Some New Classes of Graceful Lobsters Obtained by Applying Inverse and Component Moving Transformations

Debdas Mishra [1] and Pratima Panigrahi [2]
[1] Department of Mathematics
C. V. Raman College of Engineering
Bidya Nagar, Mahura, Janala
Bhubaneswar - 752054, Khurda, Orissa
[2] Department of Mathematics
Indian Institute of Technology
Kharagpur- 721302
West Midnapore, West Bengal
e-mail: *debdasmishra@gmail.com* [1] *pratima@maths.iitkgp.ernet.in* [2]

Abstract

We observe that a lobster with diameter at least five has a unique path $H = x_0x_1 \ldots x_m$ with the property that besides the adjacencies in H both x_0 and x_m are adjacent to the centers of at least one $K_{1,s}$, where $s > 0$, and each x_i, $1 \leq i \leq m-1$, is at most adjacent to the centers of some $K_{1,s}$, where $s \geq 0$. This unique path H is called the *central path* of the lobster. We call $K_{1,s}$ an *even branch* if s is nonzero even, an *odd branch* if s is odd, and a *pendant branch* if $s = 0$. In this paper we give graceful labelings to some new classes of lobsters with diameter at least five, in which the degree of each vertex x_i, $0 \leq i \leq m-1$, on the central path is even and the degree of the vertex x_m may be odd or even. The lobsters appear in [5] also possess this property. However, in the lobsters of [5], at most the vertex x_0 is attached to a combination of all three types of branches, whereas in this paper, we give graceful labelings to the lobsters in which not only the vertex x_0 but also some (or all) $x_i, 1 \leq i \leq m$, may exhibit this property.

Keywords: graceful labeling, lobster, odd and even branches, inverse transformation, component moving transformation

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1 Introduction

Definition 1.1. A *graceful labeling* of a tree T with q edges is a bijection $f : V(T) \to \{0, 1, 2, \ldots, q\}$ such that $\{|f(u) - f(v)| : \{u, v\}$ is an edge of $T\} = \{1, 2, \ldots, q\}$. A tree which has a graceful labeling is called a *graceful tree.*

Definition 1.2. A *lobster* is a tree having a path from which every vertex has distance at most two. It is easy to check that a lobster L of diameter at least five has a unique path $H = x_0, x_1, \ldots, x_m$ such that besides the adjacencies in H, each x_i, $1 \leq i \leq m-1$, is at most adjacent to the centers of some stars $K_{1,s}$, $s \geq 0$, whereas the vertices x_0 and x_m are adjacent to the center of at least one star $K_{1,s}$ with $s \geq 1$. This path H is called the *central path* of the lobster L. Throughout the paper we use H to denote the central path of a lobster with diameter at least five. For $x_i \in V(H)$, if x_i is adjacent to the center of $K_{1,s}, s \geq 0$, then we call $K_{1,s}$ an *even branch* if s is nonzero even, an *odd branch* if s is odd, and a pendant branch if $s = 0$. Furthermore, whenever we say x_i, for some $0 \leq i \leq m$, is attached to an even number of branches we mean a "non zero" even number of branches unless otherwise stated.

In 1979, Bermond [1] conjectured that all lobsters are graceful. This conjecture is a special case of the famous and unsolved "the graceful tree conjecture" of Ringel and Kotzig (1964) [8], which states that all trees are graceful. Bermond's conjecture is also open and very few classes of lobsters are known to be graceful. Ng [7], Wang et al. [9], Chen et al. [2], Morgan [6] (see [3]), and Mishra and Panigrahi [5] have given graceful labeling to some classes of lobsters. With the lobsters in [9], the lobsters in [5] and those appear in this paper share a common feature that the degree of each $x_i,\ 0 \leq i \leq m-1$, is even. However, in the lobsters of this paper and those appear in [5], the degree of x_m may be odd or even and the branches incident on each $x_i,\ 0 \leq i \leq m$, need not be of the same type. The branches incident on x_0 may be of same type, or any two types, or all three types. In the lobsters of [5], the branches incident on $x_i,\ 1 \leq i \leq m$, may be of the same type (odd or even), or any two types in which each type is odd in number, whereas in the lobsters of this paper, the branches incident on $x_i,\ 1 \leq i \leq m$, may be of the same type (odd or even), any two types, or all three types. In gross the lobsters to which we give graceful labelings in this paper have one of the following features.

(I) For some $t_1,\ 1 \leq t_1 \leq m$, each $x_i,\ 0 \leq i \leq t_1$, is attached to a combination of odd and pendant branches. If $t_1 < m$ then we have either (1) or (2).

(1) For some $t_2,\ t_1+1 \leq t_2 \leq m$, each $x_i,\ t_1+1 \leq i \leq t_2$, is attached to a combination of all three types of branches. If $t_2 < m$ then we have either (a) or (b).

(a) For some $t_3,\ t_2+1 \leq t_3 \leq m$, each $x_i,\ t_2+1 \leq i \leq t_3$, is attached to a combination of two types of branches and each of the rest of the x_i s (if any) is attached to the odd (or even) branches.

(b) Each $x_i,\ t_2+1 \leq i \leq m$, is attached to odd (or even) branches.

(2) For some $t_2,\ t_1+1 \leq t_2 \leq m$, each $x_i,\ t_1+1 \leq i \leq t_2$, is attached to a combination of two types of branches. If $t_2 < m$ then for some $t_3,\ t_2+1 \leq t_3 \leq m$, each $x_i,\ t_2+1 \leq i \leq t_3$, is attached to a combination of two types of branches and each of the rest of the x_i s (if any) is attached to odd (or even) branches.

(II) x_0 is attached to a combination of all three types of branches (respectively, odd and even branches or even and pendant branches) and satisfy the condition (1) (respectively, (2)) in (I) by setting $t_1 = 0$.

In this paper, as in [5], for a given lobster L we first form a diameter four tree $T(L)$ by identifying all the vertices on the central path of L and give a graceful labeling to $T(L)$ by using the technique of [4]. Let A be the set of all the branches incident on the center of $T(L)$. In [5], the authors applied component moving transformation on A to get a graceful labeling of L, whereas here we partition A in an appropriate manner before applying component moving transformation on it.

In order to prove the results of this paper we need some definitions, terminologies and existing results which are described in this section.

Lemma 1.3. [9], [4] If f is a graceful labeling of a tree T with n edges then the inverse transformation of f, defined as $f_n(v) = n - f(v)$, for all $v \in V(T)$, is also a graceful labeling of T.

Definition 1.4. For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a *component incident on* the vertex v. If a and b are vertices of a tree T, $u(T)$ is a component incident on a, and $b \notin u(T)$, then deleting the edge $\{a, u\}$ from T and making b and u adjacent is called *the component moving transformation*. Here we say the component $u(T)$ has been moved from a to b. Throughout the paper we write "the component u" instead of writing "the component $u(T)$"; whenever, we wish to refer to u as a vertex, we write "the vertex u". By the label of the component "$u(T)$" we mean the label of the vertex u. Moreover, we shall not distinguish between a vertex and its label.

Lemma 1.5. [4] Let f be a graceful labeling of a tree T; let a and b be two vertices of T; and let $u(T)$ and $v(T)$ be two components incident on a, where $b \notin u(T) \cup v(T)$. Then the following hold:

(i) if $f(u)+f(v) = f(a)+f(b)$ then the tree T^* obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.

(ii) if $2f(u) = f(a)+f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.

Lemma 1.6. [4] Let T be a diameter four tree with q edges. If a_0 is the center vertex and the degree of a_0 is $2k+1$ then there exists a graceful labeling f of T such that

(a) $f(a_0) = 0$ and the labelings of the neighbours of a_0 are $1, 2, \ldots, k, q, q-1, \ldots, q-k$.

(b) if n_1, n_2, and n_3 are the number of odd, even, and pendant branches incident on a_0, then from the sequence $S = (q, 1, q-1, 2, q-2, 3, \ldots, q-k+1, k, q-k)$ of vertex labels, n_1 terms from the beginning are the labels of the centers of the odd branches, the next n_2 terms are the labels of the centers of the even branches, and the rest n_3 terms are the labels of the centers of the pendant branches.

(c) for any $i = 1, 2, 3$, the n_i labels of S which are the labels of the centers of the same type of branches may be assigned in any order. However, different arrangements of branches of the same type may give different graceful labelings of the same diameter four tree without disturbing (a) and (b).

Remark 1.7. In the graceful labeling f of the diameter four tree T in Lemma 1.6, the labelings of the pendant vertices adjacent to the centers of the odd and even branches can be given by using the technique of [4].

Lemma 1.8. [5] Let $S = (t_1, t_2, \ldots, t_{2p})$ be a finite sequence of natural numbers in which the sums of consecutive terms are alternately $l+1$ and l, beginning (and ending) with the sum $l+1$. Then the sums of consecutive terms in the sequence $S_1 = (\phi_{l+1}(t_{2k+2}), \phi_{l+1}(t_{2k+3}), \ldots, \phi_{l+1}(t_{2p-2k_1-1}))$, where $\phi_n(x) = n-x$, $0 \le k, k_1 \le p-2$, and $0 \le k+k_1 \le p-2$, are alternately $l+2$ and $l+1$, beginning (and ending) with $l+2$.

2 Results

Construction 2.1. Let T be a graceful tree with q edges. Let a_0 be a non pendant vertex of T with degree $2k+1$ such that there exists a graceful labeling f of T in which a_0 gets the label 0 and the labels of the neighbours of a_0 are $1, 2, \ldots, k, q, q-1, q-2, \ldots, q-k$ (see Figure 1). Consider the sequence $S = (q, 1, q-1, 2, \ldots, k, q-k)$ of vertices adjacent to a_0 (recall that we do not distinguish between a vertex and its label). For any integer n, $n \ge 2$, if possible, we partition this sequence into n parts $A_1, A_2, \ldots, A_n$ (see Figure 1), where $A_1 = (q, 1, q-1, 2, \ldots, r_1, q-r_1)$ and $A_j = (r_{j-1}+1, q-r_{j-1}-1, r_{j-1}+2, q-r_{j-1}-2, \ldots, r_j, q-r_j)$, $2 \le j \le n$ and $0 < r_1 < r_2 < \ldots < r_n = k$.

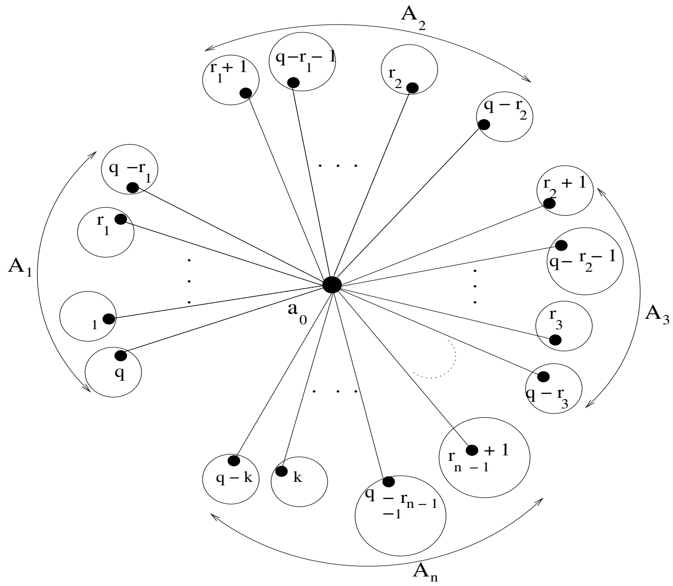

Figure 1: The tree T with vertex a_0 and its neighbours. The circles around the neighbouring vertices of a_0 represent the respective components incident on them.

We construct a tree T_1 (see Figure 2) from T by identifying the vertex y_0 of a path $H' = y_0, y_1, \ldots, y_m$, with a_0 and distributing the components (incident on the vertex a_0) in A_j, $j = 1, 2, \ldots, n$, to y_i, $i = 1, 2, \ldots, s_j$, where $0 \le s_j \le m$, in the following manner.

(1) For $0 \le i \le s_2$ we keep $2l_i^{(2)}$ components of A_2 at y_i, where $l_i^{(2)} > 0$. In particular, we retain

$2p_i+1,\ 0 \le p_i,\ 2p_i+1 < l_i^{(2)}$, components whose labels appear consecutively from the beginning of $A_2^{(i)}$, and $2l_i^{(2)} - 2p_i - 1$ components whose labels appear consecutively from the end of $A_2^{(i)}$, where $A_2^{(0)} = A_2$ and for $1 \le i \le s_2$, $A_2^{(i)}$ is obtained from $A_2^{(i-1)}$ by deleting the component which are kept at y_{i-1}.

(2) The components of A_j, $1 \le j \le n$, $j \ne 2$, are distributed to the vertices $y_1, y_2 \ldots, y_{s_j}$, in the following way:

(i) At y_0 we retain $2l_0^{(1)} + 1$, $l_0^{(1)} \ge 0$ (respectively, $2l_0^{(j)}$, $l_0^{(j)} \ge 1$, $3 \le j \le n$), components of A_1 (respectively, A_j). Among these components $2l_0^{(1)}$ (respectively, $2l_0^{(j)} - 1$) components get labels consecutively from the beginning of A_1 (respectively, A_j) and the remaining component gets the last label of A_1 (respectively, A_j). If $s_j > 0$ then we delete these terms from A_j which are kept at y_0 and name the remaining sequence as $A_j^{(1)}$.

(ii) For $1 \le j \le n$, $j \ne 2$ if $s_j > 0$, we move $2l_i^{(j)}$, $l_i^{(j)} \ge 1$, components from A_j to y_i, where $1 \le i \le s_j$. In particular, we move $2l_i^{(j)} - 1$ components whose labels appear consecutively from the beginning of $A_j^{(i)}$ and one component whose label is the last term of $A_j^{(i)}$, where, for $i > 1$, $A_j^{(i)}$ is obtained from $A_j^{(i-1)}$ by deleting the components which are moved to y_{i-1}.

For $j = 1, 2, \ldots, n$, the numbers $l_i^{(j)}$, $i = 0, 1, 2, \ldots, s_j$, are chosen in such a way that $\sum_{i=0}^{s_j} l_i^{(j)} = r_j - r_{j-1}$, where $r_0 = 0$. $\square$

In the following theorem, for a graceful tree R with n edges and a graceful labeling g we use the notation "$g(R)$" to denote the tree R with the graceful labeling g. Also, for any sequence $F = (a_1, a_2, \ldots, a_r)$, $g_n(F)$ is the sequence $(n - a_1, n - a_2, \ldots, n - a_r)$.

Theorem 2.2. The tree T_1 in Construction 2.1 is graceful.

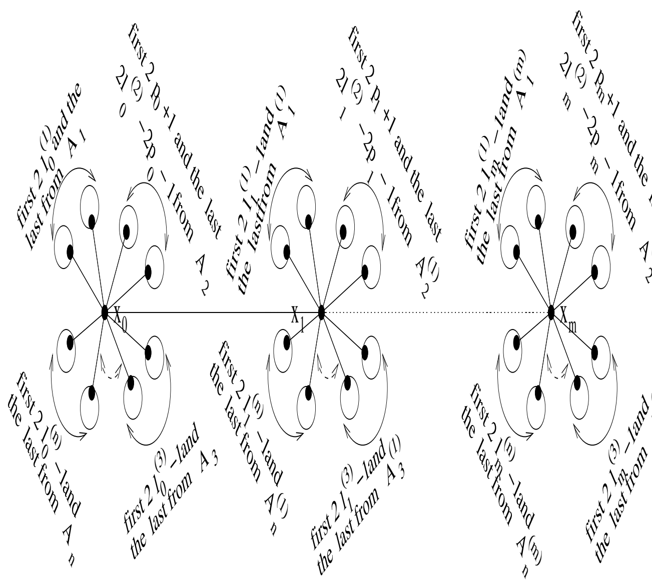

Figure 2: The tree T_1 obtained from T. Here we take $s_1 = s_2 = \ldots = s_n = m$.

Proof: We identify the vertices $a_0 \in V(T)$ and $y_0 \in V(H')$ and give the label $q+1$ to y_1. Clearly the subtree $T \cup \{y_0, y_1\}$ admits a graceful labeling $f^{(1)}$, where $f^{(1)}(x) = f(x)$ if $x \in V(T)$, and $f^{(1)}(y_1) = q+1$. Since $A_j^{(1)}$, $j = 1, 2, \ldots, n$, can be partitioned into pairs of labels whose sum is $q+1$ (consecutive terms), by Lemma 1.5(i) the tree $T^{(1)}$ obtained by moving the components in $A_j^{(1)}$, $1 \le j \le n$ (for which $s_j \ge 1$), to y_1 admits the same graceful labeling $f^{(1)}$. By Lemma 1.3, $f_{q+1}^{(1)}$ is a graceful labeling of $T^{(1)}$ and the label of y_1 in $f_{q+1}^{(1)}(T^{(1)})$ is 0. Next we give the label $q+2$ to y_2. Obviously $f^{(2)}$ is a graceful labeling of $T^{(1)} \cup \{y_1, y_2\}$, where $f^{(2)}(x) = f_{q+1}^{(1)}(x)$ if $x \in V(T^{(1)})$, and $f^{(2)}(y_2) = q+2$. We observe that the sums of consecutive terms in $A_j^{(1)}, j = 1, 2, \ldots, n$, are alternately $q+1$ and q, beginning and ending with the sum $q+1$; so by Lemma 1.8 the sums of consecutive terms in $f_{q+1}^{(1)}(A_j^{(2)})$, are alternately $q+2$ and $q+1$, beginning and ending with the sum $q+2$. Therefore, $f_{q+1}^{(1)}(A_j^{(2)})$ can be partitioned into pairs of labels whose sum is $q+2$. By Lemma 1.5(i), the tree $T^{(2)}$ obtained by moving the components in $f_{q+1}^{(1)}(A_j^{(2)}), 1 \le j \le n$, to y_2, is graceful.

Let $s^\star = max\{s_1, s_2, \ldots s_n\}$. On repeating the above procedure for $s^\star$ times we get the graceful tree $T^{(s^\star)}$ with vertex set $V(T) \cup \{y_1, \ldots, y_{s^\star}\}$ in which the vertex $y_{s^\star}$ gets the label $q + s^\star$. If $s^\star = m$, then we stop otherwise, we proceed as follows.

We apply inverse transformation to the graceful tree $T^{(s^\star)}$ so that the vertex $y_{s^\star}$ gets the label 0. Then we make the vertex $y_{s^\star+1}$ adjacent to $y_{s^\star}$ and give the label $q+s^\star+1$ to $y_{s^\star+1}$. If $s^\star+1 = m$ then we stop otherwise, we repeat this procedure until the vertex y_m gets a label. The graceful tree that is obtained on the vertex set $V(T) \cup V(H')$ is easily seen to be the tree T_1. □

Given a lobster L of the type to which we give a graceful labeling in this paper, we construct a diameter four tree, say $T(L)$, from L by successively identifying the vertices x_i, $i = 1, 2, \ldots, m$, with x_0. The vertex x_0 is the center of $T(L)$ and its degree is odd, say $2k + 1$. By Lemma 1.6, $T(L)$ has a graceful labeling in which x_0 gets the label 0 and the neighbours of x_0 get labels in the sequence S of Construction 2.1. However, we note that the manner in which we partition the sequence S and the order in which the centers of the branches incident on x_0 in $T(L)$ get labels from the sequence S plays an important role. To get back L and a graceful labeling of it we have to follow an appropriate partition and ordering, which will be clear from the proof of Theorem 2.3. Next we apply Theorem 2.2 to $T(L)$ and to the central path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of L. We get graceful labelings of lobsters that appear in Theorem 2.3 by taking $n = 2$ in Construction 2.1.

Theorem 2.3. The lobsters in Tables 3.1, 3.2 and 3.3 below are graceful.

Descriptions of Tables: In the column headings, the triple (x, y, z) represents the number of odd, even, and pendant branches, respectively, where e means any even number of branches (nonzero, unless otherwise stated), o means any odd number of branches, and 0 means no branch. For example, $(e, 0, o)$ means an even number of odd branches, no even branch, and an odd number of pendant branches. If in a triple e or o appear more than once then it does not mean that the corresponding branches are equal in number. For example, (e, e, o) does not mean that the number of odd branches is equal to the number of even branches. The symbol o^* means that $o \geq 3$.

1st column: 0 means that x_0 is attached to any one of the mentioned combinations of branches. The notation $0(r)$, $r = 1, 2$, means that x_0 is attached to the combination of branches mentioned in the column heading in which r is the superscript.

Other columns: $i \to j$ (respectively, $i \to j\ (r)$, $r = 1, 2$) means that each $x_l, i \leq l \leq j$, is attached to the mentioned combination or any one of the combinations of branches (respectively, the branches mentioned in the triple with superscript r).

Further, when some vertex x_i on the central path is attached to two combinations $(x, y, 0)$ and $(0, 0, e)$, we mean that x_i is attached to the combination (x, y, e). For example, in Table 3.1(d), x_{t_2+1} is attached to the combinations $(e, 0, 0)$ and $(0, 0, e)$, which means that x_{t_2+1} is attached to the combination $(e, 0, e)$.

Table 2.1

Lobsters ↓	$(e, 0, o)$	$(o^*, 0, o)$	(e, o, o)	$(0, o, o)$	$(o^*, o, 0)$	$(e, e, 0)$	$(e, 0, 0)^1$ or $(0, e, 0)^2$	$(0, 0, e)$
a	0	$1 \to t_1$, $t_1 < m-1$	$t_1+1 \to t_2$, $t_2 < m$	$t_2+1 \to t^\star$, $t^\star \leq m$	——	——	$t^\star + 1 \to m$ (2) if $t^\star < m$	——
b	0	$1 \to t_1$, $t_1 < m-1$	$t_1+1 \to t_2$, $t_2 < m$	——	——	$t_2+1 \to t'$, $t' \leq m$	$t' + 1 \to m$ if $t' < m$	——
c	0	$1 \to t_1$, $t_1 < m-1$	$t_1+1 \to t_2$, $t_2 < m$	——	——	——	$t_2 + 1 \to m$ (1)	$t_2+1 \to s$, $s \leq m$

d	0	$1 \to t_1,\ t_1 < m-1$	——	——	$t_1+1 \to t_2,\ t_2 < m$	$t_2+1 \to t',\ t' \leq m$	$t'+1 \to m$, if $t' < m$	——
e	0	$1 \to t_1,\ t_1 < m-1$	$t_1+1 \to t_2,\ t_2 < m$	——	——	t_2+1	$t_2+2 \to m$ (1) if $t_2 < m-1$	t_2+1

Table 2.2

Lobsters ↓	(o,o,o)	(e,o,o)	$(0,o,o)$	$(e,e,0)$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
a	0	$1 \to t,\ t < m$	$t+1 \to t^\star,\ t^\star \leq m$	——	$t^\star+1 \to m$ (2) if $t^\star < m$	——
b	0	$1 \to t,\ t < m$	——	$t+1 \to t',\ t' \leq m$	$t'+1 \to m$ if $t' < m$	——
c	0	$1 \to t,\ t < m$	——	——	$t+1 \to m$ (1)	$t+1 \to s,\ s \leq m$
d	0	$1 \to t,\ t < m$	——	$t+1$	$t+2 \to m$ (1), if $t < m-1$	$t+1$
e	0	$1 \to t,\ t < m$	$t+1 \to m-1$	——	m (2)	m

Table 2.3

Lob-sters ↓	$(e,o,0)^{(1)}$ or $(0,e,o)^{(2)}$	$(o,0,0)^{(1)}$or $(0,o,0)^{(2)}$or $(o,e,0)^{(3)}$	$(o^*,o,0)^{(1)}$or $(0,o^*,o)^{(2)}$	$(e,e,0)$	$(e,0,0)^{(1)}$or $(0,e,0)^{(2)}$	$(0,0,e)$
a	0 (2)	——	$1 \to t,\ t < m$ (2)	——	$t+1 \to m$ (2)	$t+1 \to s,\ s \leq m$
b	0 (1)	——	$1 \to t,\ t < m$ (1)	$t+1 \to t',\ t' \leq m$	$t'+1 \to m$ if $t' < m$	——
c	——	0 (1)	——	——	$1 \to m$ (1)	$0 \to s,\ s \leq m$
d	——	0 (2)	——	——	$1 \to m$ (2)	$0 \to s,\ s \leq m$
e	——	0 (3)	——	$1 \to t',\ t' \leq m$	$t'+1 \to m$ if $t' < m$	——

Proof: For every lobster L of this theorem we first construct the diameter four tree $T(L)$ corresponding to L. Let $|E(T(L))| = q$ and $deg(x_0) = 2k+1$. We give the label 0 to x_0. We partition the sequence S in Construction 2.1 into two parts, i.e. we take $n = 2$ in Construction 2.1.

Let L be a lobster of type (a) in Table 2.1. We follow the two steps given below.

1. We determine r_1 and hence A_1 and A_2 in the following manner:

Let the number of odd branches incident on x_0 be $2l_0$, that incident on each $x_i,\ i = 1,2,\ldots,t_1$, be $2l_i+1$, and that incident on each $x_i,\ i = t_1+1,\ldots,t_2$, be $2l_i$, where for $i = 0,1,\ldots,t_2,\ l_i \geq 1$.

Let $\beta_0,\ 0 \le \beta_0 < l_0$, and $\beta_i,\ 1 \le \beta_i \le l_i,\ 1 \le i \le t_1$, be arbitrarily chosen integers. We will give a labeling to $T(L)$ in such a way that among the odd branches incident on x_0 (respectively, $x_i,\ i = 1, 2, \ldots, t_1$), the centers of $2\beta_0 + 1$ (respectively, $2\beta_i$) branches get labels from the sequence A_1 and the centers of the rest of these branches get labels from A_2, whereas the the centers of all the odd branches incident on $x_i,\ i = t_1+1,\ \ldots, t_2$, get labels from A_1 only. Therefore, A_1 contains the centers of $2\beta_0 + 1 + \sum_{i=1}^{t_1} 2\beta_i + \sum_{i=t_1+1}^{t_2} 2l_i$ odd branches. We choose A_1 in such a way that it does not contain the center of any other branch. Therefore, $|A_1| = 2r_1 + 1 = 2\beta_0 + 1 + \sum_{i=1}^{t_1} 2\beta_i + \sum_{i=t_1+1}^{t_2} 2l_i$.

2. We give labelings to the branches incident on the center of $T(L)$ in the following manner:

(i) The centers of $2l_0$ odd branches incident on x_0 in L get $2\beta_0$ labels from the beginning and the last label of A_1, and $2(l_0 - \beta_0) - 1$ labels from the beginning of A_2.

(ii) For $i = 1, 2, \ldots, t_1$, the centers of $2l_i + 1$ odd branches incident on x_i in L get $2\beta_i - 1$ labels from the beginning and the last label of $A_1^{(i)}$, and $2(l_i - \beta_i) + 1$ labels from the beginning of the sequence $A_2^{(i)}$.

(iii) For $i = t_1 + 1, t_1 + 2, \ldots, t_2$, the centers of $2l_i$ odd branches incident on x_i in L get $2l_i - 1$ labels from the beginning and the last label of $A_1^{(i)}$.

(iv) For $i = t_1 + 1, t_1 + 2, \ldots, t^\star$, the centers of the even branches incident on x_i in L get labels from the beginning of $A_2^{(i)}$.

(v) For $i = 0, 1, 2, \ldots, t^\star$, the centers of the pendant branches incident on x_i in L get labels from the end of the sequence $A_2^{(i)}$, $A_2^{(0)} = A_2$.

If $t^\star < m$ then we do the following additional step.

(vi) For $i = t^\star + 1, t^\star + 2, \ldots, m$, among the even branches incident on x_i, the centers of any odd number of branches get labels from the beginning of $A_2^{(i)}$ and the centers of the rest of these branches get labels from the end of $A_2^{(i)}$.

We notice that the labeling of the centers of the branches incident on the center x_0 of $T(L)$ given in step 2 follows part (b) of Lemma 1.6. Therefore, by Lemma 1.6 there exists a graceful labeling of $T(L)$ with the above labels of the center x_0 and the centers of the branches incident on x_0. Finally, we apply Theorem 2.2, for $n = 2$, on $T(L)$ and the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of L (see example below). This approach will be the same for all the remaining cases of this theorem and hence we will just indicate the modification we make in steps 1 and 2.

Example: Figure 3 represents a lobster of type (a) in Table 2.1. We construct the graceful diameter four tree $T(L)$ shown in Figure 4. Here $|E(T(L))| = q = 84$ and $deg(x_0) = 2k + 1 = 35$. Therefore, the sequence $S = (84, 1, 83, 2, \ldots, 17, 67)$. Here $m = 6,\ t_1 = 1,\ t_2 = 3,\ t^\star = 5, l_0 = 2,\ l_1 = 1,\ l_2 = 1,\ l_3 = 1$. We take $\beta_0 = 1,\ \beta_1 = 1$. Therefore, $|A_1| = 2r_1 + 1 = 9$, i.e. $A_1 = (84, 1, 83, 2, 82, 3, 81, 4, 81)$ and $A_2 = (5, 79, 6, \ldots, 17, 67)$. Using step 2 and subsequently the technique of [4] we obtain a graceful labeling of $T(L)$ given in Figure 4. Then in Figure 5 we make x_1 adjacent to x_0, give label 85 to x_1, and move all the components in $A_j^{(1)},\ j = 1, 2, 3$, to x_1. Next we obtain the lobster in Figure 6 by applying inverse transformation to the lobster found in Figure 5, making x_2 adjacent to x_1, giving label 86 to x_2, and moving all the components in $f_{85}^{(1)}(A_j^{(2)}),\ j = 1, 2, 3$, to x_2. Continuing in this manner we finally get the graceful labeling of L presented in Figure 7.

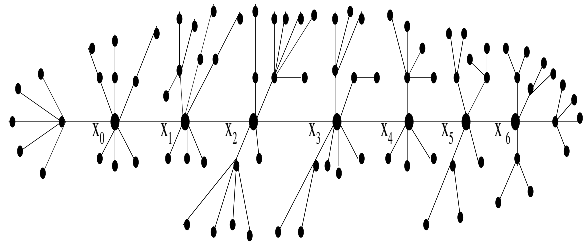

Figure 3: A lobster L of type (a) in Table 2.1. Here $m = 6,\ t_1 = 1,\ t_2 = 3$, and $t^\star = 5$.

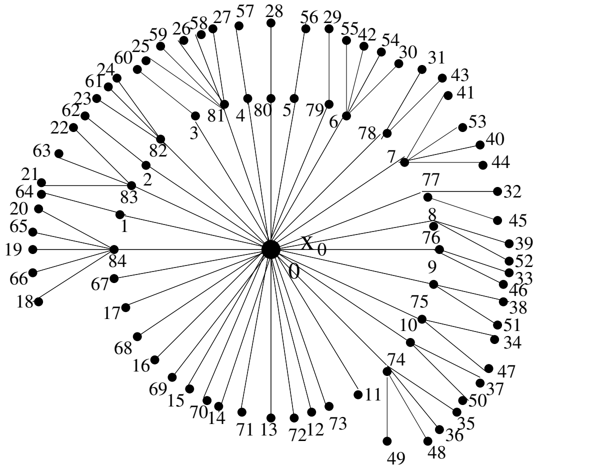

Figure 4: The tree $T(L)$ corresponding to the lobster L in Figure 3.

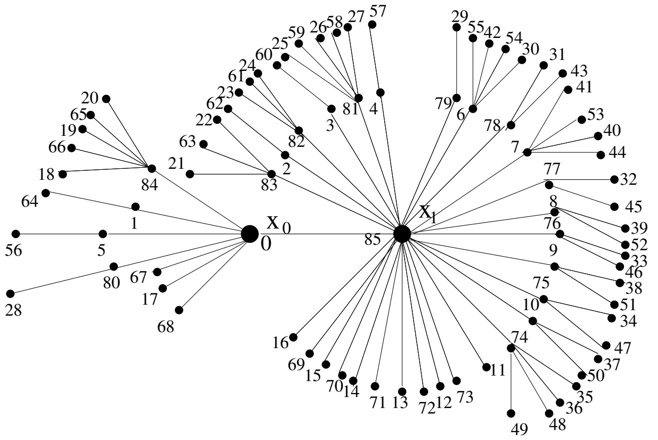

Figure 5: The graceful lobster obtained by making x_1 adjacent to x_0, giving label 85 to x_1, and moving all the branches in $A_j^{(1)}$, $j = 1, 2$, to x_1.

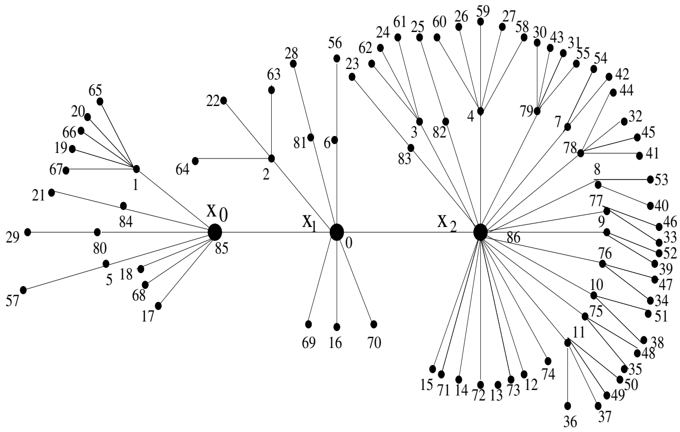

Figure 6: The graceful lobster obtained by applying inverse transformation to the lobster in Figure , making x_2 adjacent to x_1, giving label 86 to x_2, and moving all the branches in $f_{85}^{(1)}(A_j^{(2)})$, $j = 1, 2$, to x_2.

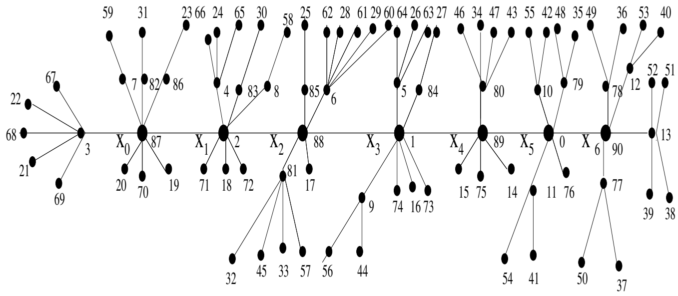

Figure 7: The lobster L with a graceful labeling.

For all lobsters of type (x), $x = b$, c, d, e, in Table 2.1, the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 2.1 by modifying steps 1 and 2. For lobsters of type (b) we first define an integer p, as $p = m$ if either $t' = m$ or $t' < m$ with each x_i, $i = t'+1, \ldots, m$, is attached to an even number of odd branches and $p = t'$ if $t' < m$ with each x_i, $i = t'+1, \ldots, m$, is attached to an even number of even branches; and this definition of p will hold henceforth in the text. Next, we set $t_2 = p$ in step 1, repeat steps 2(i) and 2(ii), set $t_2 = p$ in step 2(iii), set $t^\star = t_2$ in steps 2(iv) and 2(v), and set $t^\star = t_2$ and $m = m + t' - p$, in step 2(vi). For lobsters of type (c): set $t_2 = m$ in step 1, repeat steps 2(i) and 2(ii), set $t_2 = m$ in step 2(iii), set $t^\star = t_2$ in steps 2(iv) and 2(v), and set $t^\star = t_2$, $m = s$ and substitute even branches with pendant branches in step 2(vi). For lobsters of type (d): set $t_1 = t_2$ and $t_2 = p$ in step 1, repeat step 2(i), set $t_1 = t_2$ in step 2(ii), set $t_1 = t_2$ and $t_2 = p$ in step 2(iii), replace step 2(iv) with "for $i = t_1+1, t_1+2, \ldots, t_2$, the centers of the even branches incident on x_i in L get labels from the end of $A_2^{(i)}$", set $t^\star = t_1$ in step 2(v), and set $t^\star = t_2$ and $m = m + t' - p$ in step 2(vi). For lobsters of type (e): set $t_2 = m$ in step 1, repeat steps 2(i) and 2(ii), set $t_2 = m$ in step 2(iii), set $t^\star = t_2 + 1$ in steps 2(iv) and 2(v).

For lobsters L of type (a) in Table 2.2, the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 2.1 by modifying steps 1 and 2 in the following manner.

1. The terms of A_1 will be the labels given to the centers of the odd branches incident on $x_i, i = 0, 1, \ldots, t$. Therefore, $|A_1| = 2r_1 + 1$ is the number

of odd branches of L.

2. (i) For $i = 0, 1, 2, \ldots, t$, among the odd branches incident on x_i in L, the center of one branch gets the last label of $A_1^{(i)}$ and the centers of rest of these branches get labels from the beginning of $A_1^{(i)}$, where $A_1^{(0)} = A_1$.

(ii) For $i = 0, 1, 2, \ldots, t^\star$, the centers of the even (respectively, pendant) branches incident on x_i in L get labels from the beginning (respectively, end) of $A_2^{(i)}$, where $A_2^{(0)} = A_2$.

If $t^\star < m$ then we do the following additional step.

(iii) Repeat step 2(vi).

For lobsters of type (x), $x = b, \ldots, e$, in Table 2.2, the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 2.2 by modifying steps 1 and 2. For lobsters of type (b): set $t = p$ in steps 1 and 2(i), set $t^\star = t$ in step 2(ii), and set $t^\star = t$ and $m = m + t' - p$ in step 2(iii). For lobsters of type (c): set $t = m$ in steps 1 and 2(i), set $t^\star = t$ in step 2(ii), and set $t^\star = t$, $m = s$, and replace even branches with pendant branches in step 2(iii). For lobsters of type (d): set $t = m$ in steps 1 and 2(i), and set $t^\star = t+1$ in step 2(ii). For lobsters of type (e): repeat steps 1 and 2(i), and set $t^\star = m$ in step 2(ii).

For lobsters of type (a) (respectively, (b)) in Table 2.3, the proof follows by proceeding as the proof involving the lobsters of type (a) in Table 2.1 if we replace odd branches with even branches, set $t_1 = t$ and $t_2 = m$ in step 1, repeat step 2(i), set $t_1 = t$ in step 2(ii), set $t_1 = t$ and $t_2 = m$ in step 2(iii), set $t^\star = t$ in step 2(v), and set $t^\star = t$, $m = s$, and replace even branches with pendant branches in step 2(vi) (respectively, if we set $t_1 = t$ and $t_2 = p$ in step 1, repeat step 2(i), set $t_1 = t$ in step 2(ii), set $t_1 = t$ and $t_2 = p$ in step 2(iii), set $t^\star = t$ in step 2(v), and set $t^\star = t$, $m = m + t' - p$ in step 2(vi)).

For lobsters of type (c) (respectively, (d)) in Table 2.3, the proof follows if we proceed as the proof involving the lobsters of type (i) in Table 2.2 if we do the following changes in steps 1 and 2.
1. Repeat steps 1 and 2(i) (respectively, steps 1 and 2(i) by replacing odd branches with even branches).
2. For $i = 0, 1, 2, \ldots, s$, among the pendant branches incident on x_i in L, the centers of any odd number of branches get labels from the beginning of $A_2^{(i)}$, and the centers of the rest of these branches get labels from the end of $A_2^{(i)}$, where $A_2^{(0)} = A_2$.

For lobsters of type (e) we replace pendant branches with even branches and set $m = p$ and $s = m+t'-p$ in steps 1 and 2 in the proof involving the lobsters of type (c) in Table 2.3. □

Theorem 2.4. The lobsters in Tables 2.4, 2.5, and 2.6 below are graceful.

Description of Tables: Same as the tables in Theorem 2.3.

Table 2.4

Lobsters ↓	(e, 0, o*)	(o*, 0, o*)	(e, o, o*)	(0, o, o*)	(o*, o, 0)	(e, e, 0)	(e, 0, 0)^1 or (0, e, 0)^2	(0, 0, e)
a	0	$1 \to t_1$, $t_1 < m-2$	$t_1+1 \to t_2$, $t_2 < m-1$	$t_2+1 \to t_3$ $t_3 < m$	——	——	$t_3+1 \to m$ (2)	$t_3+1 \to s$, $s \leq m$
b	0	$1 \to t_1$, $t_1 < m-1$	$t_1+1 \to t_2$, $t_2 < m$	——	——	$t_2+1 \to t'$, $t' \leq m$	$t'+1 \to m$ if $t' < m$	$t_2+1 \to s$, $s \leq m$
c	0	$1 \to t_1$, $t_1 < m-1$	——	——	$t_1+1 \to t_2$, $t_2 < m$	$t_2+1 \to t'$, $t' \leq m$	$t'+1 \to m$, if $t' < m$	$t_1+1 \to s$, $s \leq m$

Table 2.5

Lobsters ↓	(o, o, o*)	(e, o, o*)	(0, o, o*)	(e, e, 0)	(e, 0, 0)^1 or (0, e, 0)^2	(0, 0, e)

a	0	$1 \to t_1,\ t_1 < m-1$	$t_1+1 \to t_2,\ t_2 < m$	——	$t_2+1 \to m$ (2)	$t_2+1 \to s,\ s \le m$
b	0	$1 \to t,\ t < m$	——	$t+1 \to t',\ t' \le m$	$t'+1 \to m$ if $t' < m$	$t+1 \to s,\ s \le m$
c	0	$1 \to t,\ t < m$	——	——	$t+1 \to m$ (2)	$t+1 \to s,\ s \le m$

Table 2.6

Lobsters ↓	(e, o, 0)	(o, e, 0)	(o *, o, 0)	(e, e, 0)	(e, 0, 0)[1] or (0, e, 0)[2]	(0, 0, e)
a	0	——	$1 \to t,\ t < m$	$t+1 \to t',\ t' \le m$	$t'+1 \to m$ if $t' < m$	$0 \to s,\ s \le m$
b	——	0	——	$1 \to t',\ t' \le m$	$t'+1 \to m$ if $t' < m$	$0 \to s,\ s \le m$
c	——	0	——	——	$1 \to m$	$0 \to s,\ s \le m$

proof: As in the proof of Theorem 2.3, for every lobster L of this theorem, we first construct the diameter four tree $T(L)$ corresponding to L. Let $|E(T(L))| = q$ and $deg(x_0) = 2k+1$. We give the label 0 to x_0. Here we partition the sequence S in Construction 2.1 into three parts, i.e. we take $n = 3$ in Construction 2.1.

Let L be a lobster of type in Table 2.4. We follow the steps given below:

1. We define an integer p' as $p' = t_3$ if L is of type (a), $p' = t_2$ if L is of type (b), and $p' = t_1$ if L is of type (c).

2. We determine r_1 and r_2 and hence the sequences $A_i,\ i = 1,2,3$.

(i) The integer r_1 and hence the sequence A_1 is determined by repeating step 1 in the proof involving the lobsters of type (a), (b), and (d), respectively, in Table 2.1.

(ii) Let the number of pendant branches incident on each $x_i,\ i = 0,1,\dots,p'$, be $2\alpha_i+1$, and that incident on each $x_i,\ i = p'+1,p'+2,\dots,s$, be $2\alpha_i$, where $\alpha_i \ge 1$. Let $\gamma_i,\ 0 \le \gamma_i < l_i$, be arbitrarily chosen integers. For $i = 0,1,\dots,t_3$, among the pendant branches incident on x_i, the centers of $2\gamma_i+1$ branches get labels from A_2 and the centers of the rest of these branches get labels from A_3. For $i = p'+1,\dots,s$, the centers of all the pendant branches incident on x_i get labels from A_3. Let $2r = \sum_{i=0}^{p'} 2(\alpha_i-\gamma_i) + \sum_{i=p'+1}^{s} 2\alpha_i$. We choose A_3 in such a way that it does not contain the center of any other branch. Therefore, $|A_3| = 2r$, and hence $|A_2| = 2r_2 = (2k+1)-(2r_1+1)-2r = 2(k-r_1-r)$.

3. We give labelings to the branches incident on the center of $T(L)$ in the following manner.

(i) We repeat step 2 excluding step 2(v) in the proof involving the lobsters of type (a), (b), and (d), respectively, in Table 2.1. Furthermore, if L is of type (a), then we set $t^\star = t_3$ in step 2 in the proof for the lobsters of type (a) in Table 2.1.

(ii) For $i = 0,1,2,\dots,p'$, the centers of $2\alpha_i+1$ pendant branches incident on x_i in L get $2\gamma_i+1$ labels from the end of $A_2^{(i)}$ and $2(\alpha_i-\gamma_i)-1$ labels from the beginning and the last label of $A_3^{(i)}$, where $A_2^{(0)} = A_2$ and $A_3^{(0)} = A_3$.

(iii) For $i = p'+1,p'+2,\dots,s$, the centers of $2\alpha_i$ pendant branches incident on x_i in L get $2\alpha_i-1$ labels from the beginning and the last label of $A_3^{(i)}$.

We notice that the labeling of the centers of the branches incident on the center x_0 of $T(L)$ given in step 2 follows part (b) of Lemma 1.6. Therefore, by Lemma 1.6, there exists a graceful labeling of $T(L)$ with the above labels of the center x_0 and the centers of the branches incident on x_0. Finally, we apply Theorem 2.2, for $n = 3$, on $T(L)$ and the path $H = x_0,x_1,\dots,x_m$, so as to get a graceful labeling of L.

For lobsters L in Table 2.5 , the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 2.4 by modifying steps 1 and 2 in the following manner.

1.(i) We determine A_1 by setting $t = q'$ in step 1 in the proof involving the lobsters of type (a) in Table 2.2 , where $q' = t_1$ if L is of type (a) , $q' = p$ if L is of type (b) , and $q' = t$ if L is of type (c) .

(ii) We determine A_3 and hence A_2 by setting $t_3 = q''$ in step 1(ii), where $q'' = t_2$ if L is of type (a) and $q'' = t$ if L is of type (b) or (c) .

2.(i) Set $t = q'$ in step 2(i) in the proof involving the lobsters of type (a) in Table 2.2 .

(ii) For $i = 0, 1, \ldots, q''$, the centers of the even branches incident on x_i in L get labels from the beginning of $A_2^{(i)}$, where $A_2^{(0)} = A_2$.

(iii) Define an integer q''' , where $q''' = m$, if L is of type (a) or (c) , and $q''' = m + t' - p$, if L is of type (b) . Set $t^\star = q''$ and $m = q'''$ in step 2(iii) in the proof for the lobsters of type (a) in Table 2.2 .

(iv) Set $t_3 = q''$ in steps 2(ii) and 2(iii).

For lobsters L of type (x), $x = a, b$, and c in Table 2.6 , the proof follows if we proceed as the proof involving the lobsters of type (a) in Table 2.4 by modifying steps 1 and 2 in the following manner.

1. We determine A_1 by repeating step 1 in the proof involving the lobsters of types (b) , (c) and (d) , respectively, in Table 2.3 . We take the terms of A_3 as the labels given to the centers of the pendant branches incident on the vertices x_i, $i = 0, 1, \ldots, s$, i.e. $|A_3|$ is the number of pendant branches incident on the central path of L . Therefore, $|A_2| = 2(k - r_1 - |A_3|) = 2r_2$.

2.(i) Repeat step 2 in the proof involving the lobsters of (b) , (c) , and (d) , respectively, in Table 2.3 .

(ii) For $i = 0, 1, 2, \ldots, s$, among the pendant branches incident on x_i in L , the center of one branch gets the last label of $A_3^{(i)}$ and the centers of the rest of these branches get labels from the beginning of $A_3^{(i)}$, where $A_3^{(0)} = A_3$. □

Remark 2.5. In all the lobsters to which we give graceful labelings in this paper, the vertex x_m gets the largest label and x_{m-1} gets the label 0 . Therefore, we get some more graceful lobsters by attaching a caterpillar to the vertex x_m or by attaching a suitable caterpillar (any number of pendant branches or an odd (or even) branch or the combination of both) to the vertex x_{m-1} in any of the lobsters discussed in Theorems 2.3 and 2.4.

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