Tensor product of R-Algebra and R-Homomorphism with M-Injective modules

Dr. Sumit Kumar Dekate
Associate Professor
Sagar Institute of Science and Technology (SISTec) Bhopal MP India

Abstract-In this paper we consider a natural homomorphism

$$\varnothing_M : U \otimes \mathrm{Hom}_R(N, M) \rightarrow \mathrm{Hom}_U(U \otimes N, U \otimes M)$$

where U is an R-algebra ,R=CenU and M , N are any two R-modules .Here it is shown that ,

1. If E_1 be any M-injective submodule of M, then $\varnothing_M$ is isomorphism iff $\varnothing_{E_1}$ is isomorphism,

2. If M is a left R-module then $\varnothing_{Tr_M(N)}$ is isomorphism iff $\varnothing_M$ is isomorphism

3. If N and M are two finitely generated module over an artinian ring R and U be any submodule of M such that each simple submodule of U is M-injective then $\varnothing_M$ is isomorphism iff $\varnothing_U$ is isomorphism

4. If E be M-injective module and E_1 is unique largest submodule of E such that E_1 is generated by E then $\varnothing_{E_1}$ is isomorphism iff $\varnothing_M$ is isomorphism

INTRODUCTION

Our aim is to find the relation between the groups

$U \otimes_R \mathrm{Hom}_R(N, M)$ and $\mathrm{Hom}_U(U \otimes N, U \otimes M)$

under the natural homomorphism

$$\varnothing_M : U \otimes_R \mathrm{Hom}_R(N, M) \rightarrow \mathrm{Hom}_U(U \otimes N, U \otimes M)$$

defined by

$$\emptyset_M : (a \otimes f)\ (b \otimes x)\ = ab \otimes f(x)\ ,\ a,b \in U\ ,\ x \in M\ ,\ f \in \mathrm{Hom}_R(N, M)$$

where R is any commutative ring with identity and U is an algebra .For our proposed purpose we have studied the condition under which $\emptyset_M$ is isomorphism , epimorphism or monomorphism.

Throughout this paper we assumed that all rings have unity and all modules are unitary.

Keywords- M-injective module, injective modules, artinian ring, tensor product, finitely generated module.

Theorem 1.1: Let M be any left R-module and E_1 be any M-injective submodule of M then $\emptyset_M$ is isomorphism iff $\emptyset_{E_1}$ is isomorphism.

Proof: Given that M be any left R-module and E_1 be any M-injective submodule of M then monomorphism

$$f: E_1 \to M$$

splits and it implies that E_1 is the direct summand of M then there is a module E_2 such that

$$M = E_1 \oplus E_2$$

then consider the commutative diagram

$$\emptyset_1 : U \otimes \mathrm{Hom}_R(N, M) \to \bigoplus_{i=1}^{2} U \otimes \mathrm{Hom}_R(N, E_i)$$

$$\emptyset_M \downarrow \qquad\qquad\qquad\qquad \downarrow \emptyset_{E_i}$$

$$\emptyset_2 : \mathrm{Hom}_U(U \otimes N, U \otimes M) \to \bigoplus_{i=1}^{2} \mathrm{Hom}_U(U \otimes N, U \otimes E_i)$$

Here both $\varnothing_1$ and $\varnothing_2$ are obvious isomorphism, then from above diagram $\varnothing_M$ is isomorphism iff $\varnothing_{E_1}$ is isomorphism.

Theorem 1.2: If M be any left R-module and E be any injective submodule of M, then $\varnothing_M$ is isomorphism iff $\varnothing_E$ is isomorphism.

Proof: It is an easy consequence of above theorem.

Theorem1.3: If M is left R-module then $\varnothing_{Tr_M(N)}$ is isomorphism iff then $\varnothing_M$ is isomorphism.

Proof: Let M be left R-module then consider a commutative diagram

$$\begin{array}{ccc} \varnothing_1 : U\otimes \mathrm{Hom}_R(N, \mathrm{Tr}_M(N)) & \rightarrow & U\otimes \mathrm{Hom}_R(N, M) \\ \varnothing_{Tr_M(N)}\downarrow & & \downarrow \varnothing_M \\ \varnothing_2 : \mathrm{Hom}_U(U\otimes N, U\otimes \mathrm{Tr}_M(N)) & \rightarrow & \mathrm{Hom}_U(U\otimes N, U\otimes M) \end{array}$$

them from Exercise 8.7 [1] P 112 , $\varnothing_1$ is isomorphism and from definition of tensor product of homomorphism $\varnothing_2$ is also be an isomorphism, then from above diagram $\varnothing_{Tr_M(N)}$ is isomorphism iff $\varnothing_M$ is isomorphism.

Example 1.4: Let R be any artinian ring and N, M are finitely generated left R-module if U be any submodule of M such that each simple submodule of U is M-injective, then $\varnothing_M$ is isomorphism iff $\varnothing_U$ is isomorphism.

Proof : Given that R be any artinian ring and N , M are finitely generated left R-module then left R-module M is artinian and since U is any submodule of M then

U is also be artinian , then

$$\mathrm{SocU} \trianglelefteq \mathrm{U}$$

since given that each simple submodule of U is M-injective then SocU is M-injective and since SocU $\trianglelefteq$ U, then from Exercise 16.13 [1] P 190, U is M-injective and from Theorem 1.1 , $\varnothing_M$ is isomorphism iff $\varnothing_U$ is isomorphism.

Example 1.5: Let E be M-injective module and E_1 is unique largest submodule of E such that E_1 is generated by E then $\varnothing_{E_1}$ is isomorphism iff $\varnothing_M$ is isomorphism.

Proof : Given that E be M-injective module and E_1 is unique largest submodule of E generated by M then

$$Tr_E(M) \leq E_1$$

then from Exercise 16.13 [1] P 190, E_1 is M-injective and then from Theorem 1.3, $\varnothing_{E_1}$ is isomorphism iff $\varnothing_M$ is isomorphism.

REFERENCES

[1] F. W. Anderson and K. R. Fuller, Rings and Catagories of modules , New York Springer – Verlag Inc. 1973.
[2] LOUIS HALLE ROWEN, Ring theory.
[3] T. Y. LAM, A first course in noncommutative rings Springer-Verlag.
[4] ANDOR KERTE'SZ, Lectures on artinian ring, Akademiai Kiado, Budapest 1987.