# Cocycles and Bilenear Forms

A.A.I Perera[1] and D.G.T.K. Samarasiri[2]

*Department of Mathematics, University of Peradeniya, Sri Lanka*

***Abstract***— A ( two dimensional ) cocycle $\psi$ is a mapping $\psi : G \times G \to C$ such that

$$\psi(g,h)\psi(gh,k) = \psi(g,hk)\psi(h,k) \quad \forall\, g,h,k \in G$$

where $G$ is a finite group and $C$ is a finite abelian group.

Additive form of the cocycle equation is

$$\psi(g,h) + \psi(g+h,k) = \psi(g,h+k) + \psi(h,k) \quad \forall\, g,h,k \in G$$

A cocycle naturally displays as a matrix, $M = [\psi(g,h)]_{g,h\in G}$ and this matrix is the Hadamard product of Inflation, Transgression and Coboundary matrices. In our work, we prove that the bilinear form is a cocycle and the converse is not true by giving a counter example.

***Keywords:*** *Cocycle matrix, bilinear forms.*

## I INTRODUCTION

In mathematics, a *bilinear form* on a vector space $V$ is a bilinear mapping $V \times V \to F$, where $F$ is the field of scalars. That is, a bilinear form is a function $\alpha : V \times V \to F$ which is linear in each argument separately:

(i). $\alpha(u+u',v) = \alpha(u,v) + \alpha(u',v)$,

(ii). $\alpha(u,v+v') = \alpha(u,v) + \alpha(u,v')$,

(iii) $\alpha(\lambda u,v) = \alpha(u,\lambda v) = \lambda\alpha(u,v)$.

Any bilinear form on $F^n$ can be expressed as

$$\alpha(x,y) = x^T A y = \sum_{i,j=1}^{n} a_{ij} x_i y_j$$ , where $A$ is an $n \times n$ matrix.

If $G$ is a finite group and $C$ is a finite abelian group, a 2-cocycle is a mapping

$$\varphi(g,h)\varphi(gh,k) = \varphi(g,hk)\varphi(h,k) \quad \forall g,h \in G$$

This implies

$$\varphi(g,1) = \varphi(1,h) = \varphi(1,1) \quad \forall g,h \in G$$

A cocycle $\varphi$ is naturally displayed as a cocycle matrix; that is, a square matrix whose rows and columns are indexed by the element of $G$ under some fixed ordering and whose entry in position $(g,h)$ is $\varphi(g,h)$. . The matrix $M_\varphi = [\varphi(g,h)]_{g,h\in G}$ is called a $G$-cocycle matrix over $C$. Some authors call this matrix a pure cocycle matrix. [1]

In our work, we have proved the following lemma.

**Lemma:** Every bilinear form is an additive cocycle. But the converse is not true, in general.

## II. METHODOLOGY

A Cocycle matrix over $G$ is a Hadamard product of Inflation, Transgression and Coboundary matrices. [1,2] Coboundary matrix of $G$ over $C$ can be obtained by normalizing the multiplication table of the group $G$ and the construction of Inflation and Transgression matrices are given by K.J. Horadom and W. De. Launey [2] .

## III RESULTS

It can be easily shown that every bilinear form is an additive cocycle.

Let $\alpha : G \times G \to C$ be a 2-cocycle, then

$$\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z)$$

In additive form

$$\alpha(x,y) + \alpha(x+y,z) = \alpha(x,y+z) + \alpha(y,z)$$

If $\alpha$ is bilinear

$$\alpha(x,y+z) = \alpha(x,y) + \alpha(x,z) \text{ and } \alpha(x+y,z) = \alpha(x,z) + \alpha(y,z) \quad \forall x,y,z \in G.$$

$$\therefore \alpha(x,y) + \alpha(x+y,z) = \alpha(x,y) + \alpha(x,z) + \alpha(y,z) = \alpha(x,y+z) + \alpha(y,z)\ \forall x,y,z \in G.$$

So, $\alpha$ satisfy a cocycle equation and hence it is a 2-cocycle.

We have proved that the converse of this Lemma is not true, in general, by giving the following example.

Consider the finite group $Z_2^3$ which is a $Z_2$-module. Therefore it is a vector space over $Z_2$.

Define a mapping $\alpha : Z_2^3 \times Z_2^3 \to Z_2$ such that $\alpha(x,y)\alpha(xy,z) = \alpha(y,z)\alpha(x,yz)$ for all *x,y,z* in $Z_2^3$.

If $\alpha$ is bilinear it should satisfy the following:
$\alpha(x, y+z) = \alpha(x,y) + \alpha(x,z)$; and
$\alpha(x+y, z) = \alpha(x,z) + \alpha(y,z)$ for all $x, y, z$ in $Z_2^3$.

First we will compute the cocycle matrix for $Z_2^3$.

Inflation matrix for $Z_2^3$ is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & A & 1 & A & 1 & A & 1 & A \\ 1 & 1 & B & B & 1 & 1 & B & B \\ 1 & A & B & AB & 1 & A & B & AB \\ 1 & 1 & 1 & 1 & C & C & C & C \\ 1 & A & 1 & A & C & AC & C & AC \\ 1 & 1 & B & B & C & C & BC & BC \\ 1 & A & B & AB & C & AC & BC & ABC \end{pmatrix}$$

where $A^2 = B^2 = C^2 = 1$.

Transgression matrix for $Z_2^3$ is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & K & 1 & K & 1 & K & 1 & K \\ 1 & K & 1 & K & 1 & K & 1 & K \\ 1 & L & M & LM & 1 & L & M & LM \\ 1 & L & M & LM & 1 & L & M & LM \\ 1 & KL & M & KLM & 1 & KL & M & KLM \\ 1 & KL & M & KLM & 1 & KL & M & KLM \end{pmatrix}$$

where $K^2 = L^2 = M^2 = 1$.

Coboundary matrix for $Z_2^3$ is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & P & P & Q & Q & R & R \\ 1 & P & 1 & P & S & QRS & S & QRS \\ 1 & P & P & 1 & PRS & PQS & PQS & PRS \\ 1 & Q & S & PRS & 1 & Q & S & PRS \\ 1 & Q & QRS & PQS & Q & 1 & PQS & QRS \\ 1 & R & S & PQS & S & PQS & 1 & R \\ 1 & R & QRS & PRS & PRS & QRS & R & 1 \end{pmatrix}$$

The cocycle matrix for $Z_2^3$ is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & A & P & AP & Q & AQ & R & AR \\ 1 & KP & B & BKP & S & KQRS & BS & BKQRS \\ 1 & AKP & BP & ABK & PRS & AKPQS & BPQS & ABKPRS \\ 1 & LQ & MS & LMPRS & C & CLQ & CMS & CLMPRS \\ 1 & ALQ & MQRS & ALMPQS & CQ & ACL & CMPQS & ACLMQRS \\ 1 & KLR & BMS & BKLMPQS & CS & CKLPQS & BCM & BCKLMR \\ 1 & AKLR & BMQRS & ABKMPRS & CPRS & ACKLQRS & BCMR & ABCKLM \end{pmatrix}$$

Transforming to additive entries over $Z_2 =, \{0,1\},$ one can obtain the following cocycle matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & P & A+P & Q & A+Q & R & A+R \\ 0 & K+P & B & B+K+P & S & K+Q+R+S & B+S & B+K+Q+R+S \\ 0 & A+K+P & B+P & A+B+K & P+R+S & A+K+P+Q+S & B+P+Q+S & A+B+K+P+R+S \\ 0 & L+Q & M+S & L+M+P+R+S & C & C+L+Q & C+M+S & C+L+M+P+R+S \\ 0 & A+L+Q & M+Q+R+S & A+L+M+P+Q+S & C+Q & A+C+L & C+M+P+Q+S & A+C+L+M+Q+R+S \\ 0 & K+L+R & B+M+S & B+K+L+M+P+Q+S & C+S & C+K+L+P+Q+S & B+C+M & B+C+K+L+M+R \\ 0 & A+K+L+R & B+M+Q+R+S & A+B+K+L+M+P+R+S & C+P+R+S & A+C+K+L+Q+R+S & B+C+M+R & A+B+C+K+L+M \end{pmatrix}$$

Consider $Z_2^3 = \{000, 100, 010, 110, 001, 101, 011, 111\}$. Then, we can obtain its multiplication table under the addition as follows:

$$e = 000,\ a = 100,\ b = 010,\ a+b = 110,\ d = 001,\ a+d = 101,\ b+d = 011,\ a+b+d = 111.$$

| + | e | a | b | a+b | d | a+d | b+d | a+b+d |
|---|---|---|---|---|---|---|---|---|
| e | 000 | 100 | 010 | 110 | 001 | 101 | 011 | 111 |
| a | 100 | 000 | 110 | 010 | 101 | 001 | 111 | 011 |
| b | 010 | 110 | 000 | 100 | 011 | 111 | 001 | 101 |
| a+b | 110 | 010 | 100 | 000 | 111 | 011 | 101 | 001 |
| d | 001 | 101 | 011 | 111 | 000 | 100 | 010 | 110 |
| a+d | 101 | 001 | 111 | 011 | 100 | 000 | 110 | 010 |
| b+d | 011 | 111 | 001 | 101 | 010 | 110 | 000 | 100 |
| a+b+d | 111 | 011 | 101 | 001 | 110 | 010 | 100 | 000 |

Now, $\alpha(a, b, +d) = \alpha(100, 011) = R$

$\alpha(a, b) = \alpha(100, 010) = P$

$\alpha(a, d) = \alpha(100, 001) = Q$

In general $R \neq P + Q$.

Therefore, $\alpha(a, b+d) \neq \alpha(a, b) + \alpha(a, d)$.

Therefore, $\alpha$ is not bilinear.

## IV CONCLUSION

We have proved that a bilinear form is an additive cocycle. Further, proved that the converse is not true by giving a counter example.

## REFERENCES

[1]. K.J. Horadom & W. De Launey, Cocyclic Development of Designs, *Journal of Algebraic Combinatorics* **2**: 267-290, 1993.

[2]. D.L. Flannery, Calculation of Cocyclic Matrices, Journal of Pure &Applied. Algebra: 181-190.,1996.

[3]. K.J. Horadam & A.A.I. Perera, Codes from Cocycles, *Lecture notes in Computer Science, Applied Algebra, Algebraic Algorithms and Error Correcting Codes* **1255**:151-163,1997.