Supra M-topological space and decompositions of some types of supra msets

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Abstract

The main purpose of this paper is to introduce and study the notion of supra M-topological spaces. Moreover, the notions of supra γ-operation, supra pre open msets, supra α-open msets, supra semi open msets, supra β-open msets and supra b-open msets are presented. The current notion is a generalization of the notion in [7]. The properties of the present notion are studied and the relationships between them are given. The importance of this approach is that, the class of supra M-topological spaces is wider and more general than the class of M-topological spaces.

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1 Introduction

The notion of a multiset is well established both in mathematics and computer science [1, 2, 4, 5, 8, 16, 19, 20, 22]. In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained [3, 9, 16, 17, 18, 21]. For the sake of convenience a multiset is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer. The number of occurrences of an object x in a mset A, which is finite in most of the studies that involve msets, is called its multiplicity or characteristic value, usually denoted by $m_A(x)$ or $C_A(x)$ or simply by $A(x)$. One of the most natural and simplest examples is the multiset of prime factors of a positive integer n. The number 504 has the factorization $504 = 2^3 3^2 7^1$ which gives the multiset $X = \{3/2, 2/3, 1/7\}$ where $C_X(2) = 3$, $C_X(3) = 2$, $C_X(7) = 1$.

Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. So, the only possible relation between two mathematical objects is either they are equal or they are different. The situation in science and in ordinary life is not like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. This leads to three possible relations between any two physical objects; they are different, they are the same but separate or they coincide and are identical. For the sake of definiteness we say that two physical objects are the same or equal, if they are indistinguishable, but possibly separate, and identical if they physically coincide.

Girish et. al. [10] presented multiset topologies induced by multiset relations and studied the concepts of closure operator, interior operator and neighborhood operator on multiset. In 2012 Girish et. al. [11] studied the notions of basis, sub-basis, closed sets, closure, interior and continuous mset function. Mahanta [13] introduced the concepts of semi open (SOM) and semi closed (SCM) msets in multiset topological spaces. Recently, El-Sheikh et. al. [7] introduced the notions of γ-operation, pre-open msets, α-open msets, semi open msets, b-open msets and β-open msets in M-topological space. Moreover, they studied the relationships between this types. In classical set theory, supra topological space was first studied by Mashhour [14]. Devi et. al. [6] presented supra α-open sets in topological spaces and studied some basic properties of supra α-open sets. Additionally, the definition of supra b-open sets is introduced by [15].

In this paper, some deviations between M-topology and ordinary topology are given. Moreover, the notion of supra M-topological space is introduced. Furthermore, some types of supra msets such as supra pre-open msets, supra α-open msets, supra semi open msets, supra b-open msets and supra β-open msets are defined and studied in supra M-topological spaces. The properties of the present supra msets are studied. Some important results related to this types of supra msets are given. Finally, the relationships between different types of supra msets in supra M-topological space are presented.

2 Preliminaries

In this section, we present the basic definitions and results of mset theory which will be needed in the sequel.

Definition 2.1 *[12] A mset X drawn from the set U is represented by a count function X or C_X defined as $C_X : U \to N$, where N represents the set of non negative integers.*

Here C_X (x) is the number of occurrences of the element x in the mset X. We present the mset X drawn from the set $U=\{x_1, x_2, x_3, ..., x_n\}$ as $X=\{m_1/x_1, m_2/x_2, m_3/x_3, ..., m_n/x_n\}$ where m_i is the number of occurreneces of the element x_i, $i=1,2,3,...,n$ in the mset X.

Definition 2.2 *[12] A domain U, is defined as a set of elements from which msets are constructed. The mset space $[U]^w$ is the set of all msets whose elements are in U such that no element in the mset occurs more than w times.*

The mset space $[U]^\infty$ is the set of all msets over a domain U such that there is no limit on the number of occurrences of an element in a mset. If $U = \{x_1, x_2, ..., x_k\}$, then $[U]^w = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : \text{for } i = 1,2,...,k;\ m_i \in \{0,1,2,...,w\}\}$.

Definition 2.3 *[12] Let X and Y be two msets drawn from a set U. Then:*

1. $X = Y$ if $C_X(x) = C_Y(x)$ for all $x \in U$.
2. $X \subseteq Y$ if $C_X(x) \leq C_Y(x)$ for all $x \in U$.
3. $P = X \cup Y$ if $C_P(x) = Max\{C_X(x), C_Y(x)\}$ for all $x \in U$.
4. $P = X \cap Y$ if $C_P(x) = Min\{C_X(x), C_Y(x)\}$ for all $x \in U$.
5. $P = X \oplus Y$ if $C_P(x) = Min\{C_X(x) + C_Y(x), w\}$ for all $x \in U$.
6. $P = X \ominus Y$ if $C_P(x) = Max\{C_X(x) - C_Y(x), 0\}$ for all $x \in U$ where $\oplus$ and $\ominus$ represents mset addition and mset subtraction respectively.

Definition 2.4 *[12] Let X be a mset drawn from the set U. If $C_X(x) = 0\ \forall\ x \in U$, then X is called an empty mset and denoted by ϕ, i.e., $\phi(x) = 0\ \forall x$.*

If X is an ordinary set with n distinct elements, then the power set $P(X)$ of X contains exactly 2^n elements. If X is a multiset with n elements (repetitions counted), then the power set $P(X)$ contains strictly less than 2^n elements because singleton submsets do not repeat in $P(X)$. In classical set theory, Cantor's power set theorem fails for msets. It is possible to formulate the following reasonable definition of a power mset of X for finite mset X that preserves Cantor's power set theorem.

Definition 2.5 *[1] (Power Mset) Let $X \in [U]^w$ be a mset. Then, the power mset $P(X)$ of X is the set of all submsets of X. We have $Y \in P(X)$ if and only if $Y \subseteq X$. If $Y = \phi$, then $Y \in^1 P(X)$; and if $Y \neq \phi$, then $Y \in^k P(X)$ where $k = \prod_z \binom{|[X]_z|}{|[Y]_z|}$, the product $\prod_z$ is taken over by distinct elements of z of the mset Y and $|[X]_z| = m$ iff $z \in^m X$, $|[Y]_z| = n$ iff $z \in^n Y$, then $\binom{|[X]_z|}{|[Y]_z|} = \binom{m}{n} = \frac{m!}{n!(m-n)!}$.*

The power set of a mset is the support set of the power mset and is denoted by $P^*(X)$. The following theorem was showed the cardinality of the power set of a mset.

Theorem 2.1 *[17] Let $P(X)$ be a power mset whose members drawn from the mset* $X = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ and $P^*(X)$ be the power set of a mset X. Then, Card $(P^*(X))$ = $\Pi_{i=1}^n$ $(1+m_i)$.

Definition 2.6 *[10] Let $X \in [U]^w$ and $\tau \subseteq P^*(X)$. Then, τ is called a multiset topology (for short, M-topology) of X if τ satisfies the following properties:*

1. The mset X and the empty mset ϕ are in τ.
2. The mset union of the elements of any subcollection of τ is in τ.
3. The mset intersection of the elements of any finite subcollection of τ is in τ.

Hence, (X, τ) is called a M-topological space. Each element in τ is called an open mset. Additionally, $OM(X)$ is the set of all open submsets of X.

Remark 2.1 *[11] The complement of any submset Y in a mset topological space (X, τ) is defined by :* $Y^c ⍰ = X \ominus Y$.

Definition 2.7 *[11] A submset Y of a M-topological space X is said to be closed if the mset $X \ominus Y$ is open.*

Example 2.1 *Let $X = \{2/x, 3/y, 1/z\}$ be a mset and let $\tau = \{\phi, X, \{2/x\}, \{1/y\}, \{2/x, 1/y\}\}$ be a M-topological space. Then, the complement of any submset Y of X in a M-topological space (X, τ) is shown as:*

1. If $Y = \{2/x, 1/y\}$, then $Y^c = \{2/y, 1/z\}$.
2. If $Y = \{3/y\}$, then $Y^c = \{2/x, 1/z\}$.
3. If $Y = \{3/y, 1/z\}$, then $Y^c = \{2/x\}$.
4. If $Y = \{1/x, 1/y, 1/z\}$, then $Y^c = \{1/x, 2/y\}$.
5. If $Y = X$, then $Y^c = \phi$.

It should be noted that the De Morgan laws are satisfied in multisets by Wildberger [22].

Definition 2.8 *[11] Let A be a submset of a M-topological space (X, τ). Then, the interior of A is defined as the mset union of all open msets contained in A and is denoted by $int(A)$.*

i.e., $int(A) = \cup\{G \subseteq X : G \text{ is an open mset and } G \subseteq A\}$

and $C_{int(A)}(x) = max\{C_G(x) : G \subseteq A, G \in \tau\}$.

Definition 2.9 *[11] Let A be a submset of a M-topological space (X, τ). Then, the closure of A is defined as the mset intersection of all closed msets containing A and is denoted by $cl(A)$, i.e., $cl(A) = \cap\{K \subseteq X : K \text{ is a closed mset and } A \subseteq K\}$*

and $C_{cl(A)}(x) = min\{C_K(x) : A \subseteq K, K \in \tau^c\}$.

Proposition 2.1 *If (X, τ) is a M-topological space and A, B are two submsets of X. Then, the following properties are satisfied by [10, 11]:*

- $int(A^c) = (cl(A))^c$.
- $cl(A^c) = (int(A))^c$.
- $cl(A \cup B) = cl(A) \cup cl(B)$.
- $int(A \cap B) = int(A) \cap int(B)$.

Definition 2.10 *[19] A mset X is called a M-singleton and denoted by $\{m/x\}$ if* $C_X : U \to N$ such that $C_X(x) = m$ and $C_X(x') = 0 \; \forall \; x' \in U - \{x\}$.

Note that, $x \in^k X$, means $C_X(x) = k$, so $\{k/x\}$ is called M-singleton submset of X and $\{m/x\}$ is called simple M-singleton where $0 < m < k$.

Definition 2.11 *[18] Two msets* A *and* B *are said to be similar msets if for all* x *(* $x \in A \Leftrightarrow x \in B$ *), where* x *is an object. Thus, similar msets have equal root sets, but need not be equal themselves.*

Definition 2.12 *[1] The M-point* m/x *is said to be belonging to the mset* Y*, denoted by* $m/x \in Y$*, if* $C_Y(x) = m$.

Definition 2.13 *[1] Let* X *be a mset and if* $x \in^m X$, $x \in^n X$*. Then,* $m = n$.

3 Multi-set

Theorem 3.1 *Let* A, B *be two submsets of* $[X]^w$*. If* $A \cap B = \phi$*, then* $A \subseteq B^c$.

Proof. Let $A \nsubseteq B^c$. Then, $\exists x \in X$ such that $C_A(x) > C_{B^c}(x)$. Therefore, $C_A(x) > w - C_B(x)$. Thus, $C_A(x) + C_B(x) > w$. Since, $A \cap B = \phi$. Then, $C_B(x) = 0$. Hence, $C_A(x) > w$ which is a contradiction. Then, $A \subseteq B^c$.

Remark 3.1 *In M-topological space (* X *,* τ *), Theorem 3.1 is satisfied. But the converse of Theorem 3.1 is not true as the following example.*

Example 3.1 *Let* $X = \{a,b,c\}$ *be an universe set and* A, $B \in [X]^3$ *such that* $A = \{1/a, 1/b\}$, $B = \{2/a, 1/b, 1/c\}$*. Then,* $A \subseteq B$*, but* $B^c = \{1/a, 2/b, 2/c\}$ *this implies that* $A \cap B^c = \{1/a, 1/b\} \neq \phi$.

Remark 3.2 *From Theorem 3.1 and Example 3.1 we have, for any submset* F *of M-topological space* (X, τ)*:*

1. $F \cap F^c \neq \phi$.
2. $F \cup F^c \neq X$.

Theorem 3.2 *Let* A, B *be two submsets in M-topological space* (X, τ)*. Then,* $A \subseteq B$ *if and only if* $B^c \subseteq A^c$.

Proof.

$$
\begin{aligned}
\text{Let } A \subseteq B \quad &\Leftrightarrow C_A(x) \leq C_B(x), \forall x \in X \\
&\Leftrightarrow C_X(x) - C_B(x) \leq C_X(x) - C_A(x), \forall x \in X \\
&\Leftrightarrow C_{B^c}(x) \leq C_{A^c}(x), \forall x \in X \\
&\Leftrightarrow B^c \subseteq A^c.
\end{aligned}
$$

4 Supra M-topology

Definition 4.1 *Let* μ *be a collection of submsets of* X*. Then,* $\mu \subseteq P^*(X)$ *is called a supra M-topology on* X *if the following conditions are satisfied:*

(1) $X, \phi \in \mu$.

(2) The union of any number of msets in μ belongs to μ.

The pair (X,μ) is called supra M-topological space (or supra M-spaces) over X.

Remark 4.1 *Every M-topological space is a supra M-topological space, but the converse is not true in general as shown in the following example.*

Example 4.1 *Let* $X=\{1/a,2/b,3/c\}$ *and* $\mu=\{X,\phi,\{1/b,1/c\},\{2/b\},\{2/b,1/c\}\}$. *Then,* (X,μ) *is a supra M-topology, but it is not M-topology as* $\{1/b,1/c\}\cap\{2/b\}=\{1/b\}$ *which is not belong to* μ.

Definition 4.2 *Let* (X,τ) *be a M-topological space and* (X,μ) *be a supra M-topological space. We say that* μ *is a supra M-topology associated with* τ *if* $\tau\subset\mu$.

Definition 4.3 *Let* (X,μ) *be a supra M-topological space over* X. *Then, the members of* μ *are said to be supra open msets in* X. *The set of all supra open msets over* X *is denoted by* $SOM(X,\mu)$, *or when there can be no confusion by* $SOM(X)$ *and the set of all supra closed msets is denoted by* $SCM(X,\mu)$, *or* $SCM(X)$.

Definition 4.4 *Let* (X,μ) *be a supra M-topological space. Then, a mset* Y *is said to be a supra closed mset in* (X,μ), *if its relative complement* Y^c *is a supra open mset.*

Definition 4.5 *A mset* Y *in a supra M-topological space* (X,μ) *is called a supra M-neighborhood of the M-point* $m/x\in X$ *if there exists a supra open mset* G *such that* $m/x\in G\subseteq Y$. *The supra M-neighborhood system of a M-point* m/x, *denoted by supra* $N_\mu(m/x)$, *is the family of all its supra M-neighborhoods.*

Definition 4.6 *Let* (X,μ) *be a supra M-topological space and* $G\in P^*(X)$. *Then, the supra M-interior of* G, *denoted by* $int^s(G)$ *is the union of all supra open submsets of* G. *Clearly,* $int^s(G)$ *is the largest supra open mset over* X *which contained in* G, *i.e.,*

$$int^s(G)=\cup\{H: H \text{ is a supra open mset and } H\subseteq G\}.$$

Definition 4.7 *Let* (X,μ) *be a supra M-topological space and* $F\in P^*(X)$. *Then, the supra M-closure of* F, *denoted by* $cl^s(F)$ *is the intersection of all supra closed super msets of* F. *Clearly,* $cl^s(F)$ *is the smallest supra closed mset over* X *which contains* F, *i.e.,*

$$cl^s(F)=\cap\{H: H \text{ is a supra closed mset and } F\subseteq H\}.$$

Definition 4.8 *Let* (X,μ) *be a supra M-topological space and* $G\in P^*(X)$. *Then,* $m/x\in X$ *is called a supra limit M-point of* G *if* $(G\ominus\{m/x\})\cap H\neq\varphi\ \ \forall H\in SOM(X)$. *The set of all supra limit M-points of* G *is called the supra M-derived of* G *and denoted by* $d^s(G)$ *or* G^{sd}.

The following theorem studies the main properties of the the supra M-closure and the supra M-interior which can be considered as the deviations between the current work and the previous one [11].

Theorem 4.1 *Let* (X,μ) *be a supra M-topological space and* $F,G\in P^*(X)$. *Then:*

(1) $cl^s(F)\cup cl^s(G)\subseteq cl^s(F\cup G)$.

(2) $int^s(F \cap G) \subseteq int^s(F) \cap int^s(G)$.

Proof. Immediate.

Remark 4.2 *The equality of Theorem 4.1 is not necessarily true as shown in the following example.*

Example 4.2

(1) Let $X = \{1/a, 2/b, 3/c\}$ and $\mu = \{X, \phi, \{1/a, 1/b\}, \{1/b, 1/c\}, \{1/a, 1/b, 1/c\}\}$. Then, μ defines a supra M-topology on X. Let $G = \{1/b, 3/c\}$ and $F = \{1/a, 1/b, 2/c\}$ be two submsets of X. Then, $cl^s(F \cup G) \neq cl^s(F) \cup cl^s(G)$.

(2) Let $X = \{1/a, 2/b, 3/c\}$ and $\mu = \{X, \phi, \{1/a, 1/b\}, \{1/b, 1/c\}, \{1/a, 1/b, 1/c\}\}$. Then, μ defines a supra M-topology on X. Let $G = \{1/b, 1/c\}$ and $F = \{1/a, 1/b\}$ be two submsets of X. Then, $int^s(G) \cap int^s(F) \neq int^s(G \cap F)$.

Theorem 4.2 *Let (X, μ) be a supra M-topological space and $F, G \in P^*(X)$. Then:*

1. $F \subseteq G \Rightarrow int^s(F) \subseteq int^s(G)$.
2. $F \subseteq G \Rightarrow cl^s(F) \subseteq cl^s(G)$.

Proof. Immediate.

Remark 4.3 *The converse of Theorem 4.2 is not necessarily true as shown in the following example.*

Example 4.3

(1) Let $X = \{1/a, 2/b, 3/c\}$ and $\mu = \{X, \phi, \{1/a, 1/b\}, \{1/b, 1/c\}, \{1/a, 1/b, 1/c\}\}$. Then, μ defines a supra M-topology on X. Let $F = \{1/b, 3/c\}$ and $G = \{1/a, 1/b, 2/c\}$ be two submsets of X. Then, $int^s(F) \subseteq int^s(G)$, but $F \nsubseteq G$.

(2) Let $X = \{1/a, 2/b, 3/c\}$ and $\mu = \{X, \phi, \{1/a, 1/b\}, \{1/b, 1/c\}, \{1/a, 1/b, 1/c\}\}$. Then, μ defines a supra M-topology on X. Let $F = \{1/b, 3/c\}$ and $G = \{1/a, 2/b\}$ be two submsets of X. Then, $cl^s(F) \subseteq cl^s(G)$, but $F \nsubseteq G$.

Theorem 4.3 *Let (X, μ) be a supra M-topological space and $F \in P^*(X)$. Then:*

1. $(int^s(F))^c = cl^s(F^c)$.
2. $(cl^s(F))^c = int^s(F^c)$.
3. $int^s(F) = (cl^s(F^c))^c$.
4. $cl^s(F) = (int^s(F^c))^c$.

Proof. Immediate.

5 Submsets of supra M-topological spaces

Definition 5.1 *Let (X, μ) be a supra M-topological space. A mapping*

$\gamma : P^*(X) \to P^*(X)$ *is said to be an operation on* $SOM(X)$ *if* $F \subseteq \gamma(F)\ \forall\ F \in SOM(X)$. *The collection of all supra* γ*-open msets is denoted by* $SOM(\gamma) = \{F : F \subseteq \gamma(F), F \in P^*(X)\}$. *Additionally, the complement of a supra* γ*-open mset is called a supra* γ*-closed mset, and the family of all supra* γ*-closed msets is denoted by* $SCM(\gamma)$.

Definition 5.2 *Let* (X, μ) *be a supra M-topological space. Different cases of* γ*-operations on* $P^*(X)$ *are as follows:*

(1) If $\gamma = int^s(cl^s)$, then γ is called a supra pre-open M-operator. The set of all supra pre-open msets is denoted by $SPOM(X, \mu)$, or when there can be no confusion by $SPOM(X)$ and the set of all supra pre-closed msets by $SPCM(X, \mu)$, or $SPCM(X)$.

(2) If $\gamma = int^s(cl^s(int^s))$, then γ is called a supra α-open M-operator. The set of all supra α-open msets is denoted by $S\alpha OM(X, \mu)$, or $S\alpha OM(X)$ and the set of all supra α-closed msets by $S\alpha CM(X, \mu)$, or $S\alpha CM(X)$.

(3) If $\gamma = cl^s(int^s)$, then γ is called a supra semi open M-operator. The set of all supra semi open msets is denoted by $SSOM(X, \mu)$, or $SSOM(X)$ and the set of all supra semi closed msets by $SSCM(X, \mu)$, or $SSCM(X)$.

(4) If $\gamma = cl^s(int^s(cl^s))$, then γ is called a supra β-open M-operator. The set of all supra β-open msets is denoted by $S\beta OM(X, \mu)$, or $S\beta OM(X)$ and the set of all supra β-closed msets by $S\beta CM(X, \mu)$, or $S\beta CM(X)$.

(5) If $\gamma = int^s(cl^s) \cup cl^s(int^s)$, then γ is called a supra b-open M-operator. The set of all supra b-open msets is denoted by $SBOM(X, \mu)$, or $SBOM(X)$ and the set of all supra b-closed msets by $SBCM(X, \mu)$, or $SBCM(X)$.

Theorem 5.1 *Let* (X, μ) *be a supra M-topological space and* $\gamma : P^*(X) \to P^*(X)$ *be an operation on* $SOM(X)$.

If $\gamma \in \{int^s(cl^s), int^s(cl^s(int^s)), cl^s(int^s), cl^s(int^s(cl^s)), int^s(cl^s) \cup cl^s(int^s)\}$, then:

(1) Arbitrary union of supra γ-open msets is supra γ-open msets.

(2) Arbitrary intersection of supra γ-closed msets is supra γ-closed msets.

Proof.

(1) The proof is given for the case of supra pre-open M-operator, i.e., $\gamma = int^s(cl^s)$. Let $\{F_j : j \in J\} \subseteq SPOM(X)$. Then, $F_j \subseteq int^s(cl^s(F_j))$, $\forall\ j \in J$. It follows that, $\bigcup_j F_j \subseteq \bigcup_j int^s(cl^s(F_j)) \subseteq int^s(\bigcup_j cl^s(F_j)) \subseteq int^s(cl^s(\bigcup_j F_j))$. Hence, $\bigcup_j F_j \in SPOM(X)$ $\forall\ j \in J$. The rest of the proof is similar.

(2) Immediate.

Remark 5.1 *The finite intersection of two supra pre-open (respectively, supra* β*-open, supra* α*-open, supra semi open, supra b-open) msets need not to be supra pre-open (respectively, supra* β*-open, supra* α*-open, supra semi-open, supra* b*-open), as shown in the following example.*

Example 5.1 *Let* $X = \{1/a, 2/b, 3/c\}$, *and* $\mu = \{X, \phi, \{1/a, 2/b\}, \{2/b, 2/c\}, \{1/a, 2/b, 2/c\}\}$. *Then,* μ *is a supra M-topology on* X. *We have the following cases:*

(1) Let $H = \{1/a, 2/c\}$ and $G = \{1/b, 3/c\}$. Then, it is clear that H, G are supra pre-open msets of (X, μ), but their intersection $H \cap G = \{2/c\}$ is not a supra pre-open.

(2) Let $H = \{1/a, 2/c\}$ and $G = \{1/b, 3/c\}$. Then, it is clear that H, G are supra β-open msets of (X, μ), but their intersection $H \cap G = \{2/c\}$ is not a supra β-open.

(3) Let $H = \{1/a, 2/b\}$ and $G = \{2/b, 2/c\}$. Then, it is clear that H, G are supra semi open msets of (X, μ), but their intersection $H \cap G = \{2/b\}$ is not a supra semi open.

(4) Let $H = \{1/a, 2/b\}$ and $G = \{2/b, 2/c\}$. Then, it is clear that H, G are supra α-open msets of (X, μ), but their intersection $H \cap G = \{2/b\}$ is not a supra α-open.

(5) Let $H = \{1/a, 2/c\}$ and $G = \{1/b, 3/c\}$. Then, it is clear that H, G are supra b-open msets of (X, μ), but their intersection $H \cap G = \{2/c\}$ is not a supra b-open.

6 Relationships between submsets of supra M-topological spaces

In this section, the relationships between some special submsets of a supra M-topological space (X, μ) are introduced.

Theorem 6.1 *In a supra M-topological space (X, μ), the following statements hold:*

(1) Every supra open mset is a supra pre-open (respectively, semi-open, α-open, β-open, b-open) mset.

(2) Every supra closed mset is a supra pre-closed (respectively, semi-closed, α-closed, β-closed, b-closed) mset.

Proof. The proof is given for the case of supra pre-open M-operator, i.e., $\gamma = int^s(cl^s)$, the other cases are similar.

(1) Let $F \in SOM(X)$. Then, $int^s(F) = F$. Since, $F \subseteq cl^s(F)$, then $F \subseteq int^s(cl^s(F))$. Hence, $F \in SPOM(X)$.

(2) By a similar way.

Remark 6.1 *The converse of part (1) and (2) in Theorem 6.1 is not necessarily true as shown in the following example for part (1) and we can add example for part (2).*

Example 6.1 *Let $X = \{1/a, 2/b, 3/c\}$, and $\mu = \{X, \phi, \{1/a, 2/b\}, \{2/b, 2/c\}, \{1/a, 2/b, 2/c\}\}$. Then, μ is a supra M-topology on X. Hence, the mset H which defined as follows:*

(1) $H = \{1/a, 2/c\}$ is a supra pre open mset of (X, μ), but it is not a supra open mset.

(2) $H = \{2/b, 3/c\}$ is a supra semi-open mset of (X, μ), but it is not a supra open mset.

(3) $H = \{2/b, 3/c\}$ is a supra α-open mset of (X, μ), but it is not a supra open mset.

(4) $H = \{1/b, 3/c\}$ is a supra β-open mset of (X, μ), but it is not a supra open mset.

(5) $H = \{1/a, 2/c\}$ is a supra b-open mset of (X, μ), but it is not a supra open mset.

Theorem 6.2 *Let* (X,μ) *be a supra M-topological space. Then, the following statements are hold:*

(1) Every supra α-open (respectively, α-closed) mset is a supra semi open (respectively, semi closed) mset.

(2) Every supra α-open (respectively, α-closed) mset is a supra pre-open (respectively, pre-closed) mset.

(3) Every supra semi open (respectively, semi-closed) mset is a supra b-open (respectively, b-closed) mset.

(4) Every supra pre-open (respectively, pre-closed) mset is a supra b-open (respectively, b-closed) mset.

(5) Every supra b-open (respectively, b-closed) mset is a supra β-open (respectively, β-closed) mset.

Proof. We prove the assertion in the case of open mset, the other case is similar.

(1) Let $F \in S\alpha OM(X)$. Then, $F \subseteq int^s(cl^s(int^s(F))) \subseteq cl^s(int^s(F))$. Hence, $F \in SSOM(X)$.

(2) Let $F \in S\alpha OM(X)$. Then, $F \subseteq int^s(cl^s(int^s(F))) \subseteq int^s(cl^s(F))$. Hence, $F \in SPOM(X)$.

(3) Let $F \in SSOM(X)$. Then, $F \subseteq cl^s(int^s(F)) \subseteq [cl^s(int^s(F)) \cup int^s(cl^s(F))]$. Hence, $F \in SBOM(X)$.

(4) Let $F \in SPOM(X)$. Then, $F \subseteq int^s(cl^s(F)) \subseteq [cl^s(int^s(F)) \cup int^s(cl^s(F))]$. Hence, $F \in SBOM(X)$.

(5) Let $F \in SBOM(X)$. Then, $F \subseteq [cl^s(int^s(F)) \cup int^s(cl^s(F))]$. Therefore,

$$\begin{aligned} F \subseteq\ & [cl^s(int^s(F)) \cup cl^s(int^s(cl^s(F)))] \\ & \subseteq cl^s[int^s(F) \cup int^s(cl^s(F))] \\ & \subseteq cl^s(int^s[F \cup cl^s(F)]) \\ & \subseteq cl^s(int^s(cl^s(F))). \end{aligned}$$

Hence, $F \in S\beta OM(X)$.

Remark 6.2 *The converse of Theorem 6.2 is not necessarily true as shown in the following examples.*

Examples 6.1

(1) Let $X = \{2/a, 3/b, 1/c\}$, and $\mu = \{X, \phi, \{1/a\}, \{2/b\}, \{1/a, 1/b\}, \{1/a, 2/b\}\}$. Then, μ is a supra M-topology on X. Hence, the mset $H = \{1/a, 1/c\}$ is a supra semi-open mset of (X,μ), but it is not a supra α-open mset.

(2) Let $X = \{1/a, 2/b, 3/c\}$, and $\mu = \{X, \phi, \{1/a, 2/b\}, \{2/b, 2/c\}, \{1/a, 2/b, 2/c\}\}$. Then, μ is a supra M-topology on X. Hence, the mset $H = \{1/a, 2/c\}$ is a supra pre-open mset of (X,μ), but it is not a supra α-open mset.

(3) Let $X = \{1/a, 2/b, 3/c\}$, and $\mu = \{X, \phi, \{1/a, 2/b\}, \{2/b, 2/c\}, \{1/a, 2/b, 2/c\}\}$. Then, μ is a supra M-topology on X. Hence, the mset $H = \{1/a, 2/c\}$ is a supra b-open mset of

(X,μ), but it is not a supra semi-open mset.

(4) Let $X=\{1/a,2/b,3/c\}$, and $\mu=\{X,\phi,\{1/a,1/b\},\{1/b,1/c\},\{1/a,1/b,1/c\}\}$. Then, μ is a supra M-topology on X. Hence, the mset $H=\{1/b,2/c\}$ is a supra b-open mset of (X,μ), but it is not a supra pre-open mset.

(5) Let $X=\{2/a,3/b,1/c,1/d\}$, and $\mu=\{X,\phi,\{1/a,2/b\},\{1/b,1/c\},\{1/a,2/b,1/c\}\}$. Then, μ is a supra M-topology on X. Hence, the mset $H=\{2/a\}$ is a supra β-open mset of (X,μ), but it is not supra b-open mset.

On account of Theorem 6.1, Theorem 6.2, Examples 6.1 and Examples 6.1, we have the following corollary.

Corollary 6.1 *For a supra M-topological space (X, μ), we have :*

$$\begin{tikzcd} SOM(X) \arrow[r] \arrow[d] & S\alpha OM(X) \arrow[r] \arrow[dl] & SSOM(X) \arrow[dl] \\ SPOM(X) \arrow[r] & SBOM(X) \arrow[r] & S\beta OM(X) \end{tikzcd}$$

Theorem 6.3 *Let (X,μ) be a supra M-topological space, $\gamma : P^*(X)\to P^*(X)$ be one of the operations in Definition 5.2 and $F\in P^*(X)$. Then, the following hold:*

(1) $\gamma(int^s(\mathrm{F}^c)) = \mathrm{X}\ominus\gamma(cl^s(\mathrm{F}))$.

(2) $\gamma(cl^s(\mathrm{F}^c)) = \mathrm{X}\ominus\gamma(int^s(\mathrm{F}))$.

Proof. The proof is given for the case of supra pre-open M-operator, i.e., $\gamma = int^s(cl^s)$, the other cases are similar.

(1) $Pcl^s(F)=\cap\{G : G\in SPCM(X), F\subseteq G\}$

$[Pcl^s(F)]^c=\cup\{G^c : G\in SPCM(X), F\subseteq G\}=\cup\{G^c : G^c\in SPOM(X), G^c\subseteq F^c\}$

$= Pint^s(F^c)$.

(2) Similarly.

Theorem 6.4 *Let (X,μ) be a supra M-topological space and $N,G\in P^*(X)$. Then:*

(1) $N\in SSOM(X)$ if and only if $cl^s(N)=cl^s(int^s(N))$.

(2) $N\in SSOM(X)$ if and only if there exists $G\in SOM(X)$ such that $G\subseteq N\subseteq cl^s(G)$.

(3) If $N\in SSOM(X)$ and $N\subseteq G\subseteq cl^s(N)$, then $G\in SSOM(X)$.

Proof.

(1) ($\Rightarrow$)Let $N\in SSOM(X)$. Then, $N\subseteq cl^s(int^s(N))$. Hence,

$$cl^s(N)\subseteq cl^s(int^s(N)). \qquad (1)$$

Since, $int^s(N)\subseteq N$. Then,

$$cl^s(int^s(N))\subseteq cl^s(N). \qquad (2)$$

From (1) and (2) we get, $cl^s(N)=cl^s(int^s(N))$.

($\Leftarrow$) Since, $cl^s(N)=cl^s(int^s(N))$ and $N\subseteq cl^s(N)$. Then, $N\subseteq cl^s(int^s(N))$. Thus,

$N \in SSOM(X)$.

(2) $(\Rightarrow)$Let $N \in SSOM(X)$. Then, $N \subseteq cl^s(int^s(N))$. Take $G = int^s(N) \in SOM(X)$ such that $G = int^s(N) \subseteq N \subseteq cl^s(int^s(N)) = cl^s(G)$. Therefore, there exists $G \in SOM(X)$ such that $G \subseteq N \subseteq cl^s(G)$.

$(\Leftarrow)$Since, $G \in SOM(X)$ such that $G \subseteq N$ and $int^s(N)$ is the largest supra open mset contained in N. Then, $G \subseteq int^s(N)$. Thus, $N \subseteq cl^s(G) \subseteq cl^s(int^s(N))$. Hence, $N \in SSOM(X)$.

(3) Let $N \in SSOM(X)$. Then, from part (2) $\exists H \in SOM(X)$ such that $H \subseteq N \subseteq cl^s(H)$. Hence, $H \subseteq N \subseteq G \subseteq cl^s(N) \subseteq cl^s(H)$. Thus, $G \in SSOM(X)$.

Theorem 6.5 *Let* (X, μ) *be a supra M-topological space and* $F, G \in P^*(X)$. *Then:*

(1) $F \in S\alpha OM(X)$ if and only if $\exists\, H \in SOM(X)$ such that $H \subseteq F \subseteq int^s(cl^s(H))$.

(2) If $F \in S\alpha OM(X)$ and $F \subseteq G \subseteq int^s(cl^s(F))$, then $G \in S\alpha OM(X)$.

Proof. (1) $(\Rightarrow)$ Suppose that $int^s(F) = H \in SOM(X)$. Then, $H \subseteq F \subseteq int^s(cl^s(H))$.

$(\Leftarrow)$ Let $H \subseteq F \subseteq int^s(cl^s(H))$, $H \in SOM(X)$. Then, $int^s(H) = H \subseteq int^s(F)$. It follows that, $F \subseteq int^s(cl^s(int^s(H))) \subseteq int^s(cl^s(int^s(F)))$. Thus, $F \in S\alpha OM(X)$.

(2) Let $F \in S\alpha OM(X)$. Then, $F \subseteq int^s(cl^s(int^s(F)))$.

Hence, $F \subseteq G \subseteq int^s(cl^s(int^s(cl^s(int^s(F))))) \subseteq int^s(cl^s(int^s(F))) \subseteq int^s(cl^s(int^s(G)))$. Thus, $G \in S\alpha OM(X)$.

Theorem 6.6 *Let* (X, μ) *be a supra M-topological space and* $F \in P^*(X)$. *Then:*

(1) $F \in S\alpha OM(X)$ if and only if $F \in SPOM(X) \cap SSOM(X)$.

(2) $F \in S\alpha CM(X)$ if and only if $F \in SPCM(X) \cap SSCM(X)$.

Proof.

(1) $(\Rightarrow)$ Let $F \in S\alpha OM(X)$. Then, $F \subseteq int^s(cl^s(int^s(F)))$. Hence, $F \subseteq cl^s(int^s(F))$ and $F \subseteq int^s(cl^s(F))$. Thus, $F \in SPOM(X) \cap SSOM(X)$.

$(\Leftarrow)$ Let $F \in SPOM(X) \cap SSOM(X)$. Then, $F \subseteq cl^s(int^s(F))$ and $F \subseteq int^s(cl^s(F))$. Thus, $F \subseteq int^s(cl^s(cl^s(int^s(F)))) = int^s(cl^s(int^s(F)))$. It follows that, $F \in S\alpha OM(X)$.

(2) By a similar way.

Theorem 6.7 *Let* (X, μ) *be a supra M-topological space and* $F \in P^*(X)$. *Then:*

1. $F \in SPCM(X)$ if and only if $cl^s(int^s(F)) \subseteq F$.
2. $F \in S\alpha CM(X)$ if and only if $cl^s(int^s(cl^s(F))) \subseteq F$.
3. $F \in SSCM(X)$ if and only if $int^s(cl^s(F)) \subseteq F$.

4. $F \in S\beta CM(X)$ if and only if $int^s(cl^s(int^s(F))) \subseteq F$.
5. $F \in SBCM(X)$ if and only if $int^s(cl^s(F)) \cap cl^s(int^s(F)) \subseteq F$.

Proof. Straightforward.

Theorem 6.8 *Let* (X, μ) *be a supra M-topological space. If* $F \in S\alpha OM(X)$ *and* $F^c \in SPOM(X)$*, then* $F \in SOM(X)$.

Proof. Let $F \in S\alpha OM(X)$ and $F^c \in SPOM(X)$. Then, $F \in SPCM(X)$. Hence, $cl^s(int^s(F) \subseteq F \subseteq int^s(cl^s(int^s(F))) \subseteq cl^s(int^s(F)$. This means that, $cl^s(int^s(F)) = F$. Thus, $F \subseteq int^s(cl^s(int^s(F))) = int^s(F)$. Therefore, $F \in SOM(X)$.

7 Conclusion

In this paper, the supra M-topological spaces are introduced, by dropping only the intersection condition in M-topological spaces. Every M-topological space is a supra M-topological space, but the converse is not necessarily true. Moreover, the supra M-interior and supra M-closure are presented and their properties are obtained. The supra M-closure of a union of two supra msets not equal the supra M-closure of the first supra mset union the supra M-closure of the second supra mset. It can be considered to be one the main deviation between the current results and the previous one [11]. Furthermore, different types of submsets of supra M-topological spaces are studied. Finally, comparisons between this types are introduced.

References

[1] Blizard, Wayne D., Multiset theory, Notre Dame Journal of Formal Logic 30 (1989 a) 36-66.

[2] Blizard, Wayne D., Real-valued multisets and fuzzy sets, Fuzzy Sets and Systems 33 (1989 b) 77-97.

[3] Blizard, Wayne D., The development of multiset theory, Modern Logic 1 (1991) 319-352.

[4] Clements, G. F., On multiset k-families, Discrete Mathematics 69 (1988) 153-164.

[5] Conder, M., S. Marshall and Arkadii M. Slinko, Orders on multisets and discrete cones, A Journal on the Theory of Ordered Sets and its Applications 24 (2007) 277-296.

[6] Devi, R., S. Sampathkumar and M. Caldas, On supra α-open sets and $S\alpha$-continuous functions, General Mathematics 16 (2008) 77-84.

[7] El-Sheikh, S. A., R. A-K. Omar and M. Raafat, γ-operation in M-topological space, Gen. Math. Notes, preprint.

[8] Girish, K. P. and Sunil Jacob John, Relations and functions in multiset context, Information Sciences 179 (2009) 758-768.

[9] Girish, K. P. and Sunil Jacob John, Rough multisets and information multisystems, Advances in Decision Sciences (2011) 1-17.

[10] Girish, K. P. and Sunil Jacob John, Multiset topologies induced by multiset relations, Information Sciences 188 (2012) 298-313.

[11] Girish, K. P. and Sunil Jacob John, On Multiset Topologies, Theory and Applications of Mathematics & Computer Science, 2 (2012) 37-52.

[12] Jena, S. P. , S. K. Ghosh and B. K. Tripathy, On the theory of bags and lists,

Information Sciences 132 (2001) 241-254.

[13] Mahanta, J. and D. Das, Semi-compactness in multiset topology, arXiv 1403.5642v1, (2014).

[14] Mashhour, A. S., A. A. Allam, F. S. Mahmoud and F. H. Khedr, On supratopological space, Indian J. Pure Appl. Math. 14 (1983) 502-510.

[15] Sayed, O.R. and Takashi Noiri, On supra b-open sets and supra b-continuity on topological spaces, European Journal of Pure and Applied Mathematics 3 (2010) 295-302.

[16] Singh, D., A note on the development of multiset theory, Modern Logic 4 (1994) 405-406.

[17] Singh, D. and J. N. Singh, Some combinatorics of multisets, International Journal of Mathematical Education in Science and Technology 34 (2003) 489-499.

[18] Singh, D., A. M. Ibrahim, T. Yohana and J. N. Singh, An overview of the applications of multisets, Novi Sad J. Math. 37 (2007) 73-92.

[19] Singh, D., A. M. Ibrahim, T. Yohana and J. N. Singh, Complementation in multiset theory, International Mathematical Forum 6 (2011) 1877-1884.

[20] Slapal, Josef, Relations and topologies, Czechoslovak Mathematical Journal 43 (1993) 141-150.

[21] Yagerm, R. R., On the theory of bags, Int. General Syst. 13 (1986) 23-37.

[22] Wildberger, N. J., A new look at multisets, University of New South Wales, Sydney, Australia (2003) 1-21.