P} \bigcap M = \emptyset$. $\Rightarrow a \notin \overline{P}$. $\Rightarrow a \notin P$. $\Rightarrow a \notin \mathbb{B}_2(\overline{A})$. Therefore $\mathbb{B}_2(\overline{A}) \subseteq \overline{A}$. Hence $\mathbb{B}_2(\overline{A}) = \overline{A}$. □

References

[1] Acharyulu, G.V.S.: *A Study of Sum-Ordered Partial Semirings*, Doctoral thesis, Andhra University, 1992.

[2] Arbib, M.A., Manes, E.G.: *Partially Additive Categories and Flow-diagram Semantics*, Journal of Algebra, 62, pp. 203-227, 1980.

[3] Jonathan S. Golan.: *Semirings and their Applications*, Kluwer Academic Publishers, 1999.

[4] Manes, E.G., and Benson, D.B.: *The Inverse Semigroup of a Sum-Ordered Partial Semiring*, Semigroup Forum, 31, pp. 129-152, 1985.

[5] Nanda Kumar, P.: *1-(2-) Prime Ideals in Semirings*, Kyungpook Mathematical Journal, 50, pp. 117-122, 2010.

[6] Streenstrup, M.E.: *Sum-Ordered Partial Semirings*, Doctoral thesis, Graduate school of the University of Massachusetts, Feb 1985 (Department of computer and Information Science).

[7] Srininvasa Rao, P.V.: *Ideals of Sum-Ordered Partial Semirings*, International Journal of Computational Cognition(IJCC), 7(2), pp. 59-64, June 2009.

[8] Srininvasa Rao, P.V.: *2(1)-Prime Partial Ideals of Partial Semirings*, accepted in the International Journal of Modern Sciences and Engineering Technology (IJMSET).

[9] Srininvasa Rao, P.V.: *Ideal Theory of Sum-Ordered Partial Semirings*, Doctoral thesis, Acharya Nagarjuna University, 2011.