Continuity of Gabor Frame Operators On Weighted Wiener Amalgam Spaces

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***Abstract*— We prove eight theorems for the continuity of Gabor Frame operators defied on weighted Wiener amalgam spaces, which generalize the corresponding results of Walnut (1992) on L^p-spaces.**

***Keywords*— Gabor Frames, Gabor expansions, Wiener Amalgam spaces and time frequency analysis.**

I. INTRODUCTION

Denis Gabor, in his famous paper [6] entitled "Theory of Communication" initiated the study of a signal representation as a series formed of translations and modulations of the Gaussian function $g(x)\pi^{-1/4}e^{-x^2/2}$. It is well known that the use of this function in time-frequency analysis is interesting for minimizing uncertainty principle inequality. This type of representations, using more general window functions g, play a vital role in the study of signal analysis and quantum mechanics both. In the mean time, theory of frames was introduced by Duffin and Schaeff [2]and, later on, it was observed that there was a close relationship between wavelet theory and the theory of frames. In particular, the connection between Gabor theory, signal analysis and the theory of frames in Hilbert spaces was studied in much detail b a number of research workers including A.J.E.M. Janssen, I. Daubechies, Christopher Heil, D.F. Walnut and J.J. Bendetto (for details and other references [9] and [FG 89]).

Feichtinger and Gröchenig [FG 89], on the other hand, using a more general theory of coorbit spaces, have investigated a very general concept of frames applicable to various types of Banach spaces and Hilbert spaces. Also, it is known that if the window function is well-localized in the time-frequency plane, then the Gabor representations hold good for general modulation spaces too.

Wiener amalgam spaces, as a generalization to function spaces, were introduced by Feichtinger [3]. Recently, these spaces have been found highly useful for the study of time-frequency analysis (cf. [7] and references there).

Walnut [11] has proved a number of interesting results concerning the contributing of the Gabor frames operators and inverse Gabor frames operators on L^p-spaces, $1 \le p \le \infty$, and on weighted Wiener amalgam spaces.

In the present paper we prove eight theorems, which provide generalizations to the corresponding results of Walnut (loc. cit.). The paper is divided into eleven sections. In the first four sections, we present a number of known definitions and concepts for use in the sequel. The remaining seven sections are devoted to the study of continuing properties of Gabor frame operators and their inverses on suitable Wiener amalgam spaces. This paper paves the way for the study continuing properties of Gabor frame operators and their inverses on more general modulation spaces.

II. PRELIMINARIES

Let $\{f_i\}_{i\in I}$ be a collection of vectors in a Hilbert spaces H. The set $\{f_i\}_{i\in I}$ is called a frame for the space H provided there exist positive constants A and B satisfying the condition.

$$A \parallel f \parallel^2 \le \sum_{i\in I} |<f,f_i>|^2 \le B \parallel f \parallel^2 \qquad 2.1$$

for all $f \in H$.

The numbers A and B known as lower and upper frame bounds respectively. In case $A = B$, then $\{f_i\}_{i\in I}$ is called a tight frame. The frame is said to be exact provided it ceases to be a frame when even a single element is removed from the above collection.

A sequence $\{f_i\}_{i\in I}$ satisfying the condition

$$\sum_{i\in I} |<f,f_i>|^2 < \infty$$

for all $f \in H$ is called a Bessel sequence. On account of the uniform boundedness principle, there exists an upper frame bound for a Bessel such that

$$\sum_{i\in I} |<f,f_i>|^2 \le B \parallel f \parallel^2, \quad \forall f \in H.$$

If $\{f_i\}_{i\in I}$ is a Bessel sequence, then an operator T defined by

$$T: l^2(I) \to H,$$

$$T(c_i)_{i\in I} = \sum_{i\in I} c_i f_i,$$

is a bounded and linear operator, which is known as a preframe operator, where

$$l^2(I) = \{(f_i)_{i\in I}: \sum_{i\in I} |f_i|^2 < \infty\}$$

It is well known that $l^2(I)$ is a Hilbert space with respect to the inner product

$$<\{f_i\},\{g_i\}>=\sum_{i\in I} f_i\,,\overline{g_i},$$

$\overline{g_i}$ being the conjugate of g.

The adjoint operator T^* of T is given by

$$T^*: H \to l^2(I),\quad T^*f=\{<f,f_i>_{i\in I}.$$

If S is the composition operator of T and T^*, then we have

$$S: H \to H,\quad S\,f = TT^*f.$$

$$i.e.\quad Sf=\sum_{i\in I}<f,f_i>f_i.$$

S is known as frame operator. Hence it is clear that

$$\begin{aligned}\| S \| &\le \| TT^* \| \\ &\le \| T \| \| T^* \| \\ &\le \| T \|^2 = B\end{aligned}$$

$$\begin{aligned}\text{Also, we see that} S^* &= (TT^*)^* \\ &= TT^* = S\end{aligned}$$

$\Rightarrow S$ is a self-adjoint operator.

It is well known that S is invertible and satisfies the condition

$$B^{-1}I \le S^{-1} \le\le A^{-1}I,$$

I being the identity operator.

Also, it is known that $\{S^{-1}f_i\}$ is a frame for the Hilbert space H, which is known as the dual frame of $\{f_i\}_{i\in I}$.

By virtue of the above definitions, we see that the function f can be recovered in the form

$$\begin{aligned} f &= SS^{-1}f \\ &= \sum_{i\in I} <f,S^{-1}f_i>f_i,\quad \forall\, f\in H,\end{aligned}$$

oras

$$\begin{aligned} f &= SS^{-1}f \\ &= \sum_{i\in I} <f,f_i>S^{-1}f_i,\quad \forall\, f\in H,\end{aligned}$$

the sums in the above expression being convergent in the space H.

Daubechies, on the other hand, has shown that (cd., [D 90]) if $\{f_i\}_{i\in I}$ is a frame for H with the frame operators S, then $\{S^{-1/2}f_i\}_{i\in I}$ is a tight frame found 1.

III. WEIGHTED WIENER AMALGAMS ON $\boldsymbol{R}$

Let $w: R \to R^+$ be a function such that

(i) $w(0) = 1$.

(ii) $w(x+y) \le w(x)\, w(y)$ for all $x,y \in R$

Then w is called a submultiplication weight function on R.

A locally integrable function $w: R \to R^+$ is known as moderate weight function if there exists a submultiplictive weight satisfying the condition:

$$w(x+y) \le w(x)\, w(y);\quad \forall x,y\in R.$$

Throughout this paper we assume that w is a moderate weight function on R.

We denote by $L_w^p(R)$, $1\le p\le\infty$, the Banach space of all complex-valued functions on R under the norm

$$\| f \|_{p,w} = \| f | L_w^p \| = (\int_R |f(x)^p w^p(x)dx)^{1/p} < \infty.$$

In case $p=\infty$, we denote by $L_w^\infty(R)$ the Banach space of all measurable functions f with respect to the norm

$$\| f \|_{\infty,w} = \text{esssup}_{x\in R}\{|f(x)|w(x)\} < \infty.$$

Since w is moderate, L_w^p is translation invariant (for the proof see [10], pp. 22-23).

Let Q be a fixed non-empty compact set in R. We denote by $W(L^p, L_w^p)(R)$, $1\le p\le\infty$, the Wiener amalgam space under the norm (for details see [3]);

$$\begin{aligned}\| f \|_{W(L^p,L_w^p)} &= \| f | W(L^p,L_w^p) \| = \| \| f\cdot\chi_{Q+x} \|_p \|_{p,w} \\ &= (\int_R (\int_R |f(y)^p \chi_Q(y-x)dy)^{1/p} w^p(x)dx)^{1/p},\end{aligned}$$

where χ_Q is the characteristic function of Q.

It is known that the definition of the space $W(L^p, L_w^p)(R)$ is independent of the choice of Q (cf. [3]) and , since w is moderate $W(L^p, L_w^p)(R)$ is translation invariant (for the detail see [Heil 90], p. 33).

IV. GABOR FRAME OPERATORS ON $\boldsymbol{L(R)}$

Gabor expansion of a function $f\in L^2(R)$ in terms of a single generating function $g\in L^2(R)$ by translations (time-shifts) and modulation (frequency-shifts) of g is given by

$$f(t) = \sum_{m,n\in Z} c_{m,n}\, g_{m,n}(t), \tag{4.1}$$

where

$$\begin{aligned} g_{m,n}(t) &= g(t-na)e^{2\pi imbt} \\ &= T_{na}M_{mb}g,\end{aligned}$$

Z being the set of all integers.

The Gabor coefficients $c_{m,n} = <f, g_{m,n}>$ for all $m,n\in Z$ are given the analysis mapping T_g such that

$$T_g: f \to \{<f,g_{m,n}>_{m,n}.$$

The adjoint T_g^* of T_g is given by

$$T_g^*:\{c_{m,n} \to \sum_{m,n} c_{m,n}\, g_{m,n}.$$

It is well known that T_g and T^* both are bounded linear operators.

Hence the Gabor frame operator $S\equiv S_g = T^*\circ T$ is given by

$$Sf = \sum_{m,n\in Z} <f,g_{m,n}>g_{m,n}.$$

Daubechies [D 90] has shown that the Gabor frame $\{g_{m,n}\}_{m,n\in Z}$ for $L^2(R)$ exists and its dual frame $\{\tilde{g}_{m,n}\}_{m,n\in Z}$ also is a Gabor frame, where $\tilde{g} = S^{-1}g_{m,n}$. Walnut [Wal 92, pp. 499-500] has proved that $\{S^{-/1}g_{m,n}\}_{m,n\in Z}$ frame with bound 1.

V. CONTINUITY OF GABOR FRAME OPERATORS.

In a recent paper, Walnut [Wal 92, Theorem 3.1] has proved that if $g\in W = W(L^\infty, L^1)(R)$; $a,p>0$ and $1\le p\le\infty$, then there exists a constant C depending on g, a and b such that for all $f\in C_c^\infty(R)$

$$\| Sf \|_p \le C \| f \|_p$$

and

$$\| Sf \|_{W(L^\infty,L^1)} \le C \| f \|_{W(L^\infty,L^1)},$$

where C_c^∞ is the space of functions with compact support C.

In part (b) of the above theorem, Walnut has proved amalgams results for the weighted norms $\| f \|_{p,w_p}$ and $\| f \|_{W,w_s}$, where w_s denotes the weight function $(1+|x|)^s$ or $e^{s|x|}$, $s>0$.
In this section our aim is to generalize the above results of Walnut for the Wiener amalgams norms $\| f \|_{W(L^p,L^p_w)}$, w being a moderate weight function on R. Precisely, we prove the following

Theorem 5.1 If $g \in WL^{\infty}, L^1(R)$, $a,b>0$ and $1 \le p \le \infty$, then there exists a constant C depending only on g a and b such that

$$\| Sf \|_{W(L^p,L^p_w)} \le C(b,w) \| f \|_{W(L^p,L^p_w)} \tag{5.1}$$

for all $f \in C_c^{\infty}(R)$.
It may be mentioned here that in case $w=1$ our result in (5.1) reduce to (3.1.1) of walnut, while for the local norm $\| f \|_{\infty}$ and global norm $\| f \|_1$ the result (3.1.2) of Walnut becomes a particular case of the result in (5.1). Also, it is clear that for $w=w_s$ the part (b) of the Theorem 3.1 (b) of Walnut is a particulars case of our results in Theorem 5.1.
We shall use the following lemma in the proof of our theorem:

Lemma 5.2. If $g \in W(L^{\infty},L^1)(R)$ and $a,b>0$, then for all $f \in L^2(R)$, we have

$$Sf = b^{-1} \sum_{j \in Z} T_{j/b} f \cdot G_j,$$

where G_j is the discrete auto-correlation function given by

$$G_j = \sum_{n \in Z} T_{na} (g T_{j/b} \bar{g}),$$

$\bar{g}$ being the conjugate function of g.
Also, we have $\sum_{j \in Z} w(j/b) \| G_j \|_{\infty} < \infty$ for $gw \in W(L^{\infty},L^1)$.
For the proof see [Wal 90, Lemma 2.1 and 2.2].

Proof of Theorem 5.1. Since $C_c^{\infty}(R) \subset L^2(R)$, the result in (5.2) holds true for all $f \in C_c^{\infty}(R)$. Thus we see that

$$\begin{aligned}\| Sf \|_{W(L^p,L^p_w)} &\le b^{-1} \sum_{j \in Z} \| T_{j/b} f \cdot G_j \|_{W(L^p,L^p_w)} \\ &\le b^{-1} \sum_{j \in Z} \| G_j \|_{\infty} \| T_{j/b} f \|_{W(L^p,L^p_w)} \\ &\le \| f \|_{W(L^p,L^p_w)} \, b^{-1} \sum_{j \in Z} \| G_j \|_{\infty} \cdot w(j/b).\end{aligned}$$

Walnut (loc. cit., Lemma 2.2) has shown that the series on the right-hand side is convergent. Hence, writing

$$C(b,w) = b^{-1} \sum_{j \in Z} \| G_j \|_{\infty} \cdot w(j/b).$$

The proof of the theorem follows:
In case $p=\infty$, the proof follows in a similar way.

VI. CONTINUITY IF THE FOURIER TRANSFORM OF $\boldsymbol{Sf}$.

Let f be the Fourier transform operator on the space of tempered distribution $S'(R)$ and $Ff = \hat{f}$. We denote by $FL^p(R)$ the space of all tempered distributions f such that $\dot{a}tf \in L^p(\hat{R})$, $\hat{R}$ being the dual group of R. We define the norm on $FL^p(R)$ by

$$\| f \|_{FL^p} = \| \hat{f} \|_p.$$

Since $\hat{R}$ is isometric to R, we shall write R in space of $\hat{R}$.
In this section we prove the following theorem, which provide a generalization of the corresponding results of Walnut (cf. [11], Theorem 3.2 (a) and (b)) as in the case of Theorem 5.1:

Theorem 6.1: If $\hat{g} \in W(L^{\infty},L^1)(R)$; $a,b,>0$ and $1 \le p \le \infty$, then there exists a constant C depending only on g, a and b such that

$$\| (Sf)^{\wedge} \|_{W(L^p \, L^p_w)} \le C \| \hat{f} \|_{W(L^p \, L^p_w)}$$

for all $\hat{f} \in C_c^{\infty}(R)$.
We shall use the following lemma in the proof of our theorem :
Lemma 6.2. If $\hat{g} \in W(L^{\infty},L^1)(R)$ and $a,b>0$, then

$$(Sf)^{\wedge} = a^{-1} \sum_{j \in Z} T_{j/a} \hat{f} \cdot \tilde{G}_j,$$

$$where$$

$$\tilde{G}_j = \sum_{m \in Z} T_{mb} (\hat{g} T_{j/a} \bar{\hat{g}}).$$

Also, we have

$$\sum_{j \in Z} w(j/a) \| \tilde{G}_j \|_{\infty} < \infty.$$

for $\hat{g}w \in W(L^{\infty},L^1)(R)$.
The proof follows on the lines of Walnut [Wal 92, Lemma 2.3].

Proof of Theorem 6.1. On account of the relation (6.2) and denseness of the space $C_c^{\infty}(R)$ in $L^2(R)$, we obtain

$$\begin{aligned}\| (Sf)^{\wedge} \|_{W(L^p \, L^p_w)} &\le a^{-1} \sum_{j \in Z} \| T_{j/a} \hat{f} \cdot \tilde{G} \|_{W(L^p \, L^p_w)}, \\ &\le a^{-1} \| \hat{f} \|_{W(L^p \, L^p_w)} \sum_{j \in Z} w(j/a) \| \tilde{G}_j \|_{\infty} \\ &= C \| \hat{f} \|_{W(L^p \, L^p_w)}\end{aligned}$$

$$for$$

$$C = a^{-1} \sum_{j \in Z} w(j/a) \| \tilde{G}_j \|_{\infty}.$$

This complete the proof of the theorem.

VII. APPROXIMATION OF FUNCTIONS BY THEIR GABOR EXPANSIONS.

Heil and Walnut [9] have shown that a necessary condition for the generation of a frame by $\{g_{m,n}\}_{m,n} \in Z$ is

$$A \le \sum_{n \in Z} |T_{na} g(x)|^2 \le B \quad \text{foralmostall} x \in R,$$

where $A,B>0$ are some constants.
Using an analogous for $g \in W(L^{\infty},L^1)(R)$, Walnut has proved that

$$\left\| f - \frac{2b}{B+A} Sf \right\|_p \le \frac{B-A+2C(b)}{B+A} \|f\|_p, \quad 1 \le p \le \infty,$$

where $b>0$ and $C(b)$ is a constant depending on g and a.

He has obtained three other similar approximations in the spaces $W(L^{\infty},L^1)(R)$, $L^p_{w_s}(R)$ and weighted $W(L^{\infty},L^1)(R)$ with the weight function w_s.

In this section our aim is to generalize the above mentioned result of Walnut in the Wiener amalgam space $W(L^p,L^p_w)(R)$. Precisely, we prove the following:

Theorem 7.1 If $g \in W(L^\infty, L^1)(R)$ satisfies the condition

$$A \leq \sum_{n\in Z} |T_{na} g(x)|^2 \leq B \quad \text{a.e.}\, x, a > 0, \tag{7.1}$$

then for each $b > 0$, there exists a constant $C(b)$ depending on g and a such that

$$\left\| f - \frac{2b}{B+A} Sf \right\|_{W(L^p, L^p_w)} \leq \frac{B - A + 2C(b,w)}{B+A} \|f\|_{W(L^p, L^p_w)}, \quad 1 \leq p \leq \infty, \tag{7.2}$$

where

$$\lim_{b\to 0} C\,(b,w) = 0 \tag{7.3}$$

It may be mentioned here that all the four results in Theorem 3.3 of Walnut [11] are particular cases of our theorem for special values of p and $w = ws$. The condition (7.1) ensure that Sf is well defined (cd [loc. cit., p-495).

Proof of the Theorem Using (5.2), we have

$$\left\| f - \frac{2b}{B+A} Sf \right\|_{W(L^p, L^p_w)} = \left\| f - \frac{2}{B+A} G_0 \cdot f - \frac{2}{B+A} \sum_{\substack{j\neq 0\\ j\in Z}} T_{j/b}\, f G_j \right\|_{W(L^p, L^p_w)}$$

$$\leq \left\| \left(1 - \frac{2}{B+A} G_0\right) f \right\|_{W(L^p, L^p_w)} + \frac{2}{B+A} \left\| \sum_{\substack{j\neq 0\\ j\in Z}} T_{j/b}\, f G_j \right\|_{W(L^p, L^p_w)}$$

$$\leq \left\| 1 - \frac{2}{B+A} G_0 \right\|_\infty \| f \|_{W(L^p, L^p_w)} + \frac{2}{B+A} \| f \|_{W(L^p, L^p_w)} \cdot \sum_{\substack{j\neq 0\\ j\in Z}} \| G_j \|_\infty\, w(j/b)$$

$$\leq \max\{1 - \frac{2}{B+A}, \frac{2B}{B+A} - 1\} \| f \|_{W(L^p, L^p_w)} + \frac{2B}{B+A} \| f \|_{W(L^p, L^p_w)} \sum_{\substack{j\neq 0\\ j\in Z}} w\,(j/b) \| G_j \|_\infty.$$

$$= \frac{B-A}{B+A} \| f \|_{W(L^p, L^p_w)} + \frac{2C(b)}{B+A} \| f \|_{W(L^p, L^p_w)}$$

$$= \frac{B - A + 2C(b,w)}{B+A} \| f \|_{W(L^p, L^p_w)},$$

where

$$C(b,w) = \sum_{\substack{j\neq 0\\ j\in Z}} \| G_j \|_\infty\, w(j/b).$$

Walnut [loc. cit., p. 486] has shown that for any given $\eta > 0$, there exists a $b_0 > 0$ depending only on g and η such that

$$\sum_{\substack{j\neq 0\\ j\in Z}} w\,(j/b) \| G_j \|_\infty < a^{-1}\eta$$

for every $0 \leq b \leq b_0$ and every $0 < a \leq 1$.

Now, choosing $\eta \to 0$ as $b \to 0$,, the result in (7.2) follows.

VIII.

In this section we prove the following theorem, which generalize all the four results of Walunt (cf. [11], p.499):

Theorem 8.1 If $\hat{g} = W(L^\infty, L^1)(R)$, $b < 0$ and there exist some constants $A', B' > 0$ satisfying the condition

$$A' \leq \sum_{m\in Z} |T_{mb} \hat{g}(\xi)|^2 \leq B' \quad \text{a.e.}\, \xi, \tag{8.1}$$

then, for each $a > 0$, there exists a constant $\tilde{C}(a)$ depending on g and b such that

$$\left\| \left(f - \frac{2a}{B' + A'} Sf \right)^\wedge \right\|_{W(L^p, L^p_w)} \leq \frac{B' - A' + 2\tilde{C}(a)}{B' + A'} \|\hat{f}\|_{W(L^p, L^p_w)}, \tag{8.2}$$

and

$$\lim_{a\to 0} \tilde{C}\,(a,w) = 0. \tag{8.3}$$

Proof of the Theorem: Using Lemma 6.2, we obtain

$$\left\| \left(f - \frac{2a}{B' + A'} Sf \right)^\wedge \right\|_{W(L^p, L^p_w)} = \left\| \hat{f} - \frac{2}{B' + A'} \tilde{G}_0 \cdot f - \frac{2}{B' + A'} \sum_{\substack{j\neq 0\\ j\in Z}} T_{j/b}\, \hat{f} \tilde{G}_j \right\|_{W(L^p, L^p_w)}$$

$$\leq \left\| \left(1 - \frac{2}{B' + A'} \tilde{G}_0\right) \hat{f} \right\|_{W(L^p, L^p_w)} + \frac{2}{B' + A'} \left\| \sum_{\substack{j\neq 0\\ j\in Z}} T_{j/b}\, \hat{f} \tilde{G}_j \right\|_{W(L^p, L^p_w)}$$

$$\leq \left\| 1 - \frac{2}{B' + A'} G_0 \right\|_\infty \| \hat{f} \|_{W(L^p, L^p_w)} + \frac{2}{B' + A'} \| \hat{f} \|_{W(L^p, L^p_w)} \cdot \sum_{\substack{j\neq 0\\ j\in Z}} w\,(j/b) \| \tilde{G}_j \|_\infty$$

$$\leq \max\{1 - \frac{2A'}{B' + A'}, \frac{2B'}{B' + A'} - 1\} \| \hat{f} \|_{W(L^p, L^p_w)} + \frac{2}{B' + A'} \| \hat{f} \|_{W(L^p, L^p_w)} \sum_{\substack{j\neq 0\\ j\in Z}} w\,(j/b) \| \tilde{G}_j \|_\infty.$$

$$= \frac{B' - A' + 2\tilde{C}(a,w)}{B' + A'} \parallel f \parallel_{W(L^p, L^p_w)},$$

where

$$\tilde{C}(a,w) = \sum_{\substack{j\neq 0 \\ j\in Z}} w\,(j/b) \parallel \tilde{G}_j \parallel_\infty .$$

Also, as in Lemma 2.3 of Walnut (loc. cit.), for a given $\eta > 0$, there exists $a_0 > 0$ depending only on g and η such that for every $a \le a \le a_0$ and every $a < b \le 1$, we have

$$\sum_{\substack{j\neq 0 \\ j\in Z}} w\,(j/b) \parallel \tilde{G}_j \parallel_\infty < b^{-1}\eta.$$

Now, choosing $\eta \to 0$ as $a \to 0$, we get

$$\lim_{a\to 0} \tilde{C}\,(a,w) = 0.$$

Thus the theorem holds true.

IX. CONTINUITY OF INVERSE GABOR FRAME OPERATOR

In this section our objective is to prove two theorems for the continuity of the inverse Gabor frame operators corresponding to the Theorem 5.1 and 6.1. We prove the following theorems:

Theorem 9.1 If $g \in W(L^\infty, L^1)$ and satisfying the condition (7.1), then for all $b > 0$ sufficiently small, S^{-1} exists and is continues on $W(L^p, L^p_w)(R)$, $1 \le p \le \infty$.

Theorem 9.2 If $\hat{g} \in W(L^\infty, L^1)$ and satisfies the condition (8.1) then , for all $a > 0$ sufficiently small, S^{-1} exists and is continuous on $FW(L^\infty, L^1)$,$1 \le p \le \infty$. **Proof of Theorem 9.1** Assuming $C(b,w) < A$ in Theorem 7.1, we see that

$$\frac{B - A + 2C(b,w)}{B + A} < 1.$$

Thus, for $f \in C_c^\infty$, we have

$$\left\| \left(I - \frac{2b}{B+A}S\right) f \right\|_{W(L^p, L^p_w)} \le \parallel f \parallel_{W(L^p, L^p_w)}$$

which implies that

$$\left\| \left(I - \frac{2b}{B+A}S\right) | W(L^p, L^p_w) \right\| < 1.$$

Hence we may choose d number $d > 0$ such that

$$\left\| I - \frac{2b}{B+A}S \right\| < d < 1.$$

$\rightarrow$ S is bounded and invertible on $W(L^p, L^p_w)(R)$.

$\rightarrow$ S^{-1} can be represented by the Neumann Series:

$$S^{-1} = \frac{2b}{B+A}\sum_{k=0}^{\infty}\left(I - \frac{2b}{B+A}S\right)^k$$

and its sum convergence to S^{-1} in the operator norm.

Thus we see that

$$\parallel S^{-1} f \parallel_{W(L^p, L^p_w)} = \left\| \frac{2b}{B+A}\sum_{k=0}^{\infty}\left(I - \frac{2b}{B+A}S\right) f \right\|_{W(L^p, L^p_w}$$

$$\le C \parallel f \parallel_{W(L^p, L^p_w} \left\| \frac{2b}{B+A}\sum_{k=0}^{\infty}\left(I - \frac{2b}{B+A}\right) \right\|_\infty$$

Since the series on the right hand side is convergent, the proof of the theorem is complete.

Proof of theorem 9.2 Let $\tilde{C}(a,w) < A'$ in Theorem 8.1. Then we have

$$\frac{B' - A' + 2\tilde{C}(b,w)}{B' + A'} < 1.$$

Hence, for $\hat{f} \in C_c^\infty(\hat{R})$, we obtain

$$\left\| \left[\left(I - \frac{2b}{B' + A'}S\right) f \right]^\wedge \right\|_{W(L^p, L^p_w} \le \parallel \hat{f} \parallel_{W(L^p, L^p_w)}.$$

$\Rightarrow$ $\left\| I - \frac{2b}{B'+A'}S \right\| < 1.$ $\Rightarrow$ S is bounded and invertible on $W(L^p, L^p_w)(\hat{R})$.

Hence, as in the proof of the Theorem 9.1, we have

$$\parallel S^{-1}\hat{f} \parallel_{W(L^p, L^p_w)} = \left\| \frac{2a}{B' + A'}\sum_{k=0}^{\infty}\left(I - \frac{2a}{B' + A'}S\right)\hat{f} \right\|_{W(L^p, L^p_w}$$

$$\le C$$

$$\parallel \hat{f} \parallel_{W(L^p, L^p_w} \left\| \frac{2a}{B' + A'}\sum_{k=0}^{\infty}\left(I - \frac{2a}{B' + A'}\right) \right\|_\infty$$

C being a positive constant not necessarily the same at each occurrence.

By virtue of convergence of the series on the right hand side the proof of the theorem is complete.

X. RECONSTRUCTION OF f BY ITERATION

On account if the relation (2.1) the series associated with Sf in (2.2) is unconditionally convergent for every $f \in H$. This entails that S is a positive and bounded linear operators. If I is theidentity operators, It is known that (cf. [2])

$$AI \le S \le BI.$$

Hence, as pointed out by Walnut [Wal 92, p. 480], we have

$$\frac{B-A}{B+A} I \le I - \frac{2}{B+A}S \le \frac{B-A}{B+A}I$$

$\Rightarrow \parallel I - \frac{2b}{B+A}S \parallel \le \frac{B-A}{B+A} < 1$

$\Rightarrow$ S is a bounded and invertible operators.

$\Rightarrow$ S^{-1} can be represented in the form of Neumann series:

$$S^{-1} = \frac{2}{B+A}\sum_{k=0}^{\infty}\left(I - \frac{b}{B+A}S\right)^k$$

$\Rightarrow f = \frac{2}{B+A}\sum_{k=0}^{\infty}\left(I - \frac{2}{B+A}S\right)^k Sf.$

The function f can also be recovered using the iteration process:

$$f_0 = \frac{2}{B+A}Sf, \tag{10.1}$$

$$f_{n+1} = f_n + \frac{2}{B+A}Sf - \frac{2}{B+A}Sf_N$$

It is known that $f_n \to 0$ in H as $n \to \infty$ (cf. [11], pp. 496-497).

In this section we prove the following theorem, which includes both the results of Walnut in Theorem 4.3 as particular cases:

Theorem 10.1 Let $gw \in W(L^\infty, L^1)(R)$ nd for some $a > 0$ the relation (7.1) holds with $A > 0$. If $\{f_n\}$ is a sequence defined recursively (10.1), $f \in W(L^p, L^p_w)(R)$ and $b > 0$ is sufficiently small, then

$$\lim_{n\to\infty} \| f_n - f \|_{\frac{2}{B+A}Sf} = 0, \quad 1 \le p \le \infty. \qquad (10.2)$$

Proof We assume that $C(b,w) < A$. Hence, by Theorem 7.1, we have

$$\| I - \frac{2b}{B+A} S \| < 1. \qquad (10.3)$$

We now defined an operator T on $W(L^p, L^p_w)(R)$ such that

$$T\phi = \phi + \frac{2b}{B+A} Sf - \frac{2b}{B+A} S\phi.$$

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