# Some common fixed point results for four mappings on p-cone metric type space using f-phi-psi-weakly contraction.

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***Abstract*- In this paper we prove some new theorems about common fixed point for multi-valued and single-valued mappings in p- cone metric type space satisfying a weak contractive condition. The theorems use weakly compatibility and $f-\psi-\varphi$ -weakly contraction as [1].**

***Keywords*- p-cone metric type space, common fixed point, generalized $f-\psi-\varphi$ -weakly contraction, weakly compatible mappings.**

## I. Introduction

Huang and Zhang [3] have reviewed the concept of cone metric spaces replacing the set of real numbers by an ordered Banach space. They have generalized the Banach contraction in cone metric space, and have proved many other theorems. There are many authors who have extended these results to regular cone metric space as R. H. Haghi, Sh. Rezapour[7]. In other papers is used the normality of cone metric space. Also, there are many works with common fixed point theorems in normal cone metric space as Th. Abdeljawad and E. Karapinar[6] . E. Hoxha and A. H. Ansari[1] have given some results for discolated metric spaces. In this paper, we consider cone metric type space for $p \geq 1$ which is a generalization of cone metric spaces where $p = 1$ and we have seen the results of [1] in this space. Also we have used a weakly contraction which is a generalization of [1]. We have given an example for our main result.

Now we recall some known notions, definitions and results which are used in this paper.

## II. Preliminaries

***Definition: 2.1 [3]***

Let $E$ be a real Banach space and $P$ be a subset of $E$. $P$ is called a *cone* if and only if

(i) $P$ is closed, $P \neq \emptyset$, $P \neq \{0\}$;

(ii) for all $x, y \in P \Rightarrow \alpha x + \beta y \in P$, where $\alpha, \beta \in R^+$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, int $P$ denotes the interior of $P$. From now on, it is assumed that $\operatorname{int} P \neq \Phi$.

The cone $P$ is called normal if there is a number $K > 0$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$, for all $x, y \in E$. The least positive $k$ satisfying this is called the normal constant of $P$. Sh.Rezapour and R. Hamlbarani[5] have proved that doesn't exist a cone metric space with normal constant K<1, so K > 1.

The cone $P$ is called regular if every increasing sequence which is bounded above is convergent; that is if $x_n$ is a sequence such that $x_1 \le x_2 \le \ldots \le x_n \le \ldots \le y$, for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$ .Equivalently, the cone $P$ is regular if every sequence which is bounded below is convergent.

***Lemma:2.2 [5]***

(i) Every regular cone is normal.
(ii) For each k>1, there is a normal cone with normal constant K > k.
(iii) The cone P is regular if every decreasing sequence which is bounded from below is convergent.

***Definition:2.3[3]*** Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 \le d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x=y$;
(ii) $d(x, y) = d(y, x)$for all $x, y \in X$;
(iii) $d(x, y) \le d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then, $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

***Definition:2.4[3]*** Let X be a nonempty set and $p \ge 1$. Suppose the mapping $d_p : X \times X \to E$ satisfies

(i) $0 \le d_p(x, y)$ for all $x, y \in X$,
(ii) $d_p(x, y) = d_p(y, x)$ if and only if $x = y$ ;
(iii) $d_p(x, z) \le p(d_p(x, y) + d_p(y, z))$ for all $x, y, z \in X$

Then $d_p$ is called $p$- cone metric on X, and $(X, d_p)$ is called a $p$-cone metric type space.

***Definition:2.5*** Let $(X, d_p)$ be a $p$-cone metric type space, $x \in X$ and let $\{x_n\}_{n \in N}$ be a sequence in X. Then

(i) $\{x_n\}_{n \in N}$ converges to $x$ if for every $c \in E$ with $0 \ll c$ there is a natural number $n_0$, such that $d_p(x_n, x) \ll c$ for all $n \ge n_0$. It is denoted $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ ;
(ii) $\{x_n\}_{n \in N}$ is a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there is a natural number $n_0$, such that $d_p(x_n, x_m) \ll c$ for all $n, m \ge n_0$;
(iii) $(X, d_p)$ is a complete cone metric space if every Cauchy sequence in X is convergent in X.

***Lemma:2.6[3]*** Let$(X, d_p)$ be a $p$-cone metric type space, let P be a normal cone constant K, and let$\{x_n\}_{n \in N}$ be a sequence in X. Then,

(i) The sequence $\{x_n\}_{n \in N}$ converges to $x$ if and only if $d_p(x_n, x) \to 0$ (or equivalently $\|d_p(x_n, x)\| \to 0$);
(ii) The sequence $\{x_n\}_{n \in N}$ is Cauchy if and only if $d_p(x_n, x_m) \to 0$ (or equivalently $\|d_p(x_n, x_m)\| \to 0$);
(iii) The sequence $\{x_n\}_{n \in N}$ converges to $x$ and the sequence $\{y_n\}_{n \in N}$ converges to $y$ then $d_p(x_n, y_n) \to d_p(x, y)$.

***Definition:2.7 [8]***

P is called minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in X$ .

Let $(X, d_p)$ be a p-cone metric type space. We denote the family of all nonempty, bounded subset of X by $B(X)$.

***Definition:2.8[2]***

Let $A, B \in B(X)$,then $H(A,B) = \max\{\sup_{x \in A} d_p(x,B), \sup_{y \in B} d_p(y,A)\}$, $d_p(A,B) = \inf\{d_p(x,y) : x \in A, y \in B\}$,

$\delta_p(A,B) = \inf\{d_p(x,y) : x \in A, y \in B\}$.

***Definition:2.9[10]***

The mappings $F : X \to B(X)$ and $f : X \to X$ are weakly compatible if they commute at coincidence points, so $\{t \in X / Ft = \{ft\}\} \subseteq \{t \in X / Fft = fFt\}$.

***Definition:2.10[4]***

The function $\psi : P \to P$ which satisfies the following conditions

1. $\forall t \in P, \psi(t) < t$
2. $\forall t_1, t_2 \in P, t_1 < t_2 \Rightarrow \psi(t_1) < \psi(t_2)$

is called a ultra altering function.

***Remark:2.11***

Note that in following theorem we take minihedral cone.

## III. Main results

***Definition:3.1***

The function $f : P \times P \to P$ is called C-class if it is continuous and

(i) $f(s,t) \le s$

(ii) $f(s,t) = s \Rightarrow s = 0$ or $t = 0$ for all $s, t \in P$

(iii) $f(0,0) = 0$

***Example:3.2***

$E = R^2, P = \{(x,y) \in E^2, x, y > 0\}$, $f(s,t) = s - t$, $f(s,t) = s \Rightarrow t = 0$, so *f* is a C-class function.

***Theorem:3.3***

Let $(X, d_p)$ be a complete p-cone metric space. Let $F, G : X \to B(X)$ be mappings and $I, J : X \to X$ be self-mappings. Suppose that

(1) $\psi(H(Fx, Gy)) \le f(\psi(N(x,y), \varphi(N(x,y))$ where

$N(x,y) = \max\{d_p(Ix, Jy), d_p(Ix, Fx), d_p(Jy, Gy), \frac{d_p(Ix, Gy)}{a}, \frac{d_p(Jy, Fx)}{a}\}$ for all $x, y \in P$, $a \ge p$ and $\varphi, \psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, and *f* a function of C-class

(2) $\cup G(X) \subseteq I(X), \cup F(X) \subseteq J(X)$ and either $I(X)$ or $J(Y)$ is closed,

(3) the pairs of mappings $\{F, I\}$ and $\{G, J\}$ are weakly compatible,

then $F, G, I, J$ have a unique common fixed point $u \in X$ and $Fu = Gu = \{u\}$.

***Proof:***

Let $x_0 \in X$ be an arbitrary point. By (2) there exist $x_1 \in X$ such that $Jx_1 \in Fx_0$. Furthermore, for this point $x_1$ we can choose $x_2 \in X$ such that $Ix_2 \in Gx_1$ and $d_p(Jx_1, Ix_2) \le H(Fx_0, Gx_1)$.

By condition (1) of Theorem1 and the property of function f, we have:

$$\psi(d_p(Jx_1, Ix_2) \le \psi(H(Fx_0, Gx_1)) \le f(\psi(N(x_0, x_1)), \varphi(N(x_0, x_1))) \le \psi(N(x_0, x_1))$$

Continuing this process, we can define inductively the sequence $\{x_n\}$ as follows $Jx_{2n+1} \in Fx_{2n}, Ix_{2n} \in Gx_{2n-1}$ for $n \in N$ and $d_p(Ix_{2n}, Jx_{2n+1}) \le H(Fx_{2n}, Gx_{2n-1})$.

Hence as $\psi$ is monotonically increasing we have

$$\psi(d_p(Ix_{2n}, Jx_{2n+1})) \le \psi(H(Fx_{2n}, Gx_{2n-1})) \le f(\psi(N(x_{2n}, x_{2n-1})), \varphi(N(x_{2n}, x_{2n-1})) \le \psi(N(x_{2n}, x_{2n-1}))$$

and $d_p(Ix_{2n}, Jx_{2n+1}) \le N(x_{2n-1}, x_{2n})$ .(i)

$$N(x_{2n-1}, x_{2n}) = \max\{d_p(Ix_{2n}, Jx_{2n-1}), d_p(Ix_{2n}, Fx_{2n}), d_p(Jx_{2n-1}, Gx_{2n-1}), \frac{d_p(Ix_{2n}, Gx_{2n-1})}{a}, \frac{d_p(Jx_{2n-1}, Fx_{2n})}{a}\}$$

$$\le \max\{d_p(Ix_{2n}, Jx_{2n-1}), d_p(Ix_{2n}, Jx_{2n+1}), d_p(Jx_{2n-1}, Ix_{2n}), \frac{d_p(Ix_{2n}, Ix_{2n})}{a}, \frac{d_p(Jx_{2n-1}, Jx_{2n+1})}{a}\}$$

$$N(x_{2n-1}, x_{2n}) \le \max\{d_p(Ix_{2n}, Jx_{2n-1}), d_p(Ix_{2n}, Jx_{2n+1}), d_p(Jx_{2n-1}, Ix_{2n}), \frac{p(d_p(Jx_{2n-1}, Ix_{2n})) + d_p(Ix_{2n}, Jx_{2n+1}))}{a}\}$$

$$\le \max\{d_p(Ix_{2n}, Jx_{2n+1}), d_p(Jx_{2n-1}, Ix_{2n})\}$$

If we have $\max\{d_p(Ix_{2n}, Jx_{2n+1}), d_p(Jx_{2n-1}, Ix_{2n})\} = d_p(Ix_{2n}, Jx_{2n+1}) \Rightarrow \psi(d_p(Ix_{2n}, Jx_{2n+1})) \le \psi(N(x_{2n-1}, x_{2n})) < \psi(d_p(Ix_{2n}, Jx_{2n+1}))$

which is a contradiction.

So $d_p(Ix_{2n}, Jx_{2n+1}) \le N(x_{2n-1}, x_{2n}) \le d_p(Ix_{2n}, Jx_{2n-1})$ (ii).

Also $d_p(Ix_{2n+2}, Jx_{2n+1}) \le H(Fx_{2n+2}, Gx_{2n+1})$ and

$$\psi(d_p(Ix_{2n+2}, Jx_{2n+1})) \le \psi(H(Fx_{2n}, Gx_{2n+1})) \le f(\psi(N(x_{2n}, x_{2n+1})), \varphi(N(x_{2n}, x_{2n+1}))) \le \psi(N(x_{2n}, x_{2n+1}))$$

so $d_p(Ix_{2n+2}, Jx_{2n+1}) \le N(x_{2n+1}, x_{2n})$ .(iii)

In the same way, we prove that

$d_p(Ix_{2n+2}, Jx_{2n+1}) \le N(x_{2n}, x_{2n+1}) \le d_p(Ix_{2n}, Jx_{2n+1})$ .(iv)

Let put for convenience, $y_k = \begin{cases} Ix_{2n}, & k=2n \\ Jx_{2n-1}, & k=2n-1 \end{cases}$.

By (ii) and (iv), we conclude $d_p(y_{k+1}, y_k) \le N(y_k, y_{k-1}) \le d_p(y_k, y_{k-1})$, for $k \in N$.

It implies that the sequence $\{d_p(y_k, y_{k+1})\}$ is monotone non-increasing and bounded below, so it converges to $l \ge 0$, $\lim_{k\to\infty} d_p(y_k, y_{k+1}) = \lim_{k\to\infty} N(y_k, y_{k+1}) = l$.

Since $\psi$ is continuous and $\varphi$ is lower semi-continuous, $\psi(l) = \lim_{k\to\infty} \psi(N(y_k, y_{k+1}))$ and $\varphi(l) \le \liminf_{k\to\infty} \varphi(N(y_k, y_{k+1}))$.

By (i) and (iv) and from the property of *f*, $\psi$, $\varphi$ we conclude that

$\psi(l) \le f(\psi(l), \varphi(l)) \le \psi(l) \Rightarrow f(\psi(l), \varphi(l)) = \psi(l) \Rightarrow \psi(l) = 0$ or $\varphi(l) = 0 \Rightarrow l = 0$ and so

$\lim_{k\to\infty} d_p(y_k, y_{k+1}) = \lim_{k\to\infty} N(y_k, y_{k+1}) = 0$.

Now let prove that sequence $\{y_k\}$ is Cauchy. Note $C_k = \sup\{d_p(y_i, y_j) : i, j > k\}$. This sequence is decreasing. So this sequence converges to $C \ge 0$.

For every $r \in N$, there exist $k(r), s(r) \in N, k(r), s(r) \ge r$ and $C_r - \frac{1}{r} C_0 \le d_p(y_{k(r)}, y_{s(r)}) \le C_r$, where $C_0 \in P$. Taking the limit when $r \to \infty$, we have $d_p(y_{k(r)}, y_{s(r)}) \to C$. We have to prove that $C = 0$.

Case1

If $k(r)$ is even and $s(r)$ is odd, so $k(r) = 2q$ and $s(r) = 2t-1$ then

$d_p(y_{k(r)+1}, y_{s(r)+1}) = d_p(y_{2q+1}, y_{2t}) = d_p(Jx_{2q+1}, Ix_{2t}) \le H(Fx_{2q}, Gx_{2t-1})$.

By (1), we have

$\psi(d_p(y_{k(r)+1}, y_{s(r)+1})) = \psi(d_p(y_{2q+1}, y_{2t})) = \psi(d_p(Jx_{2q+1}, Ix_{2t})) \le \psi(H(Fx_{2q}, Gx_{2t-1})) \le f(\psi(N(x_{2q}, x_{2t-1})), \varphi(N(x_{2q}, x_{2t-1})))$

So, $d_p(y_{k(r)+1}, y_{s(r)+1}) \le N(x_{2q}, x_{2t-1}) = N(x_{k(r)}, x_{s(r)})$.

$$N(x_{2q}, x_{2t-1}) = \max\{d_p(Ix_{2q}, Jx_{2t-1}), d_p(Ix_{2q}, Fx_{2q}), d_p(Jx_{2t-1}, Gx_{2t-1}), \frac{d_p(Ix_{2q}, Gx_{2t-1})}{a}, \frac{d_p(Jx_{2t-1}, Fx_{2q})}{a}\}$$

$$\le \max\{d_p(Ix_{2q}, Jx_{2t-1}), d_p(Ix_{2q}, Jx_{2q+1}), d_p(Jx_{2t-1}, Ix_{2t}), \frac{d_p(Ix_{2q}, Ix_{2t})}{a}, \frac{d_p(Jx_{2t-1}, Jx_{2q+1})}{a}\}$$

$$= \max\{d_p(y_{k(r)}, y_{s(r)}), d_p(y_{k(r)}, y_{k(r)+1}), d_p(y_{s(r)}, y_{s(r)+1}), \frac{d_p(y_{k(r)}, y_{s(r)+1})}{a}, \frac{d_p(y_{s(r)}, y_{k(r)+1})}{a}\}$$

Taking limit when $r \to \infty$, we have $\lim_{r\to\infty} N(x_{2q}, x_{2t-1}) = \lim_{r\to\infty} N(y_{k(r)}, x_{s(r)}) = \max\{C, 0, 0, \frac{C}{a}, \frac{C}{a}\} = C$

Taking limit when $r \to \infty$ in inequality

$\psi(d_p(y_{k(r)+1},y_{s(r)+1}))=\psi(d_p(y_{2q+1},y_{2t}))=\psi(d_p(Jx_{2q+1},Ix_{2t}))\leq\psi(H(Fx_{2q},Gx_{2t-1}))\leq f(\psi(N(x_{2q},x_{2t-1})),\varphi(N(x_{2q},x_{2t-1})))$

since $\psi$ is continuous and $\varphi$ lower semi-continuous, we have $\psi(C)\leq f(\psi(C),\varphi(C))\leq\psi(C)\Rightarrow\psi(C)=0$ or $\varphi(C)=0\Rightarrow C=0$.

So, $d_p(y_{k(r)},y_{s(r)})\xrightarrow[r\to\infty]{}0$, when $k(r)$ is even and $s(r)$ is odd. Similarly, if $k(r)$ is odd and $s(r)$ is even.

Case2. If $k(r)$ and $s(r)$ are even, so $k(r)=2q$ and $s(r)=2t$ we have

$$d_p(y_{k(r)},y_{s(r)})=d_p(y_{2q},y_{2t})=d_p(Ix_{2q},Ix_{2t})$$
$$\leq p(d_p(Ix_{2q},Jx_{2q+1})+d_p(Jx_{2q+1},Ix_{2t}))=p(d_p(y_{k(r)},y_{k(r)+1})+d_p(y_{k(r)+1},y_{s(r)}))$$

Taking $r\to\infty$ in this inequality, from Case1 we have that $\lim_{r\to\infty}d_p(y_{k(r)},y_{s(r)})=0$. Similarly, if $k(r)$ and $s(r)$ are odd. So the sequence $\{y_k\}_{k\in N}$ is Cauchy. Since $(X,d_p)$ is complete, $\{y_k\}_{k\in N}$ is convergent to $y$ So $\lim_{n\to\infty}Ix_{2n}=\lim_{n\to\infty}Jx_{2n-1}=y$ and $\{Fx_{2n}\},\{Gx_{2n+1}\}$ converge at $\{y\}$. Since condition 2 of theorem, there is a point $z\in X$ such that $Jz=y$. From (1) we have $\psi(d_p(Fx_{2n},Gz))\leq\psi(H(Fx_{2n},Gz))\leq f(\psi(N(x_{2n},z)),\varphi(N(x_{2n},z)))$.

Now we see

$$N(x_{2n},z)=\max\{d_p(Ix_{2n},Jz),d_p(Ix_{2n},Fx_{2n}),d_p(Jz,Gz),\frac{d_p(Ix_{2n},Gz)}{a},\frac{d_p(Jz,Fx_{2n})}{a}\}$$
$$=\max\{d_p(Ix_{2n},y),d_p(Ix_{2n},Fx_{2n}),d_p(y,Gz),\frac{d_p(Ix_{2n},Gz)}{a},\frac{d_p(y,Fx_{2n})}{a}\}.$$

Taking the limit of both sides we have

$$\lim_{n\to\infty}N(x_{2n},z)=\max\{d_p(y,y),d_p(y,\{y\}),d_p(y,Gz),\frac{d_p(y,Gz)}{a},\frac{d_p(y,\{y\})}{a}\}=d_p(y,Gz).$$

Also if we take the limit when $n\to\infty$ in $\psi(d_p(Fx_{2n},Gz))\leq\psi(H(Fx_{2n},Gz))\leq f(\psi(N(x_{2n},z)),\varphi(N(x_{2n},z)))$ we obtain

$$\psi(d_p(y,Gz))\leq f(\psi(d_p(y,Gz)),\varphi(d_p(y,Gz)))\leq\psi(d_p(y,Gz))$$
$$\Rightarrow\psi(d_p(y,Gz))=0 \text{ or } \varphi(d_p(y,Gz))=0\Rightarrow d_p(y,Gz)=0\Rightarrow Gz=\{y\}=\{Jz\}$$

But the pair of maps $\{G,J\}$ are weakly compatible, thus $GJz=JGz, Gy=\{Jy\}$.

Now we prove that $y$ is a fixed point of $G$ and $J$.

$$N(x_{2n},y)=\max\{d_p(Ix_{2n}Jy),d_p(Ix_{2n},Fx_{2n}),d_p(Jy,Gy),\frac{d_p(Ix_{2n},Gy)}{a},\frac{d_p(Jy,Fx_{2n})}{a}\}.$$

We see

$$\lim_{n\to\infty} N(x_{2n}, y) = \max\{d_p(y,Gy), d_p(Jy,\{Jy\}), \frac{d_p(y,Gy)}{a}, \frac{d_p(y,Gy)}{a}\} = d_p(y,Gy).$$

By (1) we have

$$\psi(d_p(y,Gy)) \le \psi(H(y,Gy)) \le f(\psi(d_p(y,Gy)),\varphi(d_p(y,Gy))) \le \psi(d_p(y,Gy))$$
$$\Rightarrow \psi(d_p(y,Gy)) = 0 \text{ or } \varphi(d_p(y,Gy)) = 0 \Rightarrow d_p(y,Gy) = 0 \Rightarrow Gy = \{y\} = \{Jy\}.$$

So $y$ is a fixed point of $G$ and $J$. Now from $\cup G(X) \subseteq I(X)$ implies that exists $w \in X$ such that $Gy = Iw$. Hence, $\{y\} = Gy = \{Jy\} = Iw$ and $H(Fw,y) = H(Fw,Gu)$. Using (1) we have

$$\psi(H(Fw,y)) = \psi(H(Fw,Gy)) \le f(\psi(N(w,y)),\varphi(N(w,y)))$$

$$N(w,y) = \max\{d_p(Iw,Jy), d_p(Iw,Fw), d_p(Jy,Gy), \frac{d_p(Iw,Gy)}{a}, \frac{d_p(Jy,Fw)}{a}\} = d_p(y,Fw)$$

So,

$$\psi(H(Fw,y) \le f(\psi(d_p(Fw,y),\varphi(d_p(Fw,y))) \le \psi(d_p(Fw,y))$$
$$\Rightarrow \psi(d_p(Fw,y)) = 0 \text{ or } \varphi(d_p(Fw,y)) = 0 \Rightarrow d_p(Fw,y) = 0 \Rightarrow Fy = \{y\} = \{Iy\}.$$

Hence, $\{y\} = Gy = \{Jy\} = Fw = \{Iw\}$. Since the pair of maps $\{F, I\}$ are compatible, $FIw = IFw,$ so $Fy = \{Ty\}$.

Moreover

$$\psi(d_p(Fy,y)) = \psi(H(Fy,y)) = \psi(H(Fy,Gy)) \le f(\psi(N(y,y)),\varphi(N(y,y)))$$

$$N(y,y) = \max\{d_p(Iy,Jy), d_p(Iy,Fy), d_p(Jy,Gy), \frac{d_p(Iy,Gy)}{a}, \frac{d_p(Jy,Fy)}{a}\} = d_p(y,Fy)$$

Since

$$\psi(d_p(Fy,y)) \le \psi(H(Fy,y)) \le f(\psi(d_p(Fy,y),\varphi(d_p(Fy,y))) \le \psi(d_p(Fy,y))$$
$$\Rightarrow \psi(d_p(Fy,y)) = 0 \text{ or } \varphi(d_p(Fy,y)) = 0 \Rightarrow d_p(Fy,y) = 0 \Rightarrow Fy = \{Iy\} = \{y\} = Gy = \{Jy\}$$

So $y$ is a common fixed point for $F, G, J, I$.

Now we prove that $y$ is a unique fixed point of $F, G, J, I$. Suppose that there exist another fixed point $z$ of $F, G, J, I$.

$$\psi(d_p(y,z)) = \psi(H(y,z)) = \psi(H(Fy,Gz)) \le f(\psi(N(y,z)),\varphi(N(y,z)))$$

$$N(y,z) = \max\{d_p(Iy,Jz), d_p(Iy,Fy), d_p(Jz,Gz), \frac{d_p(Iy,Gz)}{a}, \frac{d_p(Jz,Fy)}{a}\} = d_p(y,z)$$

So

$$\psi(d_p(y,z)) \le f(\psi(d_p(y,z),\varphi(d_p(y,z))) \le \psi(d_p(y,z))$$
$$\Rightarrow \psi(d_p(y,z)) = 0 \text{ or } \varphi(d_p(y,z)) = 0 \Rightarrow d_p(y,z) = 0 \Rightarrow y = z$$

***Example:3.4***

Let $X=[0,1], E=C_R^1([0,1])$ with the norm $\|f\|=\|f\|_\infty+\|f'\|_\infty$ and consider the cone $P=\{f\in E: f(t)\geq 0\}$ which is not normal.

Define $d: X\times X\to E, d(x,y)(t)=\begin{cases} e^t\max(x,y), x\neq y \\ 0, \quad x=y\end{cases}, 0\leq t\leq 1$.

Now we take $F,G: X\to B(X), F(x)=[0,\frac{x}{4}], Gy=\{\frac{y}{4}\}$ and $I,J: X\to X, I(x)=\begin{cases} x, x\leq \frac{1}{2} \\ 2x, x>\frac{1}{2}\end{cases}, J(y)=y, \psi: P\to P, \psi(x)=\frac{x}{2}$

and $\varphi: P\to P, \varphi(x)=\frac{x}{2}$, $f: P\times P\to P, f(s,t)=\frac{s}{1+t}$.

It's clear that the second and third conditions of theorem are true. Now let prove the first condition.

$H(Fx,Gy)=e^t\max\{\frac{x}{4},\frac{y}{4}\}$.

For $0\leq x\leq\frac{1}{2}, x<y, Ix=x, d(Ix,Jy)=ye^t, d(Ix,Fx)=xe^t, d(Ix,Gy)=xe^t, N(x,y)=ye^t$

$\psi(N(x,y))=\frac{1}{2}ye^t=\varphi(N(x,y))$, $f(\psi(N(x,y)),\varphi(N(x,y)))=f(\frac{y}{2}e^t,\frac{y}{2}e^t)=\frac{\frac{y}{2}e^t}{1+\frac{y}{2}e^t}=\frac{ye^t}{2+ye^t}$

$H(Fx,Gy)=e^t\frac{y}{4}, \psi(H(Fx,Gy))=\frac{y}{8}e^t$.

Since $2+ye^t<8$ for $0<t<1, 0\leq y\leq\frac{1}{2}$

we have $\psi(H(Fx,Gy))\leq f(\psi(N(x,y)),\varphi(N(x,y)))$.

For $0\leq x\leq\frac{1}{2}, x>y, Ix=x, N(x,y)=ye^t, \psi(N(x,y))=\frac{1}{2}xe^t=\varphi(N(x,y))$ $f(\psi(N(x,y)),\varphi(N(x,y)))=\frac{xe^t}{2+xe^t}$,

$\psi(H(Fx,Gy))=\frac{x}{8}e^t$.

We have $\psi(H(Fx,Gy))\leq f(\psi(N(x,y)),\varphi(N(x,y)))$.

For $\frac{1}{2}<x\leq 1, 2x\geq y, Ix=2x$, $d(Ix,Jy)=2xe^t, d(Ix,Fx)=2xe^t, d(Ix,Gy)=2xe^t$, $N(x,y)=2xe^t$

$\psi(N(x,y))=\frac{1}{2}2xe^t=xe^t=\varphi(N(x,y))$, $f(\psi(N(x,y)),\varphi(N(x,y)))=f(xe^t,xe^t)=\frac{xe^t}{1+xe^t}=\frac{xe^t}{1+xe^t}$.

$H(Fx,Gy)=\max\{\frac{x}{4},\frac{y}{4}\}e^t$, $\psi(H(Fx,Gy))=\frac{1}{2}\max\{\frac{x}{4},\frac{y}{4}\}e^t<\frac{x}{4}e^t$. Since $1+xe^t<4$ for $0<t<1,\frac{1}{2}<y\le 1$,we have $\psi(H(Fx,Gy))\le f(\psi(N(x,y)),\varphi(N(x,y)))$.

So $F,G,I,J$ have a unique fixed point $x=0$.

***Corollary:3.4***

Let $(X,d_p)$ be a complete p-cone metric space. Let $F,G:X\to B(X)$ be mappings and $I,J:X\to X$ be self-mappings. Suppose that

(1) $\psi(H(Fx,Gy))\le\psi(N(x,y))-\varphi(N(x,y))$ where

$N(x,y)=\max\{d_p(Ix,Jy),d_p(Ix,Fx),d_p(Jy,Gy),\frac{d_p(Ix,Gy)}{a},\frac{d_p(Jy,Fx)}{a}\}$ for all $x,y\in P$, $a\ge p$ and $\varphi,\psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, and *f* a function of C-class

(2) $\cup G(X)\subseteq I(X),\cup F(X)\subseteq J(X)$ and either $I(X)$ or $J(Y)$ is closed,

(3) the pairs of mappings $\{F,I\}$ and $\{G,J\}$ are weakly compatible,

then $F,G,I,J$ have a unique common fixed point $u\in X$ and $Fu=Gu=\{u\}$.

***Proof.*** Taking $f(s,t)=s-t$, we are in condition of theorem 3.3.

***Corollary:3.5***

Let $(X,d_p)$ be a complete p-cone metric space. Let $F,G:X\to B(X)$ be mappings and $I,J:X\to X$ be self-mappings. Suppose that

(1) $H(Fx,Gy)\le N(x,y)-\varphi(N(x,y))$ where

$N(x,y)=\max\{d_p(Ix,Jy),d_p(Ix,Fx),d_p(Jy,Gy),\frac{d_p(Ix,Gy)}{a},\frac{d_p(Jy,Fx)}{a}\}$ for all $x,y\in P$, $a\ge p$ and $\varphi$ are ultra-altering, $\varphi$ is lower semi continuous function, and *f* a function of C-class

(2) $\cup G(X)\subseteq I(X),\cup F(X)\subseteq J(X)$ and either $I(X)$ or $J(Y)$ is closed,

(3) the pairs of mappings $\{F,I\}$ and $\{G,J\}$ are weakly compatible,

then $F,G,I,J$ have a unique common fixed point $u\in X$ and $Fu=Gu=\{u\}$.

***Proof:***

Taking $f(s,t)=s-t$ and $\psi:P\to P,\psi(s)=s$, we are in condition of theorem 3.3.

***Theorem:3.6***

Let $(X, d_p)$ be a complete p-cone metric space. Let $F, G : X \to B(X)$ be mappings and $J : X \to X$ be self-mapping. Suppose that

(1) $\psi(H(Fx, Gy)) \le f(\psi(N(x, y), \varphi(N(x, y))$ where

$N(x, y) = \max\{d_p(Jx, Jy), d_p(Jx, Fx), d_p(Jy, Gy), \frac{d_p(Jx, Gy)}{a}, \frac{d_p(Jy, Fx)}{a}\}$ for all $x, y \in P$, $a \ge p$ and $\varphi, \psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, and *f* a function of C-class

(2) $\cup G(X) \subseteq J(X), \cup F(X) \subseteq J(X)$ and $J(X)$ is closed,

(3) the pairs of mappings $\{F, J\}$ and $\{G, J\}$ are weakly compatible,

then $F, G, J$ have a unique common fixed point $u \in X$ and $Fu = Gu = \{u\}$.

***Proof***:

We see that $F, G, J$ are in conditions of theorem1 where $I = J$, so there exists a unique common fixed point $u \in X$ and $Fu = Gu = \{u\}$.

***Theorem:3.7***

Let $(X, d_p)$ be a complete p-cone metric space. Let $F, G : X \to B(X)$ be mappings. Suppose that

(1) $\psi(H(Fx, Gy)) \le f(\psi(N(x, y), \varphi(N(x, y))$ where

$N(x, y) = \max\{d_p(x, y), d_p(x, Fx), d_p(y, Gy), \frac{d_p(x, Gy)}{a}, \frac{d_p(y, Fx)}{a}\}$ for all $x, y \in P$, $a \ge p$ and $\varphi, \psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, and *f* a function of C-class

then $F, G$ have a unique common fixed point $u \in X$ and $Fu = Gu = \{u\}$.

***Proof:***

We see that $F, G$ are in conditions of theorem1 where $I = J = I_X$, $\cup G(X) \subseteq I(X) = X$, $\cup F(X) \subseteq J(X) = X$ and the pairs $\{F, I\}$ and $\{G, J\}$ are weakly compatible, so there exists a unique common fixed point $u \in X$ and $Fu = Gu = \{u\}$.

***Theorem:3.8***

Let $(X, d_p)$ be a complete p-cone metric space. Let $F, G, I, J : X \to X$ be self-mappings. Suppose that

(1) $\psi(H(Fx, Gy)) \le f(\psi(N(x, y), \varphi(N(x, y))$ where

$N(x, y) = \max\{d_p(Ix, Jy), d_p(Ix, Fx), d_p(Jy, Gy), \frac{d_p(Ix, Gy)}{a}, \frac{d_p(Jy, Fx)}{a}\}$ for all $x, y \in P$, $a \ge p$ and $\varphi, \psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, and *f* a function of C-class

(2) $G(X) \subseteq I(X), F(X) \subseteq J(X)$ and either $I(X)$ or $J(Y)$ is closed,

(3) the pairs of mappings $\{F, I\}$ and $\{G, J\}$ are weakly compatible,

then $F,G,I,J$ have a unique common fixed point $u \in X$ .

***Proof:***

We see that $F,G,I,J$ are in conditions of theorem1 where $Gx = \{Gx\}$ and $Fx = \{Fx\}$, so there exists a unique common fixed point $u \in X$ .

***Theorem:3.9***

Let $(X, d_p)$ be a complete p-cone metric space. Let $F,G : X \to X$ be mappings. Suppose that

(1) $\psi(H(Fx,Gy)) \le f(\psi(N(x,y),\varphi(N(x,y))$ where

$N(x,y) = \max\{d_p(x,y), d_p(x,Fx), d_p(y,Gy), \frac{d_p(x,Gy)}{a}, \frac{d_p(y,Fx)}{a}\}$ for all $x,y \in P$ , $a \ge p$ and $\varphi,\psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, and *f* a function of C-class

then $F,G$ have a unique common fixed point $u \in X$ and $Fu = Gu = u$ .

***Proof:***

We see that $F,G$ are in conditions of theorem 3 where $I = J = I_X$ , $Gx = \{Gx\}$ and $Fx = \{Fx\}$, so there exists a unique common fixed point $u \in X$ and $Fu = Gu = u$ .

**Corollary:3.10**

Let $(X, d_p)$ be a complete p-cone metric space. Let $F,G : X \to X$ be mappings. Suppose that

(1) $\psi(H(Fx,Gy)) \le \psi(N(x,y)) - \varphi(N(x,y))$ where

$N(x,y) = \max\{d_p(x,y), d_p(x,Fx), d_p(y,Gy), \frac{d_p(x,Gy)}{a}, \frac{d_p(y,Fx)}{a}\}$ for all $x,y \in P$ , $a \ge p$ and $\varphi,\psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, then $F,G$ has a unique fixed point $u \in X$ .

***Proof:***

Taking $f(s,t) = s - t$, we are in conditions of theorem 5, so $F,G$ has a common unique fixed point $u \in X$ .

***Theorem:3.11***

Let $(X, d_p)$ be a complete p-cone metric space. Let $F : X \to X$ be a mapping. Suppose that

(1) $\psi(d(Fx,Fy)) \le f(\psi(N(x,y)),\varphi(N(x,y)))$ where

$N(x,y) = \max\{d_p(x,y), d_p(x,Fx), d_p(y,Fy), \frac{d_p(x,Fy)}{a}, \frac{d_p(y,Fx)}{a}\}$ for all $x,y \in P$ , $a \ge p$ and $\varphi,\psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, and *f* a function of C-class

then $F$ has a unique fixed point $u \in X$ .

***Proof:***

We see that $F$ are in conditions of theorem 3, where $I = J = I_X$ , $Fx = Gx = \{Fx\}$, so $F$ has a unique fixed point $u \in X$ .

***Corollary:3.12***

Let $(X, d_p)$ be a complete p-cone metric space. Let $F: X \to X$ be a mapping. Suppose that

(1) $\psi(d(Fx,Fy)) \le \psi(N(x,y)) - \varphi(N(x,y))$ where

$N(x,y) = \max\{d_p(x,y), d_p(x,Fx), d_p(y,Fy), \dfrac{d_p(x,Fy)}{a}, \dfrac{d_p(y,Fx)}{a}\}$ for all $x, y \in P$, $a \ge p$ and $\varphi, \psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, then $F$ has a unique fixed point $u \in X$.

***Proof:***

Taking $f(s,t) = s - t$, we are in condition of theorem 6, so $F$ has a unique fixed point $u \in X$.

**Corollary:3.13**

Let $(X, d_p)$ be a complete p-cone metric space. Let $F: X \to X$ be a mapping. Suppose that

(1) $d(Fx,Fy) \le N(x,y) - \varphi(N(x,y))$ where

$N(x,y) = \max\{d_p(x,y), d_p(x,Fx), d_p(y,Fy), \dfrac{d_p(x,Fy)}{a}, \dfrac{d_p(y,Fx)}{a}\}$ for all $x, y \in P$, $a \ge p$ and $\varphi, \psi$ are ultra-altering, $\psi$ is continuous, $\varphi$ is lower semi continuous function, then $F$ has a unique fixed point $u \in X$.

***Proof:***

Taking $f(s,t) = s - t$ and $\psi : P \to P, \psi(s) = s$, we are in condition of theorem 6, so $F$ has a unique fixed point $u \in X$.

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