A Prey-Predator Model with Third order Interaction

Bakare E.A[1,2], Adekunle Y.A[3], Kadiri K.O[4]

[1] Department of Computer and Info Sc
Lead City University, Ibadan, Oyo State, Nigeria.
[2] Department of Mathematics
University of Ibadan, Ibadan, Oyo State, Nigeria.
[3] Department of Computer and Mathematics
Babcock University, Ilishan-Remo, Ogun State, Nigeria.
[4] Department of Electrical/Electronics Engineering
Federal Polythecnic, offa, Kwara State, Nigeria.

***Abstract*: We present a mathematical model for prey-predator model. We assume that the interaction is non-linear and of third order. We discuss in detail the stability of the critical points.**

***Keywords*: Prey-Predator model, non-linear, critical points.**

I. INTRODUCTION

When species interact, the population dynamics of each species is affected. In general there is a whole web of interacting species, called a trophic web, which makes for structurally complex communities. We consider here systems involving two or more species, concentrating particularly on two-species systems. There are three main types of interaction. (i)If the growth rate of one population is decreased and the other increased the populations are in a predator-prey situation. (ii)If the growth rate of each population is decreased, then it is competition.

(iii)If each population's growth rate is enhanced, then it is called mutualism or symbiosis [8].

Some mathematical models have been developed in this area.In 1926, Volterra [13] first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic. This model was based on four assumptions. First, the prey grows unboundedly in a Malthusian way in the absence of any predation. Secondly; the effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the prey and the predator populations. Thirdly, in the absence of any prey for sustenance the predator's death rate results in exponential decay. Fourthly, the prey's contribution to the predator's growth is proportional to the available prey as well as the size of the predator population. The model, is

$$\frac{dN}{dt} = N(a - bp) \quad \text{and} \quad \frac{dP}{dt} = P(cN - d)$$

When N is the prey population and P is the predator population. This model also called Lokta-Volterra model was analyzed. Murray [8] modified the Lokta-Volterra Model by changing of the assumptions made by Volterra. The model he obtained is:

$$\frac{dN_1}{dt} = r_1 N_1 \left[1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_2}\right],$$

$$\frac{dN_2}{dt} = r_2 N_2 \left[1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right]$$ where r_1, k_1, r_2 $, k_2, b_{12}, b_{21}$ are all positive constants. This model was analyzed and the conditions for stability established.Beddington et al [2] presented some results on the dynamic complexity of coupled predator-prey systems. Dunbar [3,4] studied in detail a modified Lotka-Volterra system with logistic growth of the prey and with both predator and prey dispersing by diffusion. "Predator-Prey models are arguably the building blocks of the bio and ecosystems as biomasses are grown out of their resource masses. Species compete, evolve and

disperse simply for the purpose of seeking resources to sustain their struggle for their very existence. Depending on their specific settings of applications, they can take the forms of resource- consumer, plant-herbivore, parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions, etc. They deal with the general loss-win interactions and hence may have applications outside of ecosystems. When seemingly competitive interactions are carefully examined, they are often in fact some forms of predator-prey interaction in disguise" [5]. Another approach to modeling the interaction between prey and predators was developed to account as well for organisms (such as bacteria) taking up nutrients and this is called Jacob-Mond Model. This model was discovered independently in the several diverse applications. It is akin to the Haldane-Briggs Model and Michaelis-Menten Model in Biochemistry the Jacob-Mond Model in microbial ecology and the Beverton-Holt model in fisheries. It serves as one of the important building blocks in studies of complex biochemical reactions and in ecology [12].In this work, we consider the classical predator-prey problem. We study an ecological situation involving two similar species competing for a limited food supply for example, two species of fish in a pond that do not prey on each other but do compete for the available food. Let R_1 and R_2 be the populations of the two species at time t. The objective of this study is to present a mathematical model for prey-predator model(Lokta-Volterra equations). We assume that the interaction is non-linear and of third order. We discuss the stability of the critical points.

II. RESULTS AND DISCUSSION

Consider the general prey-predator model for an n-species system

$$\frac{dR_i}{dt} = R_i\left[b_i + \sum_{i=1}^{n} a_{ij}R_i\right]$$

i=1, 2 …n (1)

The equation represent multispecies prey-predator cases where R_i's represents the population of different species at time t where $a_i>0$, $b_i>0$ are constants and represents a given finite source of food.

One species:

In this case equation (1) is reduced to one species competing for a given finite source of food:

$$\frac{dR}{dt} = R(b + aR)$$

(2)

Where a>0, b>0 are constants and R (0)>0.This equation has an exact solution.

$$R(t) = \frac{be^{bt}}{\frac{b + aR(0)}{R(0)} - ae^{bt}} \qquad \text{for } b \neq 0$$

(3)

$$R(t) = \frac{R(0)}{1 - aR(0)t} \qquad \text{for } b = 0$$

(4)

Two species: (of the third order) competing for a common ecological niche. The predator-prey model for this case takes the following form

$$\frac{dR_1}{dt} = R_1b_1 + a_{11}R_1^3 + a_{12}R_1R_2$$

(5)

$$\frac{dR_2}{dt} = R_2b_2 + a_{22}R_2^3 + a_{21}R_1R_2$$

(6)

From (5) $R_1(b_1 + a_{11}R_1^2 + a_{12}R_2) = 0$

$R_1 = 0$ and $b_1 + a_{11}R_1^2 + a_{12}R_2 = 0$ critical point (0, 0)

$$b_1 + a_{11}R_1^2 + a_{12}R_2 = 0$$

$a_{12}R_2 = \frac{-b_1 - a_{11}R_1^2}{a_{12}}$ since $R_1 = 0$ then $R_2 = \frac{-b_1}{a_{12}}$

$\therefore$ $R_1 = 0$ and $R_2 = \frac{-b_1}{a_{12}}$

From (6): $R_2(b_2 + a_{22}R_2^2 + a_{21}R_1) = 0$

$R_2 \neq 0$ and $b_2 + a_{22}R_2^2 + a_{21}R_1 = 0$

since $R_1 = 0$ and $R_2(b_2 + a_{22}R_2^2) = 0$

$R_2 \neq 0 \quad \Rightarrow \quad R_2^2 = \frac{-b_2}{a_{22}}$

Since $R_2 = 0$, $R_1 = \frac{-b_2}{a_{21}}$, $R_2^2 = \frac{b_1^2}{a_{12}^2} = \frac{-b_2}{a_{22}}$

Hence, the critical points are (0, 1) if $b_2 = -1$, $b_1 = -1$, $a_{12} = a_{22}$ then $R_2 = -1$.

$\frac{dR_1}{dt} = R_1 b_1 + a_{11} R_1^3 + a_{12} R_1 R_2$

$\frac{dR_2}{dt} = R_2 b_2 + a_{22} R_2^3 + a_{21} R_1 R_2$

Similarly, $R_1 \neq 0$ and $R_2 = 0$

$$R_2(b_2 + a_{22} R_2^2 + a_{21} R_1) = 0$$

If $R_2(b_2 + a_{21} R_1) = 0$, $R_1 = \frac{-b_2}{a_{21}}$

$R_1(1 + a_{11} R_1^2)) = 0$ then

$$R_1^2 = \frac{-1}{a_{11}} = (\frac{-b_2}{a_{21}})^2$$

$$\Rightarrow \frac{-1}{a_{11}} = \frac{b_2^2}{a_{21}^2}$$

e.g. $a_{11} = -1$, $b_2 = 1$, $a_{21} = 1$, $R_1 = -1$ critical point (-1, 0).

Hence, the critical point are (0, 0), (-1, 0), (0, -1)

critical solution (0, 0).

$$\begin{pmatrix} \frac{dR_1}{dt} \\ \frac{dR_2}{dt} \end{pmatrix} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} a_{11} R_1^3 + a_{12} R_1 R_2 \\ a_{22} R_2^3 + a_{21} R_1 R_2 \end{pmatrix}$$

Let A = $\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$

Then $|A - \lambda I|$=0 implies that

$$\begin{vmatrix} b_1 - \lambda & 0 \\ 0 & b_2 - \lambda \end{vmatrix} = 0$$

$(b_1 - \lambda)(b_2 - \lambda) = 0$

$\therefore \lambda_1 = b_1$ and $\lambda_2 = b_2$

Hence the critical solution (0, 0) is

Case1: If $b_1 < 0$ *and* $b_2 < 0$ then (0, 0) is asymptotically stable.

Case 2: If b_1 *or* $b_2 > 0$ then (0, 0) is unstable.

(ii) Critical solution (-1, 0)

$\frac{dR_1}{dt} = R_1 + R_1^3 + R_1 R_2$

$\frac{dR_2}{dt} = R_2 + R_2^3 + R_1 R_2$

$$\begin{pmatrix} \frac{dR_1}{dt} \\ \frac{dR_2}{dt} \end{pmatrix} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} R_1^3 + R_1 R_2 \\ R_2^3 + R_1 R_2 \end{pmatrix}$$

Let A = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $|A - \lambda I|$=0 $\Rightarrow$

$$\begin{vmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0$$

$(1 - \lambda_1)(1 - \lambda_2) = 0$

$\therefore \lambda_1 = 1$ and $\lambda_2 = 1$

Hence the critical solution (-1, 0) is unstable since λ_1 and $\lambda_2 > 0$.

(iii) Critical solution (0, -1)

$\frac{dR_1}{dt} = R_1 - R_1^3 + R_1 R_2$

$\frac{dR_2}{dt} = -R_2 + R_2^3 + R_1 R_2$

$$\begin{pmatrix} \frac{dR_1}{dt} \\ \frac{dR_2}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} R_1 R_2 - R_1^3 \\ R_2^3 + R_1 R_2 \end{pmatrix}$$

Let A = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $|A - \lambda I|$=0 $\Rightarrow$

$$\begin{vmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0$$

$(1 - \lambda_1)(-1 - \lambda_2) = 0$

$\therefore \lambda_1 = 1$ and $\lambda_2 = -1$

Hence the critical solution (-1, 0) is not asymptotically stable and it is not stable since $\lambda_1 > 0$ and $\lambda_2 < 0$.

III. NUMERICAL SIMULATION

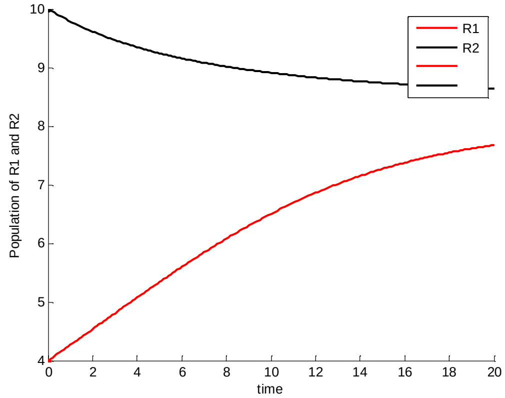

Fig1:The above shows the population of R1 and R2 over a given period of time.

III. CONCLUSION

In this paper we have been able to discuss the result of the mathematical model for prey-predator model with third order interaction. We also discuss in details the stability of the critical points.

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