Queueing Systems – A Numerical Approach

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Abstract - Queueing Systems [1] is one of the real life application technique between a customer and server who is approaching for a service facility. Numerical applications is one of the way to find a solution to any queueing model. Our review covers only general methods that can be applied for a wide range of time dependent queues. In this paper we try to investigate the potential and limitations of numerical methods to evaluate the service quality of time-dependent single facility delay systems.

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I INTRODUCTION

For Markovian [5] queues, transient solutions can be obtained by solving the Kalmogorov differential equations P'(t) = P(t) Q(t). Again as long as the Q matrix is of finite dimension, numerical techniques for solving these first order differential equations can be employed. Numerical integration methods such as the Euler, Runge kutta method have long been employed in solving systems of differential equations. Also another three methods that is particularly well suited for queueing models are Randomization method, Discrete Time modeling Approach method and Closure Approximation. We illustrate some of these techniques in the following sections.

II RUNGE KUTTA METHOD

The Runge kutta method [3] is general in the sense that it applies to any continuous time Markov chain (CTMC) but its computation time is highly dependent on the number of system states that are included and other problem parameters. Consider the Non-stationary $M(t) / \Box / s(t) / \infty$ Queueing systems. Define the state of the system N(t) as the total number of customers in the system either being served or waiting in the queue and let $P_i(t) Pr\{(N(t) = i\}$. The state probabilities evolve according to the following kolmogorov forward equations.

$$\frac{d P_0(t)}{dt} = \mu P_1(t) - \lambda(t) P_0(t)$$

$$\frac{d P_i(t)}{dt} = \lambda(t) P_{i-1}(t) + (i+1) \mu P_{i+1}(t) - (\lambda(t) + i\mu) P_i(t), \quad i = 1, 2....(s(t) - 1)$$

$$\frac{d P_i(t)}{dt} = \lambda(t) P_{i-1}(t) + s(t) \mu P_{i+1}(t) - (\lambda(t) + s(t) P_i(t), \quad i = s(t), \quad s(t) + 1....$$

To solve the above equations numerically make the infinite set of equations finite by setting a limit to a total number of customers K. Initially set

 $K = \max \{100, \max \{s(t); t \in (0, T)\}\}$ and check whether $\pi_K(t) \le 10^{-6}$ for all t. The method of Runge-kutta was successfully employed to solve a variety of Queueing problems. The numerical experience reported by the authors cited indicates that Runge kutta is a viable method for calculating transient solutions. Grassmann states that Runge kutta is must faster than simulation. Navid Izady [11] also found that Runge kutta method gives the better accuracy.

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III EULER METHOD

Euler's method is a special Runge kutta method with K = 1. It is also another general approach for solving ODEs. This method have been used by several authors to find transient solutions in queueing systems.

Consider
$$P'(t) = P(t) Q(t)$$

For small values of h

$$\lim_{h \to 0} \frac{P(t+h) - P(t)}{h} = P(t) Q(t)$$

(ie) $P(t+h) \simeq P(t) (Q(t) h + I)$

Or equivalently $P(t + h) \simeq P(t) \pi(t)$

Where $\pi(t) = Q(t) h + I$

For $\pi(t)$ to be a transition matrix, all elements must be between zero and one and rows also must add up to one. Navid Izady suggest that setting $h = \frac{1}{2v}$ to achieve three digits accuracy for service quality values. Where $V = \sup_{0 \le t \le T} \left| \max_{K} |V_{K}(t)| \right|$ [11]. Having chosen an appropriate values for h, one can easily approximate the state probability vector P(t + h) by the product P(t) $\pi(t)$ recursively starting from t = 0. The numerical experience reported by the authors cited indicates that Euler method is the fastest method whose accuracy is not as good as randomization, but it is very close and seems to be good enough for many practical purposes.

IV RANDOMIZATION METHOD

The randomization algorithm [2] was originally suggested for transient analysis of homogeneous CTMCs. To solve the differential equations, P'(t) = P(t)Q, consider a finite birth-death process with a Q matrix given as follows.

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots & 0 \\ \mu_1 & -\lambda_1 - \mu_1 & \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & -\lambda_2 - \mu_2 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & \dots & \mu_{N-} \mu_N \end{pmatrix}$$

The negative diagonal elements of Q cause high round off errors when calculating P(t). It is now possible to find P(t) by a Sieries expansion of a matrix P which contains no negative elements. For that let $P = \frac{Q}{2} + I$. Here \wedge is an arbitrary number $\geq |a_{ii}|$. Consequently P is a stochastic matrix P(t) $= \sum_{n=0}^{m-1} q P^n \left[(\wedge t)^n \exp(-\wedge t)/n! \right] + R_m$. The method given by the above formula is called randomization consequently $\left[(\wedge t)^n \exp(-\wedge t)/n! \right] q P^n$ gives the probability vector of having n Poisson-events. The truncation error R_m can be estimated with reasonable accuracy, where m as proposed by Grassmann [5] is chosen as $m = \wedge t + 4\sqrt{\Lambda t} + 5$.

The randomization method shares many characteristics with the exact method it is applicable to any system that can be modeled as a homogenous CTMC, the computation times are highly dependent on model structure and parameter values. In order to use the randomization algorithm for more complex case, Ingolfsson propose replacing the time-dependent arrival rate

International Journal of Mathematics Trends and Technology- Volume 20 Number 1- April 2015

function with a piece wise constant function and using the randomization method for segments of time where arrival rate and number of servers remain constants. Reibman and Trivedi [9] reported computation times for the randomization method that were on the order of 25% of those for the Runge kutta method, for homogenous CTMC instances.

V DISCRETE TIME MODELLING APPROACH

The discrete time modelling (DTM) approach has much in common with matrix geometric methods, which have been applied to obtain analytic expressions and numerical results for the steady-state behaviour of a variety of discrete and continuous time queuing systems with Markovian arrival processes and phase type services. The discrete $M_t / GI / s(t) / K$ queueing system where M_t has a phases and the discrete service time distribution has finite support on the range $\{1, 2, ..., m\}$. The natural state-space for the DTM approach would be

 $\{n, s_1, s_2, \dots, s_m\}$

where n = 0, 1, 2, ..., K, $\sum_{i=1}^{m} s_i = \min(n, s(t)))$

Brahimi and Worthington [6] reported that matching the first two moments of the continuous distribution produce results which are accurate enough for most practical purposes. As noted in Worthington and Wall [7] the size of the state space $s = \frac{s}{(i+m-1)!} + \frac{(s+m-1)!}{(s+m-1)!}$ which can be write large

defined is $\sum_{i=0}^{s} \frac{(i+m-1)!}{i! (m-1)!} + (k-s) \frac{(s+m-1)!}{s! (m-1)!}$ which can be quite large.

Empirical results using this method shows that it will give approximate results that are more than accurate enough for most practical purposes, for both steady- state and time dependent behaviour. The method is also computationally feasible for a wide range of service time distributions. In software form these models can be applied directly to practical problems to provide time-dependent results only previously available by simulation, but without its associated weaknesses. Navid Izady [11] states that DTM approach is substantially slower than the other three methods.

VI CLOSURE APPROXIMATION

Closure Approximation method occasionally produced sharp fluctuations in the service level that were not caused by changes in the arrival rate or the number of servers.

Here the number of customers assumed to follow a Polya Eggenberger (PE) distribution with atleast one free server also the number of customers is assumed to follow a shifted PE distribution with all servers being busy.

We can express the state probabilities as

$$P_r(N(t)i) = P_i(t) = \begin{cases} \eta(i; n_1, p, \alpha_1) E_1^{(0)}, \text{ for } i < s(t) \\ \eta(i - s(t); n_2, p_2, \alpha_2) E_2^{(0)}, \text{ for } i \ge s(t) \end{cases}$$

where η (i; n, p, α) is a PE probability mass function with parameters n, p and α . And $E_1^{(0)}$ is the probability that atleast one server is idle and $E_2^{(0)}$ is the probability that all servers are busy. The probabilities $E_1^{(0)} = i = 1, 2$ are calculated by solving

$$E_{1}^{(0)^{1}} = -\lambda P_{s-1} + \mu s P_{s}$$

$$E_{1}^{(1)^{1}} = -\lambda \left\{ E_{1}^{(0)} - s P_{s-1} \right\} + \mu \left\{ -E_{1}^{(1)} + (s-1) s P_{s} \right\}$$

$$E_{1}^{(2)^{1}} = -\lambda \left\{ E_{1}^{(0)} + 2 E_{1}^{(1)} - s^{2} P_{s-1} \right\} + \mu \left\{ E_{1}^{(1)} - 2 E_{1}^{2} + (s-1)^{2} s P_{s} \right\}$$

$$E_{2}^{(0)^{1}} = \lambda P_{s-1} - \mu s P_{s}$$

$$E_{2}^{(1)^{1}} = \lambda \left\{ E_{2}^{(0)} - s P_{s-1} \right\} + s \mu \left\{ -E_{2}^{(2)} - (s-1) P_{s} \right\}$$

$$E_{2}^{(2)^{1}} = -\lambda \left\{ E_{2}^{(0)} + 2E_{2}^{(1)} + s^{2} P_{s-1} \right\} + s \mu \left\{ E_{2}^{(0)} - 2E_{2}^{1} - (s-1)^{2} P_{s} \right\}$$

We set the parameter n_1 is equal to s(t) - 1 and n_2 is equal to k- s(t).

Clark's recommendation to set $n_2 = [6.6 E_2^{(1)} / E_2^{(0)} + 0.5]$ to approximate infinite waiting space. Ingolfsson [4] found that the closure approximation to be most challenging one to implement of all the methods.

VII CONCLUSION

In this paper, we discuss how the numerical methods applied in Queueing systems. The accuracy and flexibility of these methods will always be constrained to some content by computational limitations. To solve these methods an algorithm has been implemented in MATLAB ODE Software. The Runge kutta method has frequently been used in the literature as an exact method for service quality evaluation of time-dependent single facility Queues. The randomization method is proposed as an efficient method for transient analysis of stationary queueing models and can be easily extended to non-stationary systems as in Ingolfsson. The DTH approach is a practical method for service quality evaluation of non-stationary single facility queues with general service time distributions. The Eular method has been used in a number of research papers on time-dependent queues as by Massey and Whitt. Closure approximation would have a computational advantage for systems that were sufficiently large. We were not able to find any category where closure approximation was consistently faster than Randomization method.

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