A Brief Study on Paradox in Transportation Problem

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Abstract

Paradox occurs in a linear transportation problem, but it is related to the classical transportation problem. This paper is an attempt to show that after obtaining optimal solution of transportation problem we can increase the quantity of transportation at lesser cost. Sufficient condition for existence of paradox is proved.

Keywords: Transportation Problem, Initial Basic Feasible Solution, Optimal Solution

1. Introduction

A certain class of linear programming problem know as transportation problems arises very frequently in practical applications. The classical transportation problem received its name because it arises naturally in the contacts of determining optimum shipping pattern. For example: A product may be transported from factories to retail stores. The factories are the sources and the store are the destinations. The amount of products that is available is known and the demands are also known. The problem is that different legs of the network joining the sources to the destination have different costs associated with them. The aim is to find the minimum cost routing of products from the supply point to the destination. The general transportation problem can be formulated as: A product is available at each of m origin and it is required that the given quantities of the product be shipped to each of n destinations. The minimum cost of shipping a unit of the product from any origin to any destination is known. The shipping schedule which minimizes the total cost of shipment is to be determined. The problem can be formulated as:

$$\text{Min } Z=\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, i = 1,2,...m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, j = 1,2,...n$$

$$x_{ij} \geq 0, \text{ for all } i, j$$

For each supply point $i,(i=1,2,...m)$ and demand point $j,(j=1,2,...n)$
c_{ij}=unit transportation cost from i^{th} source to j^{th} destination
x_{ij}=amount of homogeneous product transported from i^{th} source to j^{th} destination
a_i=amount of supply at i^{th} source.
b_j=amount of demand at j^{th} destination.
where a_i and b_j are given non-negative numbers and assumed that total supply is equal to total demand, i.e. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then transportation problem is called balanced otherwise it is called unbalanced. The aim is to minimize the objective function satisfying the above mentioned constraints. In the classical transportation problem of linear programming, the traditional objective is one of minimizing the total cost.

Because of the special structure of the transportation model, the problem can also be represented as Table 1.

Table 1: Tabular representation of model (α)

Destination $\rightarrow$ source $\downarrow$	D_1	D_2	$\dots$	D_n	supply(a_i)
S_1	c_{11}	c_{12}	$\dots$	c_{1n}	a_1
S_2	c_{21}	c_{22}	$\dots$	c_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
S_m	c_{m1}	c_{m2}	$\dots$	c_{mn}	a_m
Demand (b_j)	b_1	b_2	$\dots$	b_n	

But in some cases, we obtain more flow with lesser cost then the flow corresponding to the optimum cost then we say paradox occurs. The paradox is,however, hardly mentioned at all in any number of the great number of textbooks and teaching materials where the transportation problem is treated. The paradox was first observed by whom no one knows and has been part of the folklore known to some but unknown to the majority of workers in the field of optimization.

2. Proposed method

Theorem 1. The sufficient condition for the existence of paradox: *It states that if $\exists at least one cell (r,s) \notin B$ in the optimum table of transportation problem where a_r and b_s are replaced by $a_r + l$ and $b_s + l$ respectively $(l > 0)$ then $(u_r + v_s) < 0$.*

Proof:*Let Z^0 be the value of the objective function and F^0 be the optimum flow corresponding to the optimum solution (X^0) of given transportation problem.The dual variables u_i and v_j are given by $u_i + v_j = c_{ij}\ \forall (i,j) \in B$.*

Then $Z^0 = \sum_i \sum_j c_{ij} x^0_{ij} = \sum_i \sum_j (u_i + v_j) x^0_{ij}$

$$= \sum_i (\sum_j x^0_{ij}) u_i + \sum_j (\sum_i x^0_{ij}) v_j$$

$$= \sum_i a_i u_i + \sum_j b_j v_j$$

And
$F^0 = \sum_i a_i = \sum_j b_j$
Nowlet∃atleastonecell(r,s)∉ B, where a_r and b_s are respectively replaced by $a_r + l$ and $b_s + l$ in such a way that the optimum basis remains same, then the value of the objective function is given by

$$z^{\Lambda} = [\sum_{i \neq r} a_i u_i + \sum_{j \neq s} b_j v_j + u_r(a_r + l) + v_s(b_s + l)]$$

$$= [Z^0 + l(u_r + v_s)]$$

The new flow F^{Λ} is given by

$$F^{\Lambda} = \sum_i a_i + l = \sum_j b_j + l = F^0 + l$$

$$F^{\Lambda} - F^0 = l > 0$$

.
Therefore for the existence of paradox we must have

$$Z^{\Lambda} - Z^0 < 0$$

.
Hence sufficient condition is proved.

2.1. Algorithm of Proposed Method
To proceed with proposed method the given steps are followed:

step 1. Represent the given TP into the form of cost matix as Table 1

step 2. Balance the given TP, if it is not balanced by adding dummy row/column according to requirement of supply/demand

step 3. Find optimal solution of the given TP.

step 4. Find all cells $(r,s) \notin B$ with $(u_r + v_s) < 0$.

step 5. Find minimum flow for l=1 which enter into the existing basis whose corresponding cost is minimum.

3. Numerical Examples

Numerical example: Mr. Rajan has seven factories manufacturing machines and Mr. Singh requires these machines at six different destinations.The transportation cost, supply and demand are shown in the

Table.
Input data and optimal solution obtained by applying MODI method is given in table 2

Table 2: Input data and optimal solution

Ex.	Input Data	Obtained Allocations by Modi Method	Obtained Cost
1	$[c_{ij}]_{6\times7}$=[7 5 1 4 6 12 8; 6 4 2 10 2 3 3; 3 5 1 1 5 10 2; 3 3 3 5 4 5 7; 5 1 3 9 5 6 2; 3 4 4 1 4 5 1]; $[a_i]_{6\times1}$=[60, 39, 33, 20, 13, 8]; $[b_j]_{1\times7}$=[49, 42, 34, 20, 17, 6, 5]	$x_{12} = 13$, $x_{13} = 34$, $x_{14} = 13$, $x_{22} = 16$, $x_{25} = 17$, $x_{26} = 6$, $x_{31} = 26$, $x_{34} = 7$, $x_{41} = 20$, $x_{52} = 13$, $x_{61} = 3$,$x_{67} = 5$	439 of 173 items

Now we cheek the sign of and we obtain for the non basic cell (5,3), the sign is (-ve). For l=1, for the cell (5,3) the next table is given below

Table 3: Input data and optimal solution

Ex.	Input Data	Obtained Allocations	Obtained Cost
1	$[c_{ij}]_{6\times7}$=[7 5 1 4 6 12 8; 6 4 2 10 2 3 3; 3 5 1 1 5 10 2; 3 3 3 5 4 5 7; 5 1 3 9 5 6 2; 3 4 4 1 4 5 1]; $[a_i]_{6\times1}$=[60, 39, 33, 20, 14, 8]; $[b_j]_{1\times7}$=[49, 42, 35, 20, 17, 6, 5]	$x_{12} = 13$, $x_{13} = 34$, $x_{14} = 13$, $x_{22} = 16$, $x_{25} = 17$, $x_{26} = 6$, $x_{31} = 26$, $x_{34} = 7$, $x_{41} = 20$, $x_{52} = 14$, $x_{61} = 3$,$x_{67} = 5$	422 of 174 items

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