A Fixed point approach to the stability of a mixed quadratic-quartic(QQ) functional equation in quasi$-\beta-$normed spaces

K. Balamurugan[1], **M. Arunkumar**[2] **and P. Ravindiran**[3]

[1,2] *Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.*
e-mail:[1] *balamurugankaliyamurthy@yahoo.com*
e-mail:[2] *annarun2002@yahoo.co.in*
[3] *Department of Mathematics, Arignar Anna Government Arts College, Villupuram - 605 602, Tamilnadu, India.*
e-mail:[3] *p.ravindiran@gmail.com*

Abstract: In this paper we prove the generalized Hyers-Ulam stability of the following mixed quadratic-quartic functional equation

$$\begin{aligned} &f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)\\ &=72[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+8[f(y+z)+f(y-z)]\\ &\quad+24f(2x)+4f(2y)-240f(x)-160f(y)-48f(z) \end{aligned}$$

in the quasi$-\beta-$normed spaces via fixed point method. Counterexamples for non-stability cases are also discussed.
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1 Introduction

The stability problem of functional equations is originated from a question of Ulam [19] concerning the stability of group homomorphisms. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of all the above stability results was obtained by Găvruţa [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. Kim [11] solved the general solutions and proved the Hyers-Ulam-Rassias stability for the mixed type of quartic and quadratic functional equation:

$$\begin{aligned} &f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)+4f(x)+4f(y)+4f(z)\\ &\quad=2f(x+y)+2f(x-y)+2f(x+z)+2f(x-z)+2f(y+z)+2f(y-z). \end{aligned} \tag{1.1}$$

Eshaghi Gordji et al.[6] introduced another mixed type of quartic and quadratic functional equation:

$$f(nx+y)+f(nx-y)=n^2f(x+y)+n^2f(x-y)+2n^2(n^2-1)f(x)-2(n^2-1)f(y) \tag{1.2}$$

for all fixed integer n with $n\neq 0,\pm 1$. They established the general solutions and proved the Hyer-Ulam-Rassias stability of this equation in quasi-Banach spaces. Also, for the case $n=2$, they established the general solution and investigated generalized Hyers - Ulam stability for the following equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+2f(2x)-8f(x)-6f(y) \tag{1.3}$$

with $f(0)=0$ in RN-spaces(see [7]).

Arunkumar and Agilan [14] introduced and investigated the generalized Hyers - Ulam stability for the following mixed type of quadratic and additive functional equation via fixed point method:

$$\begin{aligned} &f(x+2y+3z)+f(x-2y+3z)+f(x+2y-3z)+f(x-2y-3z)\\ &\qquad=4f(x)+8[f(y)+f(-y)]+18[f(z)+f(-z)] \end{aligned}$$

Balamurugan et al.[4](see also [3])introduced and investigated the generalized Hyers-Ulam stability for the following mixed additive-cubic functional equation:

$$\begin{aligned} &f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)\\ &=24[f(x+y)+f(x-y)]+6[f(x+z)+f(x-z)]+16f(2x)-80f(x) \end{aligned}$$

Zhou Xu et al. [22] achieved the general solution and proved the stability of the following quintic functional equation

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y)=120f(y)$$

and the sextic functional equation

$$f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y)-6f(x-2y)+f(x-3y)=720f(y)$$

in the quasi$-\beta-$normed spaces via fixed point method. The stability of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [5], [10], [12], [15], [18], [20])

In this paper, we prove the generalized Hyers-Ulam stability for the following mixed quadratic-quartic functional equation:

$$\begin{aligned} &f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)+240f(x)+160f(y)+48f(z)\\ &=72[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+8[f(y+z)+f(y-z)]+24f(2x)+4f(2y) \end{aligned} \tag{1.4}$$

It is easy to see that the mapping $f(x)=ax^2+bx^4$ is a solution of the functional equation (1.4).

2 Preliminary results on quasi$-\beta-$normed spaces

In this section, we present some preliminary results concerning to quasi-β-normed spaces.

We fix a real number β with $0<\beta\leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let X be a linear space over $\mathbb{K}$. *A quasi-β-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following:

(i) $\|x\|\geq 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$.

(ii) $\|\lambda x\| =|\lambda|^{\beta}\cdot\|x\|$ for all $\lambda\in\mathbb{K}$ and all $x\in X$.

(iii) There is a constant $K\geq 1$ such that $\|x+y\|\leq K(\|x\|+\|y\|)$ for all $x,y\in X$.

The pair $(X,\|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$.

Definition 2.2. A *quasi-β-Banach space* is a complete quasi-β-normed space.

Definition 2.3. A quasi-β-norm $\|\cdot\|$ is called a *(β,p)-norm* $(0<p\leq 1)$ if

$$\|x+y\|^p\leq\|x\|^p+\|y\|^p$$

for all $x,y\in X$. In this case, a quasi-β-Banach space is called a *(β,p)-Banach space*.

More details, one can refer [16, 17, 21, 22] for the concepts of quasi-normed spaces and p-Banach space.

Now, we present the following theorem due to Margolis and Diaz [23] for fixed point Theory.

Theorem 2.1. *[23] Suppose that for a complete generalized metric space (Ω,d) and a strictly contractive mapping $T:\Omega\to\Omega$ with Lipschitz constant L. Then, for each given $x\in\Omega$, either*

$$d(T^nx,T^{n+1}x)=\infty \qquad \forall \qquad n\geq 0,$$

or there exists a natural number n_0 such that
(i) $d(T^nx,T^{n+1}x)<\infty$ for all $n\geq n_0$;
(ii) The sequence (T^nx) is convergent to a fixed point y^* of T;
(iii) y^* is the unique fixed point of T in the set $\Delta=\{y\in\Omega: d(T^{n_0}x,y)<\infty\}$;
(iv) $d(y^*,y)\leq\frac{1}{1-L}d(y,Ty)$ for all $y\in\Delta$.

Throughout this paper, unless otherwise explicitly stated, we will assume that X is a linear space, and Y is a $(\beta,p)-$Banach space space with (β,p) norm $\|.\|_Y$. Let K be the modulus of concavity of $\|.\|_Y$. For notational convenience, given a mapping $f:X\to Y$, we define the difference operator

$$\begin{aligned} Df(x,y,z)&=f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)\\ &\quad-72[f(x+y)+f(x-y)]-18[f(x+z)+f(x-z)]-8[f(y+z)+f(y-z)]\\ &\quad-24f(2x)-4f(2y)+240f(x)+160f(y)+48f(z) \end{aligned}$$

for all $x,y,z\in X$.

3 Stability of Eq. (1.4): Quadratic case

we will use the following lemma in this paper.

Lemma 3.1. *[22] Let $j \in \{-1, 1\}$ be fixed, $s, a \in \mathbb{N}$ with $a \geq 2$ and $\psi : X \to [0, \infty)$ a function such that there exists an $L < 1$ with $\psi(a^j x) \leq a^{js\beta} L\psi(x)$ for all $x \in X$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(ax) - a^s f(x)\|_Y \leq \psi(x) \tag{3.1}$$

for all $x \in X$. Then there exists a uniquely determined mapping $F : X \to Y$ such that $F(ax) = a^s F(x)$ and

$$\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta} \mid 1 - L^j \mid} \psi(x) \tag{3.2}$$

for all $x \in X$.

Proof. Consider the set

$$\Omega = \{g \mid g : X \to Y\} \tag{3.3}$$

and introduce the generalized metric on Ω,

$$d(g, h) = \inf\{\mu > 0 \mid \|g(x) - h(x)\|_Y \leq \mu\psi(x) \quad \forall x \in X\}. \tag{3.4}$$

It is easy to show that (Ω, d) is a complete generalized metric space (see [24, 25, 26]). Define a function $J : \Omega \to \Omega$ by $Jg(x) = a^{-js} g(a^j x)$ for all $x \in X$. Let $g, h \in \Omega$ be given such that $d(g, h) < \epsilon$, by the definition,

$$\|g(x) - h(x)\|_Y \leq \epsilon\psi(x), \quad \forall x \in X. \tag{3.5}$$

Hence

$$\|Jg(x) - Jh(x)\|_Y = a^{-js\beta}\|g(a^j x) - h(a^j x)\|_Y \leq a^{-js\beta}\epsilon\psi(a^j x) \leq L\epsilon\psi(x) \tag{3.6}$$

forall $x \in X$. By definition, $d(Jg, Jh) < L\epsilon$. Therefore,

$$d(Jg, Jh) \leq Ld(g, h), \quad \forall g, h \in \Omega. \tag{3.7}$$

This means that J is a strictly contractive self-mapping of Ω with Lipschitz constant L. It follows from (3.1) that

$$d(f, Jf) \leq \begin{cases} \frac{1}{a^{s\beta}}\psi(x), & \text{if j=1,} \\ \frac{L}{a^{s\beta}}\psi(x), & \text{if j=-1,} \end{cases} \tag{3.8}$$

for all $x \in X$. Therefore, by Theorem 1.3 of [25], J has a unique fixed point $F : X \to Y$ in the set $\Delta = \{g \in \Omega : d(g, f) < \infty\}$. This implies that $F(ax) = a^s F(x)$ and

$$F(x) = \lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} a^{-jns} f(a^{jn} x) \tag{3.9}$$

for all $x \in X$. Moreover,

$$d(f, F) \leq \frac{1}{1 - L} d(f, Jf) \leq \frac{1}{a^{s\beta} \mid 1 - L^j \mid}. \tag{3.10}$$

This implies that the inequality (3.2) holds.

To prove the uniqueness of the mapping F, assume that there exists another mapping $G : X \to Y$ which satisfies (3.2) and $G(ax) = a^s G(x)$ for all $x \in X$. Fix $x \in X$. Clearly, $F(a^{jn}x) = a^{jsn}F(x)$ and $G(a^{jn}x) = a^{jsn}G(x)$ for all $n \in \mathbb{N}$. Thus

$$\|F(x) - G(x)\| \leq K \left\| \frac{F(a^{jn}x)}{a^{jsn}} - \frac{f(a^{jn}x)}{a^{jsn}} \right\|_Y \leq K \left\| \frac{G(a^{jn}x)}{a^{jsn}} - \frac{f(a^{jn}x)}{a^{jsn}} \right\|_Y \leq \frac{2KL^n}{a^{s\beta} \mid 1 - L^j \mid} \psi(x). \tag{3.11}$$

Since, for every $x \in X, \lim_{n\to\infty}(\ (2KL^n)/(a^{s\beta} \mid 1 - L^j \mid \ ) \ )\psi(x) = 0$, we get $G = F$. This completes the proof. ☐

Theorem 3.2. *Let $j \in \{-1, 1\}$ be fixed, and let $\psi_b : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi_b(4^j x, 4^j y, 4^j z) \leq 16^{j\beta} L\psi_b(x, y, z)$for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x, y, z)\|_Y \leq \psi_b(x, y, z) \tag{3.12}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{1}{16^\beta \mid 1 - L^j \mid} \tilde{\psi}_b(x) \tag{3.13}$$

for all $x \in X$*, where*

$$\tilde{\psi_b}(x) = K[M_b(x,2x,x) + KM_b(x,x,x) + \left(\frac{11}{2}\right)^{\beta} K^2 M_b(x,0,x) + 20^{\beta} K^3 M_b(x,0,0) + K^4 \left(\frac{1}{2}\right)^{\beta} M_b(0,x,0) + K^4 \left(\frac{1}{3}\right)^{\beta} M_b(0,0,x)] \quad \textit{for all } x \in X.$$

Proof. Replacing (x,y,z) by $(x,2x,x),(x,x,x),(x,0,x),(x,0,0),(0,x,0)$ and $(0,0,x)$ in (3.12), respectively, we get the following inequalities

$$\|f(8x) + f(6x) - 4f(4x) - 80f(3x) + 118f(2x) + 280f(x) - 72f(-x) + f(-2x)\|_Y \leq \psi_b(x,2x,x), \quad \forall x \in X. \tag{3.14}$$

$$\|f(6x) + f(4x) - 125f(2x) + 448f(x)\|_Y \leq \psi_b(x,x,x), \quad \forall x \in X. \tag{3.15}$$

$$\|2f(4x) - 40f(2x) + 136f(x) - 8f(-x)\|_Y \leq \psi_b(x,0,x), \quad \forall x \in X. \tag{3.16}$$

$$\|4f(3x) - 24f(2x) + 60f(x)\|_Y \leq \psi_b(x,0,0), \quad \forall x \in X. \tag{3.17}$$

$$\|-2f(2x) + 72f(x) - 72f(-x) + 2f(-2x)\|_Y \leq \psi_b(0,x,0), \quad \forall x \in X. \tag{3.18}$$

$$\|24f(x) - 24f(-x)\|_Y \leq \psi_b(0,0,x), \quad \forall x \in X. \tag{3.19}$$

Let $g, \tilde{\psi_b}(x) : X \to Y$ be mappings defined by $g(x) = f(2x) - 16f(x)$ for all $x \in X$ and

$$\tilde{\psi_b}(x) = K[M_b(x,2x,x) + KM_b(x,x,x) + \left(\frac{11}{2}\right)^{\beta} K^2 M_b(x,0,x) + 20^{\beta} K^3 M_b(x,0,0) + K^4 \left(\frac{1}{2}\right)^{\beta} M_b(0,x,0) + K^4 \left(\frac{1}{3}\right)^{\beta} M_b(0,0,x)] \tag{3.20}$$

for all $x \in X$. It follows from (3.14) – (3.20) that

$$\|f(8x) - 16f(4x) - 16f(2x) + 256f(x)\|_Y \leq \xi_b(x), \quad \forall x \in X. \tag{3.21}$$

Therefore (3.21) means

$$\|g(4x) - 4g(x)\|_Y \leq \tilde{\psi_b}(x) \tag{3.22}$$

for all $x \in X$. By Lemma 3.1, there exists a unique mapping $B : X \to Y$ such that $B(4x) = 16B(x)$ and

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{1}{16^{\beta} \mid 1 - L^j \mid} \tilde{\psi_b}(x) \tag{3.23}$$

for all $x \in X$. It remains to show that B is a quadratic map. By (3.12), we have

$$\left\| \frac{1}{16^{nj}} Df(4^{nj}x, 4^{nj}y, 4^{nj}z) \right\|_Y \leq \frac{1}{16^{nj\beta}} \psi_b(4^{nj}x, 4^{nj}y, 4^{nj}z) \leq \frac{1}{16^{nj\beta}} (16^{j\beta} L)^n \psi_b(x,y,z) = L^n \psi_b(x,y,z) \tag{3.24}$$

for all $x,y,z \in X$ and all positive integers n. so

$$\|DB(x,y,z)\|_Y = 0 \tag{3.25}$$

for all $x,y,z \in X$. Thus the mapping $B : X \to Y$ is quadratic, as desired □

Throughout this paper, we will assume that X is a quasi$-\alpha-$normed space with quasi$-\alpha-$norm $\|.\|_X$ in all the corollaries. The following corollaries are immediate consequence of Theorem 3.2.

Corollary 3.3. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that r, s and t are all $\neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y,z)\|_Y \leq \begin{cases} \nu, & \\ \nu \|x\|_X^r, & r > 0, s = 0, t = 0; \\ \nu \|y\|_X^s, & r = 0, s > 0, t = 0; \\ \nu \|z\|_X^t, & r = 0, s = 0, t > 0; \\ \nu \left\{\|x\|_X^r + \|y\|_X^s + \|z\|_X^t\right\}, & r > 0, s > 0, t > 0; \end{cases} \tag{3.26}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \alpha_b, & \\ \beta_b(x), & r > 0, s = 0, t = 0; \\ \gamma_b(x), & r = 0, s > 0, t = 0; \\ \delta_b(x), & r = 0, s = 0, t > 0; \\ \zeta_b(x), & r > 0, s > 0, t > 0; \end{cases} \tag{3.27}$$

for all $x \in X$, where

$$\begin{aligned} \alpha_b =& K\nu \left\{ \frac{1 + K + \left(\frac{11}{2}\right)^\beta K^2 + 20^\beta K^3 + \left[\left(\frac{1}{2}\right)^\beta + \left(\frac{1}{3}\right)^\beta\right] K^4}{|16^\beta - 1|} \right\}, \\ \beta_b(x) =& K\nu \left(\frac{4^{\alpha r}}{16^\beta}\right) \left\{ \frac{1 + K + \left(\frac{11}{2}\right)^\beta K^2 + 20^\beta K^3}{|16^\beta - 4^{\alpha r}|} \right\} \|x\|_X^r, \\ \gamma_b(x) =& K\nu \left(\frac{4^{\alpha s}}{16^\beta}\right) \left\{ \frac{2^{\alpha s} + K + \left(\frac{1}{2}\right)^\beta K^4}{|16^\beta - 4^{\alpha s}|} \right\} \|x\|_X^s, \\ \delta_b(x) =& K\nu \left(\frac{4^{\alpha t}}{16^\beta}\right) \left\{ \frac{1 + K + \left(\frac{11}{2}\right)^\beta K^2 + \left(\frac{1}{3}\right)^\beta K^4}{|16^\beta - 4^{\alpha t}|} \right\} \|x\|_X^t \qquad \text{and} \\ \zeta_b(x) =& \beta_b(x) + \gamma_b(x) + \delta_b(x) \quad \text{for all } x \in X. \end{aligned}$$

Corollary 3.4. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y,z)\|_Y \leq \begin{cases} \nu \left\{\|x\|_X^r \|y\|_X^s \|z\|_X^t\right\} \\ \nu \left\{\|x\|_X^r \|y\|_X^s \|z\|_X^t + \|x\|_X^\lambda + \|y\|_X^\lambda + \|z\|_X^\lambda\right\} \end{cases} \tag{3.28}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \rho_b(x), \\ \tau_b(x) \end{cases} \tag{3.29}$$

for all $x \in X$, where

$$\begin{aligned} \rho_b(x) =& K\nu \left(\frac{4^{\alpha\lambda}}{16^\beta}\right) \left\{ \frac{2^{\alpha s} + K}{|16^\beta - 4^{\alpha\lambda}|} \right\} \|x\|_X^\lambda \quad \text{and} \\ \tau_b(x) =& K\nu \left(\frac{4^{\alpha\lambda}}{16^\beta}\right) \left\{ \frac{2 + 2^{\alpha s} + 2^{\alpha\lambda} + 4K + 2\left(\frac{11}{2}\right)^\beta K^2 + 20^\beta K^3 + \left[\left(\frac{1}{2}\right)^\beta + \left(\frac{1}{3}\right)^\beta\right] K^4}{|16^\beta - 4^{\alpha\lambda}|} \right\} \|x\|_X^\lambda \end{aligned}$$

for all $x \in X$.

A counter example to illustrate the non stability in Condition (v) of Corollary 3.3.

Example 3.5. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\alpha(x) = \begin{cases} \mu x^2, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_b : \mathbb{R} \to \mathbb{R}$ by

$$f_b(x) = f(2x) - 16f(x) = \sum_{n=0}^{\infty} \frac{\alpha(4^n x)}{16^n} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f_b satisfies the functional inequality

$$|Df_b(x,y,z)| \leq \frac{676 \cdot 16^3}{15}\,\mu\left(|x|^2+|y|^2+|z|^2\right) \tag{3.30}$$

for all $x,y,z\in\mathbb{R}$. Then there do not exist a quadratic mapping $B:\mathbb{R}\to\mathbb{R}$ and a constant $c>0$ such that

$$|f_b(x)-B(x)| \leq c|x| \qquad for\ all \quad x\in\mathbb{R}. \tag{3.31}$$

Proof. Now

$$|f_b(x)| \leq \sum_{n=0}^{\infty}\frac{|\alpha(4^n x)|}{|16^n|} = \sum_{n=0}^{\infty}\frac{\mu}{16^n} = \frac{16}{15}\,\mu.$$

Therefore we see that f_b is bounded. We are going to prove that f_b satisfies (3.30)).

If $|x|^2+|y|^2+|z|^2=0$ or $|x|^2+|y|^2+|z|^2\geq\frac{1}{16}$ then

$$|Df_b(x,y,z)| \leq \frac{676\cdot 16}{15} \leq \frac{160\cdot 16^2}{15}(|x|^2+|y|^2+|z|^2). \tag{3.32}$$

Now suppose that $0<|x|^2+|y|^2+|z|^2<\dfrac{1}{16}$. Then there exists a non-negative integer k such that

$$\frac{1}{16^{k+2}} \leq |x|^2+|y|^2+|z|^2 < \frac{1}{16^{k+1}}, \tag{3.33}$$

so that $4^k x<\frac{1}{4}$, $4^k y<\frac{1}{4}$, $4^k z<\frac{1}{4}$ and consequently $4^{k-1}(3x\pm 2y\pm z), 4^{k-1}(x\pm y)$,

$$4^{k-1}(x\pm z), 4^{k-1}(2x), 4^{k-1}(x), 4^{k-1}(y), 4^{k-1}(z), 4^n(x), 4^n(y), 4^n(z)\in(-1,1).$$

Therefore for each $n=0,1,\ldots,k-1$, we have

$$4^n(3x\pm 2y\pm z), 4^n(x\pm y), 4^n(x\pm z), 4^n(2x), 4^n(x), 4^n(y), 4^n(z)\in(-1,1)$$

and $Df_b(4^n x,4^n y,4^n z)=0$. From the definition of f and (3.33), we obtain that

$$\begin{aligned}
\|Df_b(x,y,z)\| \leq{}& \sum_{n=0}^{\infty}\frac{1}{16^n}\Big|\alpha(4^n(3x+2y+z))+\alpha(4^n(3x+2y-z))+\alpha(4^n(3x-2y+z))\\
&+\alpha(4^n(3x-2y-z))-72[\alpha(4^n(x+y))+\alpha(4^n(x-y))]-18\alpha(4^n(x+z))\\
&-18\alpha(4^n(x-z))-8[\alpha(4^n(y+z))+\alpha(4^n(y-z))]-24\alpha(4^n(2x))\\
&-4\alpha(4^n(2y))+240\alpha(4^n(x))+160\alpha(4^n(y))+48\alpha(4^n(z))\Big|\\
\leq{}& \sum_{n=0}^{\infty}\frac{1}{16^n}\Big|\alpha(4^n(3x+2y+z))+\alpha(4^n(3x+2y-z))+\alpha(4^n(3x-2y+z))\\
&+\alpha(4^n(3x-2y-z))-72[\alpha(4^n(x+y))+\alpha(4^n(x-y))]-18\alpha(4^n(x+z))\\
&-18\alpha(4^n(x-z))-8[\alpha(4^n(y+z))+\alpha(4^n(y-z))]-24\alpha(4^n(2x))\\
&-4\alpha(4^n(2y))+240\alpha(4^n(x))+160\alpha(4^n(y))+48\alpha(4^n(z))\Big|\\
\leq{}& \sum_{n=k}^{\infty}\frac{1}{16^n}676\mu = 676\,\mu\times\frac{16}{15\cdot 16^k} = \frac{676\cdot 16^3}{15}\,\mu\left(|x|^2+|y|^2+|z|^2\right).
\end{aligned}$$

Thus f_b satisfies (3.30) for all $x,y,z\in\mathbb{R}$ with $0<|x|^2+|y|^2+|z|^2<\dfrac{1}{16}$.

We claim that the functional equation (1.4) is not stable for $r=s=t=2$ in condition (v) Corollary 3.3 ($\alpha=\beta=p=1$). Suppose on the contrary that there exist a quadratic mapping $B:\mathbb{R}\to\mathbb{R}$ and a constant $\eta>0$ satisfying (3.31). Since f_b is bounded and continuous for all $x\in\mathbb{R}$, B is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.2, B must have the form $B(x)=cx^2$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f_b(x)| \leq (\eta+|c|)\,|x|^2. \tag{3.34}$$

But we can choose a positive integer m with $m\mu>\eta+|c|$.

If $x\in\left(0,\frac{1}{4^{m-1}}\right)$, then $4^n x\in(0,1)$ for all $n=0,1,\ldots,m-1$. For this x, we get

$$f_b(x)=\sum_{n=0}^{\infty}\frac{\alpha(4^n x)}{16^n} \geq \sum_{n=0}^{m-1}\frac{\mu(4^n x)^2}{16^n} = m\mu x^2 > (\eta+|c|)\,x^2$$

which contradicts (3.34). □

4 Stability of Eq. (1.4): Quartic Case

Theorem 4.1. *Let $j \in \{-1, 1\}$ be fixed, and let $\psi_d : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi_d(4^j x, 4^j y, 4^j z) \leq 256^{j\beta} L \psi_d(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x, y, z)\|_Y \leq \psi_d(x, y, z) \tag{4.1}$$

for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{1}{256^\beta \mid 1 - L^j \mid} \tilde{\psi_d}(x) \tag{4.2}$$

for all $x \in X$, where

$$\tilde{\psi_d}(x) = K[\psi_d(x, 2x, x) + K\psi_d(x, x, x) + \left(\frac{1}{2}\right)^\beta K^2 \psi_d(x, 0, x) + 20^\beta K^3 \psi_d(x, 0, 0) + \left(\frac{1}{2}\right)^\beta K^4 \psi_d(0, x, 0) + \left(\frac{5}{3}\right)^\beta K^4 \psi_d(0, 0, x)] \quad \text{for all } x \in X$$

Proof. Similar to the proof of Theorem 3.2, we have

$$\|f(8x) - 4f(4x) - 256f(2x) + 1024f(x)\|_Y \leq \tilde{\psi_d}(x) \tag{4.3}$$

for all $x \in X$, where

$$\tilde{\psi_d}(x) = K[\psi_d(x, 2x, x) + K\psi_d(x, x, x) + \left(\frac{1}{2}\right)^\beta K^2 \psi_d(x, 0, x) + 20^\beta K^3 \psi_d(x, 0, 0) + \left(\frac{1}{2}\right)^\beta K^4 \psi_d(0, x, 0) + \left(\frac{5}{3}\right)^\beta K^4 \psi_d(0, 0, x)] \quad \text{for all } x \in X$$

Let $h : X \to Y$ be a mapping defined by $h(x) = f(2x) - 4f(x)$, then (4.3) means

$$\|h(4x) - 256h(x)\|_Y \leq \tilde{\psi_d}(x) \tag{4.4}$$

for all $x \in X$. By Lemma 3.1, there exists a unique mapping $D : X \to Y$ such that

$$D(4x) = 256D(x) \tag{4.5}$$

and

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{1}{256^\beta \mid 1 - L^j \mid} \tilde{\psi_b}(x) \tag{4.6}$$

for all $x \in X$. It remains to show that D is a quartic map. By (4.1), we have

$$\begin{aligned} \left\| \frac{1}{256^{nj}} Df(4^{nj}x, 4^{nj}y, 4^{nj}z) \right\|_Y &\leq \frac{1}{256^{nj\beta}} \psi_d(4^{nj}x, 4^{nj}y, 4^{nj}z) \\ &\leq \frac{1}{256^{nj\beta}} (256^{j\beta}L)^n \psi_d(x, y, z) = L^n \psi_d(x, y, z) \end{aligned} \tag{4.7}$$

for all $x, y, z \in X$ and all positive integers n. so

$$\|DD(x, y, z)\|_Y = 0 \tag{4.8}$$

for all $x, y, z \in X$. Thus the mapping $D : X \to Y$ is quartic, as desired □

The following corollaries are immediate consequence of Theorem 4.1.

Corollary 4.2. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that r, s and t are all $\neq \frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.26) *for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \begin{cases} \alpha_d, & \\ \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases} \tag{4.9}$$

for all $x \in X$, *where*

$$
\begin{aligned}
\alpha_d =&K\nu \left\{ \frac{1+K+\left(\frac{1}{2}\right)^{\beta}K^2+20^{\beta}K^3+\left[\left(\frac{1}{2}\right)^{\beta}+\left(\frac{5}{3}\right)^{\beta}\right]K^4}{|256^{\beta}-1|} \right\},\\
\beta_d(x) =&K\nu \left(\frac{4^{\alpha r}}{256^{\beta}}\right)\left\{ \frac{1+K+\left(\frac{1}{2}\right)^{\beta}K^2+20^{\beta}K^3}{|256^{\beta}-4^{\alpha r}|}\right\}\|x\|_X^r,\\
\gamma_d(x) =&K\nu \left(\frac{4^{\alpha s}}{256^{\beta}}\right)\left\{ \frac{2^{\alpha s}+K+\left(\frac{1}{2}\right)^{\beta}K^4}{|256^{\beta}-4^{\alpha s}|}\right\}\|x\|_X^s,\\
\delta_d(x) =&K\nu \left(\frac{4^{\alpha t}}{256^{\beta}}\right)\left\{ \frac{1+K+\left(\frac{1}{2}\right)^{\beta}K^2+\left(\frac{5}{3}\right)^{\beta}K^4}{|256^{\beta}-4^{\alpha t}|}\right\}\|x\|_X^t \quad \text{and}\\
\zeta_d(x) =&\beta_d(x)+\gamma_d(x)+\delta_d(x) \qquad \text{for all } x\in X.
\end{aligned}
$$

Corollary 4.3. *Let* $\nu \geq 0$ *and* r, s, t *which are all* > 0 *be real numbers such that* $\lambda = r+s+t \neq \frac{4\beta}{\alpha}$. *Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (3.28) *for all* $x, y, z \in X$. *Then there exists a unique quartic mapping* $D : X \to Y$ *such that*

$$\|f(2x)-4f(x)-D(x)\|_Y \leq \left\{ \begin{array}{l} \rho_d(x), \\ \tau_d(x) \end{array}\right. \tag{4.10}$$

for all $x \in X$, *where*

$$
\begin{aligned}
\rho_d(x) =&K\nu \left(\frac{4^{\alpha\lambda}}{256^{\beta}}\right)\left\{ \frac{2^{\alpha s}+K}{|256^{\beta}-4^{\alpha\lambda}|}\right\}\|x\|_X^{\lambda} \quad \text{and}\\
\tau_d(x) =&K\nu \left(\frac{4^{\alpha\lambda}}{256^{\beta}}\right)\left\{ \frac{2+2^{\alpha s}+2^{\alpha\lambda}+4K+2\left(\frac{1}{2}\right)^{\beta}K^2+20^{\beta}K^3+\left[\left(\frac{1}{2}\right)^{\beta}+\left(\frac{5}{3}\right)^{\beta}\right]K^4}{|256^{\beta}-4^{\alpha\lambda}|}\right\}\|x\|_X^{\lambda}
\end{aligned}
$$

for all $x \in X$.

A counter example to illustrate the non stability in Condition (v) of Corollary 4.2.

Example 4.4. *Let* $\alpha : \mathbb{R} \to \mathbb{R}$ *be a function defined by*

$$\alpha(x) = \left\{ \begin{array}{ll} \mu x^4, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{array}\right.$$

where $\mu > 0$ *is a constant, and define a function* $f_d : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$f_d(x) = \sum_{n=0}^{\infty} \frac{\alpha(4^n x)}{256^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_d *satisfies the functional inequality*

$$|Df_d(x,y,z)| \leq \frac{676\cdot 256^3}{255}\mu\left(|x|^4+|y|^4+|z|^4\right) \tag{4.11}$$

for all $x, y, z, w \in \mathbb{R}$. *Then there do not exist a quartic mapping* $D : \mathbb{R} \to \mathbb{R}$ *and a constant* $c > 0$ *such that*

$$|f_d(x) - D(x)| \leq c|x|^4 \qquad for\ all \quad x \in \mathbb{R}. \tag{4.12}$$

Proof. Now

$$|f_d(x)| \leq \sum_{n=0}^{\infty}\frac{|\alpha(4^n x)|}{|256^n|} = \sum_{n=0}^{\infty}\frac{\mu}{256^n} = \frac{256}{255}\mu.$$

Therefore we see that f_d is bounded. We are going to prove that f_d satisfies (4.11)).

If $|x|^4+|y|^4+|z|^4 = 0$ or $|x|^4+|y|^4+|z|^4 \geq \frac{1}{256}$ then

$$|Df_d(x,y,z)| \leq \frac{676\cdot 256}{255} \leq \frac{160\cdot 256^2}{63}(|x|^4+|y|^4+|z|^4). \tag{4.13}$$

Now suppose that $0 < |x|^4+|y|^4+|z|^4 < \frac{1}{256}$. Then there exists a non-negative integer k such that

$$\frac{1}{256^{k+2}} \leq |x|^4+|y|^4+|z|^4 < \frac{1}{256^{k+1}}, \tag{4.14}$$

so that $4^k|x| < \frac{1}{4}$, $4^k|y| < \frac{1}{4}$, $4^k|z| < \frac{1}{4}$ and consequently $4^{k-1}(3x \pm 2y \pm z), 4^{k-1}(x \pm y),$

$$4^{k-1}(x \pm z), 4^{k-1}(2x), 4^{k-1}(x), 4^{k-1}(y), 4^{k-1}(z), 4^n(x), 4^n(y), 4^n(z) \in (-1,1).$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$4^n(3x \pm 2y \pm z), 4^n(x \pm y), 4^n(x \pm z), 4^n(2x), 4^n(x), 4^n(y), 4^n(z) \in (-1,1)$$

and $Df_d(4^n x, 4^n y, 4^n z) = 0$. From the definition of f_d and (4.14), we obtain that

$$\begin{aligned}\|Df_d(x,y,z)\| \le \sum_{n=0}^{\infty} \frac{1}{256^n} \Big| & \alpha(4^n(3x+2y+z)) + \alpha(4^n(3x+2y-z)) + \alpha(4^n(3x-2y+z)) \\ & + \alpha(4^n(3x-2y-z)) - 72[\alpha(4^n(x+y)) + \alpha(4^n(x-y))] - 18\alpha(4^n(x+z)) \\ & - 18\alpha(4^n(x-z)) - 8[\alpha(4^n(y+z)) + \alpha(4^n(y-z))] - 24\alpha(4^n(2x)) \\ & - 4\alpha(4^n(2y)) + 240\alpha(4^n(x)) + 160\alpha(4^n(y)) + 48\alpha(4^n(z)) \Big|\end{aligned}$$

$$\begin{aligned}\le \sum_{n=0}^{\infty} \frac{1}{256^n} \Big| & \alpha(4^n(3x+2y+z)) + \alpha(4^n(3x+2y-z)) + \alpha(4^n(3x-2y+z)) \\ & + \alpha(4^n(3x-2y-z)) - 72[\alpha(4^n(x+y)) + \alpha(4^n(x-y))] - 18\alpha(4^n(x+z)) \\ & - 18\alpha(4^n(x-z)) - 8[\alpha(4^n(y+z)) + \alpha(4^n(y-z))] - 24\alpha(4^n(2x)) \\ & - 4\alpha(4^n(2y)) + 240\alpha(4^n(x)) + 160\alpha(4^n(y)) + 48\alpha(4^n(z)) \Big| \\ \le \sum_{n=k}^{\infty} \frac{1}{256^n} 676\mu = 676\, \mu \times & \frac{16}{15 \cdot 16^k} = \frac{676 \cdot 256^3}{255}\, \mu \left(|x|^4 + |y|^4 + |z|^4\right).\end{aligned}$$

Thus f_d satisfies (4.11) for all $x, y, z \in \mathbb{R}$ with $0 < |x|^4 + |y|^4 + |z|^4 < \frac{1}{4}$.

We claim that the functional equation (1.4) is not stable for $r = s = t = 4$ in condition (v) Corollary 4.2 $(\alpha = \beta = p = 1)$. Suppose on the contrary that there exist a quartic mapping $D : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ satisfying (4.12). Since f_d is bounded and continuous for all $x \in \mathbb{R}$, D is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 4.1, D must have the form $D(x) = cx^4$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f_d(x)| \le (\eta + |c|)\, |x|^4. \tag{4.15}$$

But we can choose a positive integer m with $m\mu > \eta + |c|$.

If $x \in \left(0, \frac{1}{4^{m-1}}\right)$, then $4^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_d(x) = \sum_{n=0}^{\infty} \frac{\alpha(4^n x)}{256^n} \ge \sum_{n=0}^{m-1} \frac{\mu(4^n x)^4}{256^n} = m\mu x^4 > (\eta + |c|)\, x^4$$

which contradicts (4.15). □

5 Stability of Eq. (1.4): Mixed Case

Theorem 5.1. *Let $j \in \{-1, 1\}$ be fixed, and let $\psi : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi(4^j x, 4^j y, 4^j z) \le 16^{j\beta} L \psi(x,y,z)$ and $\psi(4^j x, 4^j y, 4^j z) \le 256^{j\beta} L \psi(x,y,z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y,z)\|_Y \le \psi(x,y,z) \tag{5.1}$$

for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that

$$\|f(x) - B(x) - D(x)\|_Y \le \frac{K}{12 \cdot 256^\beta \mid 1 - L^j \mid} [16^\beta \tilde{\psi_b}(x) + \tilde{\psi_d}(x)] \tag{5.2}$$

for all $x \in X$, where $\tilde{\psi_b}(x)$ and $\tilde{\psi_d}(x)$ are defined as in Theorems 3.2 and 4.1 respectively.

Proof. Since $\psi(4^jx,4^jy,4^jz)\le 16^{j\beta}L\psi(x,y,z)$ and $\psi(4^jx,4^jy,4^jz)\le 256^{j\beta}L\psi(x,y,z)$ for all $x,y,z\in X$, by Theorems 3.2 and 4.1, there exist a quadratic mapping $B_0: X\to Y$ and a quartic mapping $D_0: X\to Y$ such that

$$\|f(2x)-16f(x)-B_0(x)\|_Y\le\frac{1}{16^\beta\mid 1-L^j\mid}\tilde{\psi}_b(x)\qquad\text{and}$$
$$\|f(2x)-4f(x)-D_0(x)\|_Y\le\frac{1}{256^\beta\mid 1-L^j\mid}\tilde{\psi}_b(x)$$

for all $x\in X$. Therefore the result follows from the last two inequalities. □

Theorem 5.2. *Let $j\in\{-1,1\}$ be fixed, and let $\psi: X^3\to[0,\infty)$ be a mapping such that there exists an $L<1$ with $\psi(4^jx,4^jy,4^jz)\le 16^{j\beta}L\psi(x,y,z)$ and $\psi(4^jx,4^jy,4^jz)\le 256^{-j\beta}L\psi(x,y,z)$ for all $x,y,z\in X$. Let a mapping $f: X\to Y$ with $f(0)=0$ satisfies the inequality (5.1) for all $x,y,z\in X$. Then there exists a quadratic mapping $B: X\to Y$ and a quartic mapping $D: X\to Y$ such that*

$$\|f(x)-B(x)-D(x)\|_Y\le\frac{K}{12\cdot 256^\beta\mid 1-L^j\mid}[16^\beta\tilde{\psi}_b(x)+\tilde{\psi}_d(x)]\tag{5.3}$$

for all $x\in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ are defined as in Theorems 3.2 and 4.1 respectively.

Proof. Since $\psi(4^jx,4^jy,4^jz)\le 16^{j\beta}L\psi(x,y,z)$ and $\psi(4^jx,4^jy,4^jz)\le 256^{-j\beta}L\psi(x,y,z)$ for all $x,y,z\in X$, by Theorems 3.2 and 4.1, there exist a quadratic mapping $B_0: X\to Y$ and a quartic mapping $D_0: X\to Y$ such that

$$\|f(2x)-16f(x)-B_0(x)\|_Y\le\frac{1}{16^\beta\mid 1-L^j\mid}\tilde{\psi}_b(x)\qquad\text{and}$$
$$\|f(2x)-4f(x)-D_0(x)\|_Y\le\frac{1}{256^\beta\mid 1-L^j\mid}\tilde{\psi}_b(x)$$

for all $x\in X$. Therefore the result follows from the last two inequalities. □

Corollary 5.3. *Let $\nu\ge 0$ and r,s,t which are all >0 be real numbers such that r,s and t are all $\ne\frac{2\beta}{\alpha}$ and $\frac{4\beta}{\alpha}$. Suppose that a mapping $f: X\to Y$ with $f(0)=0$ satisfies the inequality (3.26) for all $x,y,z\in X$. Then there exist a quadratic mapping $B: X\to Y$ and a quartic mapping $D: X\to Y$ such that*

$$\|f(x)-B(x)-D(x)\|_Y\le\frac{K}{12}\begin{cases}\alpha_b+\alpha_d, & \\ \beta_b(x)+\beta_d(x), & r>0,s=0,t=0;\\ \gamma_b(x)+\gamma_d(x), & r=0,s>0,t=0;\\ \delta_b(x)+\delta_d(x), & r=0,s=0,t>0;\\ \zeta_b(x)+\zeta_d(x), & r>0,s>0,t>0;\end{cases}$$

for all $x\in X$, where $\alpha_b,\alpha_d,\beta_b(x),\beta_d(x),\gamma_b(x),\gamma_d(x),\delta_b(x),\delta_d(x),\zeta_b(x)$ and $\zeta_d(x)$ are defined as in Corollaries 3.3 and 4.2

Corollary 5.4. *Let $\nu\ge 0$ and r,s,t which are all >0 be real numbers such that $\lambda=r+s+t\ne\frac{2\beta}{\alpha}$ and $\frac{4\beta}{\alpha}$. Suppose that a mapping $f: X\to Y$ with $f(0)=0$ satisfies the inequality (3.28) for all $x,y,z\in X$. Then there exist a quadratic mapping $B: X\to Y$ and a quartic mapping $D: X\to Y$ such that*

$$\|f(x)-B(x)-D(x)\|_Y\le\frac{K}{12}\begin{cases}\rho_b(x)+\rho_d(x),\\ \tau_b(x)+\tau_d(x)\end{cases}\tag{5.4}$$

for all $x\in X$, where $\rho_b(x),\rho_d(x),\tau_b(x),\tau_d(x)$ are defined as in Corollaries 3.4 and 4.3

Example 5.5. *Let $\alpha:\mathbb{R}\to\mathbb{R}$ be a function defined by*

$$\alpha(x)=\begin{cases}\mu(x^2+x^4), & \text{if } |x|<1\\ \mu, & \text{otherwise}\end{cases}$$

where $\mu>0$ is a constant, and define a function $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\sum_{n=0}^{\infty}\frac{\alpha(4^nx)}{4^n}\qquad for\ all\quad x\in\mathbb{R}.$$

Then f satisfies the functional inequality

$$|Df(x,y,z)| \leq \frac{1352\cdot 16^3}{15}\,\mu\left(|x|^2+|y|^2+|z|^2\right) \tag{5.5}$$

for all $x,y,z,w \in \mathbb{R}$. Then there do not exist a quadratic mapping $B:\mathbb{R}\to\mathbb{R}$ and a quartic mapping $D:\mathbb{R}\to\mathbb{R}$ and a constant $\eta>0$ such that

$$|f(x)-B(x)-D(x)| \leq \eta|x| \qquad for\ all \quad x\in\mathbb{R}. \tag{5.6}$$

6 Stability of Eq. (1.4) using various substitutions

In this section, the generalized Hyers-Ulam stability of (1.4) using various substitutions is investigated. The proofs of the following theorems and corollaries are similar to that of the Theorems 3.2, 4.1,and 5.1 and the corollaries 3.3, 3.4, 4.2, 4.3,5.3 and 5.4. Hence the details of the proofs are omitted.

Theorem 6.1. *Let $j\in\{-1,1\}$ be fixed, and let $\psi_b : X^3\to[0,\infty)$ be a mapping such that there exists an $L<1$ with $\psi_b(3^jx,3^jy,3^jz)\leq 9^{j\beta}L\psi_b(x,y,z)$for all $x,y,z\in X$. Let a mapping $f:X\to Y$ with $f(0)=0$ satisfies the inequality* (3.12) *for all $x,y,z\in X$. Then there exists a unique quadratic mapping $B:X\to Y$ such that*

$$\|f(2x)-16f(x)-B(x)\|_Y \leq \frac{1}{9^\beta\mid 1-L^j\mid}\tilde{\psi_b}(x) \tag{6.1}$$

for all $x\in X$, where

$$\tilde{\psi_b}(x) = K[M_b(x,x,x)+K\left(\frac{1}{2}\right)M_b(x,0,x)+K^2 4^\beta M_b(x,0,0)+K^2\left(\frac{1}{6}\right)^\beta M_b(0,0,x)]$$

for all $x\in X$.

Corollary 6.2. *Let $\nu\geq 0$ and r,s,t which are all >0 be real numbers such that r,s and t are all $\neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f:X\to Y$ with $f(0)=0$ satisfies the inequality* (3.26) *for all $x,y,z\in X$. Then there exists a unique quadratic mapping $B:X\to Y$ such that*

$$\|f(2x)-16f(x)-B(x)\|_Y \leq \begin{cases} \alpha_b, & \\ \beta_b(x), & r>0, s=0, t=0; \\ \gamma_b(x), & r=0, s>0, t=0; \\ \delta_b(x), & r=0, s=0, t>0; \\ \zeta_b(x), & r>0, s>0, t>0; \end{cases}$$

for all $x\in X$, where

$$\begin{aligned} \alpha_b =& K\nu\left\{\frac{1+\left(\frac{1}{2}\right)^\beta K+K^2\left[4^\beta+\left(\frac{1}{6}\right)^\beta\right]}{|9^\beta-1|}\right\}, \\ \beta_b(x) =& K\nu\left(\frac{3^{\alpha r}}{9^\beta}\right)\left\{\frac{1+\left(\frac{1}{2}\right)^\beta K+4^\beta K^2}{|9^\beta-3^{\alpha r}|}\right\}\|x\|_X^r, \\ \gamma_b(x) =& K\nu\left(\frac{3^{\alpha s}}{9^\beta}\right)\left\{\frac{1}{|9^\beta-3^{\alpha s}|}\right\}\|x\|_X^s, \\ \delta_b(x) =& K\nu\left(\frac{3^{\alpha t}}{9^\beta}\right)\left\{\frac{1+\left(\frac{1}{2}\right)^\beta K+\left(\frac{1}{6}\right)^\beta K^2}{|9^\beta-3^{\alpha t}|}\right\}\|x\|_X^t \qquad \text{and} \\ \zeta_b(x) =& \beta_b(x)+\gamma_b(x)+\delta_b(x) \quad \text{for all } x\in X. \end{aligned}$$

Corollary 6.3. *Let $\nu\geq 0$ and r,s,t which are all >0 be real numbers such that $\lambda=r+s+t\neq\frac{2\beta}{\alpha}$. Suppose that a mapping $f:X\to Y$ with $f(0)=0$ satisfies the inequality* (3.28) *for all $x,y,z\in X$. Then there exists a unique quadratic mapping $B:X\to Y$ such that*

$$\|f(2x)-16f(x)-B(x)\|_Y \leq \begin{cases} \rho_b(x), \\ \tau_b(x) \end{cases}$$

for all $x\in X$, where

$$\begin{aligned} \rho_b(x) =& K\nu\left(\frac{3^{\alpha\lambda}}{9^\beta}\right)\left\{\frac{1}{|9^\beta-3^{\alpha\lambda}|}\right\}\|x\|_X^\lambda \qquad \text{and} \\ \tau_b(x) =& K\nu\left(\frac{3^{\alpha\lambda}}{9^\beta}\right)\left\{\frac{4+2\left(\frac{1}{2}\right)^\beta K+K^2\left[4^\beta+\left(\frac{1}{6}\right)^\beta\right]}{|9^\beta-3^{\alpha\lambda}|}\right\}\|x\|_X^\lambda \qquad \text{for all } x\in X. \end{aligned}$$

Theorem 6.4. *Let $j \in \{-1,1\}$ be fixed, and let $\psi_d : X^3 \to [0,\infty)$ be a mapping such that there exists an $L < 1$ with $\psi_d(3^j x, 3^j y, 3^j z) \leq 81^{j\beta} L\psi_d(x,y,z)$for all $x,y,z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (4.1) for all $x,y,z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{1}{81^\beta \mid 1 - L^j \mid} \tilde{\psi}_d(x) \tag{6.2}$$

for all $x \in X$, where

$$\tilde{\psi}_d(x) = K[M_d(x,x,x) + \left(\frac{1}{2}\right)^\beta M_d(x,0,x)K + M_d(x,0,0)K^2 + \left(\frac{1}{6}\right)^\beta M_d(0,0,x)K^2]$$

for all $x \in X$.

Corollary 6.5. *Let $\nu \geq 0$ and r,s,t which are all > 0 be real numbers such that r, s and t are all $\neq \frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.26) for all $x,y,z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \begin{cases} \alpha_d, & \\ \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases} \tag{6.3}$$

for all $x \in X$, where

$$\begin{aligned} \alpha_d &= K\nu \left\{ \frac{1 + \left(\frac{1}{2}\right)^\beta K + \left[1 + \left(\frac{1}{6}\right)^\beta\right] K^2}{|81^\beta - 1|} \right\}, \\ \beta_d(x) &= K\nu \left(\frac{3^{\alpha r}}{81^\beta}\right) \left\{ \frac{1 + \left(\frac{1}{2}\right)^\beta K + K^2}{|81^\beta - 3^{\alpha r}|} \right\} \|x\|_X^r, \\ \gamma_d(x) &= K\nu \left(\frac{3^{\alpha s}}{81^\beta}\right) \left\{ \frac{1}{|81^\beta - 3^{\alpha s}|} \right\} \|x\|_X^s, \\ \delta_d(x) &= K\nu \left(\frac{3^{\alpha t}}{81^\beta}\right) \left\{ \frac{1 + \left(\frac{1}{2}\right)^\beta K + \left(\frac{1}{6}\right)^\beta K^2}{|81^\beta - 3^{\alpha t}|} \right\} \|x\|_X^t \quad \text{and} \\ \zeta_d(x) &= \beta_d(x) + \gamma_d(x) + \delta_d(x) \quad \text{for all } x \in X. \end{aligned}$$

Corollary 6.6. *Let $\nu \geq 0$ and r, s and t which are all > 0 be real numbers such that $\lambda = r + s + t \neq \frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.28) for all $x,y,z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - B(x)\|_Y \leq \begin{cases} \rho_d(x), \\ \tau_d(x) \end{cases} \tag{6.4}$$

for all $x \in X$, where

$$\begin{aligned} \rho_d(x) &= K\nu \left(\frac{3^{\alpha\lambda}}{81^\beta}\right) \left\{ \frac{1}{|81^\beta - 3^{\alpha\lambda}|} \right\} \|x\|_X^\lambda \quad \text{and} \\ \tau_d(x) &= K\nu \left(\frac{3^{\alpha\lambda}}{81^\beta}\right) \left\{ \frac{4 + 2\left(\frac{1}{2}\right)^\beta K + K^2 \left[1 + \left(\frac{1}{6}\right)^\beta\right]}{|81^\beta - 3^{\alpha\lambda}|} \right\} \|x\|_X^\lambda \quad \text{for all } x \in X. \end{aligned}$$

Theorem 6.7. *Let $j \in \{-1,1\}$ be fixed, and let $\psi : X^3 \to [0,\infty)$ be a mapping such that there exists an $L < 1$ with $\psi(3^j x, 3^j y, 3^j z) \leq 9^{j\beta} L\psi(x,y,z)$ and $\psi(3^j x, 3^j y, 3^j z) \leq 81^{j\beta} L\psi(x,y,z)$ for all $x,y,z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (5.1) for all $x,y,z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12 \cdot 81^\beta \mid 1 - L^j \mid} [9^\beta \tilde{\psi}_b(x) + \tilde{\psi}_d(x)] \tag{6.5}$$

for all $x \in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ are defined as in Theorems 6.1 and 6.4 respectively.

Theorem 6.8. *Let $j \in \{-1, 1\}$ be fixed, and let $\psi : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi(3^j x, 3^j y, 3^j z) \leq 9^{j\beta} L\psi(x, y, z)$ and $\psi(3^j x, 3^j y, 3^j z) \leq 81^{-j\beta} L\psi(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (5.1) *for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12 \cdot 81^\beta \mid 1 - L^j \mid}[9^\beta \tilde{\psi_b}(x) + \tilde{\psi_d}(x)] \tag{6.6}$$

for all $x \in X$, where $\tilde{\psi_b}(x)$ and $\tilde{\psi_d}(x)$ are defined as in Theorems 6.1 and 6.4 respectively.

Corollary 6.9. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that r, s and t are all $\neq \frac{2\beta}{\alpha}$ and $\frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.26) *for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12}\begin{cases} \alpha_b + \alpha_d, & \\ \beta_b(x) + \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_b(x) + \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_b(x) + \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_b(x) + \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases}$$

for all $x \in X$, where $\alpha_b, \alpha_d, \beta_b(x), \beta_d(x), \gamma_b(x), \gamma_d(x), \delta_b(x), \delta_d(x), \zeta_b(x)$ and $\zeta_d(x)$ are defined as in Corollaries 6.2 and 6.5

Corollary 6.10. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq \frac{2\beta}{\alpha}$ and $\frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.28) *for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12}\begin{cases} \rho_b(x) + \rho_d(x), \\ \tau_b(x) + \tau_d(x) \end{cases} \tag{6.7}$$

for all $x \in X$, where $\rho_b(x), \rho_d(x), \tau_b(x), \tau_d(x)$ are defined as in Corollaries 6.3 and 6.6

Theorem 6.11. *Let $j \in \{-1, 1\}$ be fixed, and let $\psi_b : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi_b(2^j x, 2^j y, 2^j z) \leq 4^{j\beta} L\psi_b(x, y, z)$for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.12) *for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that*

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{1}{4^\beta \mid 1 - L^j \mid}\tilde{\psi_b}(x) \tag{6.8}$$

for all $x \in X$, where

$$\tilde{\psi_b}(x) = K[M_b(x, x, x) + \left(\frac{1}{4}\right)^\beta KM_b(2x, 0, 0) + 3^\beta K^2 M_b(x, 0, x) + K^2 M_b(0, 0, x)] \quad \text{for all } x \in X.$$

Corollary 6.12. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that r, s and t are all $\neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.26) *for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that*

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \alpha_b, & \\ \beta_b(x), & r > 0, s = 0, t = 0; \\ \gamma_b(x), & r = 0, s > 0, t = 0; \\ \delta_b(x), & r = 0, s = 0, t > 0; \\ \zeta_b(x), & r > 0, s > 0, t > 0; \end{cases}$$

for all $x \in X$, where

$$\alpha_b = K\nu\left\{\frac{1 + \left(\frac{1}{4}\right)^\beta K + 3^\beta K^2 + K^2}{|4^\beta - 1|}\right\},$$

$$\beta_b(x) = K\nu\left(\frac{2^{\alpha r}}{4^\beta}\right)\left\{\frac{1 + \left(\frac{1}{4}\right)^\beta 2^{\alpha r} K + 3^\beta K^2}{|4^\beta - 2^{\alpha r}|}\right\}\|x\|_X^r,$$

$$\gamma_b(x) = K\nu\left(\frac{2^{\alpha s}}{4^\beta}\right)\left\{\frac{1}{|4^\beta - 2^{\alpha s}|}\right\}\|x\|_X^s,$$

$$\delta_b(x) = K\nu\left(\frac{2^{\alpha t}}{4^\beta}\right)\left\{\frac{1 + 3^\beta K^2 + K^2}{|4^\beta - 2^{\alpha t}|}\right\}\|x\|_X^t \qquad \text{and}$$

$$\zeta_b(x) = \beta_b(x) + \gamma_b(x) + \delta_b(x) \quad \text{for all } x \in X.$$

Corollary 6.13. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.28) *for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that*

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \rho_b(x), \\ \tau_b(x) \end{cases}$$

for all $x \in X$, where

$$\rho_b(x) = K\nu\left(\frac{2^{\alpha\lambda}}{4^\beta}\right)\left\{\frac{1}{|4^\beta - 2^{\alpha\lambda}|}\right\}\|x\|_X^\lambda \quad \text{and}$$
$$\tau_b(x) = K\nu\left(\frac{2^{\alpha\lambda}}{4^\beta}\right)\left\{\frac{4 + 2^{\alpha\lambda}\left(\frac{1}{4}\right)^\beta K + 2\cdot 3^\beta K^2 + K^2}{|4^\beta - 2^{\alpha\lambda}|}\right\}\|x\|_X^\lambda$$

for all $x \in X$.

Theorem 6.14. *Let $j \in \{-1, 1\}$ be fixed, and let $\psi_d : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi_d(2^j x, 2^j y, 2^j z) \leq 16^{j\beta} L\psi_d(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (4.1) *for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{1}{16^\beta \mid 1 - L^j \mid}\tilde{\psi}_d(x) \tag{6.9}$$

for all $x \in X$, where

$$\tilde{\psi}_d(x) = K[M_d(x, x, x) + \left(\frac{1}{4}\right)^\beta KM_d(2x, 0, 0) + 3^\beta K^2 M_d(x, 0, x) + K^2 M_d(0, 0, x)] \quad \text{for all } x \in X.$$

Corollary 6.15. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that r, s and t are all $\neq \frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.26) *for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \begin{cases} \alpha_d, \\ \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases} \tag{6.10}$$

for all $x \in X$, where

$$\alpha_d = K\nu\left\{\frac{1 + \left(\frac{1}{4}\right)^\beta K + 3^\beta K^2 + K^2)}{|16^\beta - 1|}\right\},$$
$$\beta_d(x) = K\nu\left(\frac{2^{\alpha r}}{16^\beta}\right)\left\{\frac{1 + 2^{\alpha r}\left(\frac{1}{4}\right)^\beta K + 3^\beta K^2 + K^2)}{|16^\beta - 2^{\alpha r}|}\right\}\|x\|_X^r,$$
$$\gamma_d(x) = K\nu\left(\frac{2^{\alpha s}}{16^\beta}\right)\left\{\frac{1}{|16^\beta - 2^{\alpha s}|}\right\}\|x\|_X^s,$$
$$\delta_d(x) = K\nu\left(\frac{2^{\alpha t}}{16^\beta}\right)\left\{\frac{1 + 3^\beta K^2 + K^2)}{|16^\beta - 2^{\alpha t}|}\right\}\|x\|_X^t \quad \text{and}$$
$$\zeta_d(x) = \beta_d(x) + \gamma_d(x) + \delta_d(x) \quad \text{for all } x \in X.$$

Corollary 6.16. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq \frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.28) *for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that*

$$\|f(2x) - 4f(x) - D(x)\|_Y \leq \begin{cases} \rho_d(x), \\ \tau_d(x) \end{cases} \tag{6.11}$$

for all $x \in X$, where

$$\rho_d(x) = K\nu\left(\frac{2^{\alpha\lambda}}{16^\beta}\right)\left\{\frac{1}{|16^\beta - 2^{\alpha\lambda}|}\right\}\|x\|_X^\lambda \quad \text{and}$$
$$\tau_d(x) = K\nu\left(\frac{2^{\alpha\lambda}}{16^\beta}\right)\left\{\frac{4 + 2^{\alpha\lambda}\left(\frac{1}{4}\right)^\beta K + 2\cdot 3^\beta K^2 + K^2}{|16^\beta - 2^{\alpha\lambda}|}\right\}\|x\|_X^\lambda \quad \text{for all } x \in X.$$

Theorem 6.17. *Let $j \in \{-1, 1\}$ be fixed, and let $\psi : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi(2^j x, 2^j y, 2^j z) \le 4^{j\beta} L\psi(x, y, z)$ and $\psi(2^j x, 2^j y, 2^j z) \le 16^{j\beta} L\psi(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (5.1) *for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \le \frac{K}{12 \cdot 16^\beta \mid 1 - L^j \mid}[4^\beta \tilde{\psi}_b(x) + \tilde{\psi}_d(x)] \tag{6.12}$$

for all $x \in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ are defined as in Theorems 6.11 and 6.14 respectively.

Theorem 6.18. *Let $j \in \{-1, 1\}$ be fixed, and let $\psi : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi(2^j x, 2^j y, 2^j z) \le 4^{j\beta} L\psi(x, y, z)$ and $\psi(2^j x, 2^j y, 2^j z) \le 8^{-j\beta} L\psi(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (5.1) *for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \le \frac{K}{12 \cdot 16^\beta \mid 1 - L^j \mid}[4^\beta \tilde{\psi}_b(x) + \tilde{\psi}_d(x)] \tag{6.13}$$

for all $x \in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ are defined as in Theorems 6.11 and 6.14 respectively.

Corollary 6.19. *Let $\nu \ge 0$ and r, s, t which are all > 0 be real numbers such that r, s and t are all $\ne \frac{2\beta}{\alpha}$ and $\frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.26) *for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that*

$$\|f(x) - B(x) - D(x)\|_Y \le \frac{K}{12}\begin{cases} \alpha_b + \alpha_d, & \\ \beta_b(x) + \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_b(x) + \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_b(x) + \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_b(x) + \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases}$$

for all $x \in X$, where $\alpha_b, \alpha_d, \beta_b(x), \beta_d(x), \gamma_b(x), \gamma_d(x), \delta_b(x), \delta_d(x), \zeta_b(x)$ and $\zeta_d(x)$ are defined as in Corollaries 6.12 and 6.15

Corollary 6.20. *Let $\nu \ge 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \ne \frac{2\beta}{\alpha}$ and $\frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality* (3.28) *for all $x, y, z \in X$. Then there exists a unique quartic mapping $B : X \to Y$ satisfying*

$$\|f(x) - B(x) - D(x)\|_Y \le \frac{K}{12}\begin{cases} \rho_b(x) + \rho_d(x), \\ \tau_b(x) + \tau_d(x) \end{cases} \tag{6.14}$$

for all $x \in X$, where $\rho_b(x), \rho_d(x), \tau_b(x), \tau_d(x)$ are defined as in Corollaries 6.13 and 6.16

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