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Multifunction between Soft Topological Spaces

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Abstract — **In this paper, we define the notion of a multifunction between soft topological spaces and study several properties of these multifunctions such as image, upper and lower inverse image, upper and lower continuity etc. Finally, we made an application to the problem of medical diagnosis in medical expert systems on soft multifunction.**

Keywords—**Soft sets, soft topology, the multifunction between soft topological spaces, soft continuity.**

I. Introduction

Many complex problems in economics, engineering, environmental science, social science, medical science etc. cannot be dealt with by classical methods, because classical methods have inherent difficulties. So, in 1999, Molodtsov [1] introduced the concept of soft set theory as a new approach for coping with uncertainties and these problems. Also he and et al. [2] presented the basic results of the soft set theory. He successfully applied the soft set theory into several directions such as smoothness of functions, theory of probability, game theory, Riemann Integration, Perron Integration, theory of optimization, theory of measurement etc. Then Maji and Roy [3] applied soft sets theory in a multicriteria decision making problems. Then, Shabir and Naz [4] introduced the notions of soft topological spaces. Çağman et al [5] defined a soft topological space. After that, Kharal and Ahmad [6] defined a mapping on soft classes and studied properties of these mappings.

We [9] previously defined the soft multifunction from an ordinary topological space (X,τ) to a soft topological space (Y,σ,K) as the image of every point of X is soft set in Y. But, in this paper, we redefine the notion of a multifunction between soft topological spaces (X,τ,E) and (Y,σ,K) as the image of every soft point of X is soft set in Y. Then, we study several properties of these multifunctions such as image, upper and lower inverse image, upper and lower continuity etc. Finally, we made an application to the problem of medical diagnosis in medical expert systems on soft multifunction.

II. Preliminaries

Definition 1. [1] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A pair (F,A) is called a soft set over X, where F is a mapping given by $F: A \rightarrow P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F,A).

Definition 2. [7] A soft set (F,A) over X is called a null soft set, denoted by Φ, if $e \in A$, $F(e) = \emptyset$. If $A = E$, then the null soft set is called universal null soft set, denoted by Φ.

Definition 3. [7] A soft set (F,A) over X is called an absolute soft set, denoted by A, if $e \in A$, $F(e) = X$. If $A = E$, then the absolute soft set is called universal soft set, denoted by $\tilde{X}$.

Definition 4. [4] Let Y be a non-empty subset of X, then Y denotes the soft set (Y,E) over X for which $Y(e) = Y$, for all $e \in E$.

Definition 5. [7] The union of two soft sets of (F,A) and (G,B) over the common universe X is the soft set (H,C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B \\ G(e), & \text{if } e \in B - A \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

We write $(F,A) \tilde{\cup} (G,B) = (H,C)$.

Definition 6. [7] The intersection (H,C) of two soft sets (F,A) and (G,B) over a common universe X, denoted $(F,A) \tilde{\cap} (G,B)$, is defined as $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for all $e \in C$.

Definition 7. [7] Let (F,A) and (G,B) be two soft sets over a common universe X. $(F,A) \tilde{\subset} (G,B)$, if $A \subset B$ and $F(e) \subset G(e)$ for all $e \in A$.

Definition 8. [8] For a soft set (F, A) over X the relative complement of (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c: A \to P(X)$ is a mapping given by $F^c(\alpha) = X - F(\alpha)$ for all $\alpha \in A$.

Proposition 1. [10] Let $(G, A), (H, A), (S, A), (T, A)$ be soft sets in X. Then the following are true;
i) If $(G, A) \cap (H, A) = \Phi$, then $(G, A) \tilde{\subseteq} (H, A)^c$
ii) $(G, A) \tilde{\cup} (G, A)^c = \tilde{X}$
iii) If $(G, A) \tilde{\subseteq} (H, A)$ and $(H, A) \tilde{\subseteq} (S, A)$, then $(G, A) \tilde{\subseteq} (S, A)$
iv) If $(G, A) \tilde{\subseteq} (H, A)$ and $(S, A) \tilde{\subseteq} (T, A)$ then $(G, A) \tilde{\cap} (S, A) \tilde{\subseteq} (H, A) \tilde{\cap} (T, A)$
v) $(G, A) \tilde{\subseteq} (H, A)$ if and only if$(H, A)^c \tilde{\subseteq} (G, A)^c$

Definition 9. The soft set (G, E) over X is called a soft point in X, denoted by $E_e{}^x$, if for $e \in E$ there exist $x \in X$ such that $G(e) = \{x\}$ and $G(e') = \emptyset$ for all $e' \in E - \{e\}$.

Definition 10. The soft point $E_e{}^x$ is said to be in the soft set (H, E), denoted by $E_e{}^x \tilde{\in} (H, E)$, if $x \in H(e)$.

Proposition 2. [10] Let $E_e{}^x$ be a soft point and (H, E) be a soft set in X. If $E_e{}^x \tilde{\in} (H, E)$, then $E_e{}^x \tilde{\notin} (H, E)^c$.

Definition 11. [4] Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X if satisfies the following axioms.
(1) Φ, X belong to τ,
(2) the union of any number of soft sets in τ belongs to τ,
(3) the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X. Let (X, τ, E) be a soft topological space over X, then the members of τ are said to be soft open sets in X. A soft set (F, A) over X is said to be a soft closed set in X, if its relative complement $(F, A)^c$ belongs to τ.

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Definition 12. [9] A soft multifunction F from an ordinary topological space (X, τ) into a soft topological space (Y, σ, E) assings to each point x in X a soft set $F(x)$ over Y. A soft multifunction will be denoted by $F: (X, \tau) \to (Y, \sigma, E)$.

In this paper, we redefined the soft multifunction as follows:

Let $S(X, E)$ and $S(Y, K)$ be two soft classes. Let $u: X \to Y$ be multifunction and $p: E \to K$ be mapping. Then a soft multifunction $F: S(X, E) \to S(Y, K)$ is defined as follows:

For a soft set (G, E) in (X, E), $(F(G, E), K)$ is a soft set in (Y, K) given by $F(G, E)(k) = \cup u(G(e))$ for $k \in K$.
$(F(G, E), K)$ is called a soft image of a soft set (G, E).
Moreover, $F(G, E) = \tilde{\cup} \{F(E_e{}^x): E_e{}^x \tilde{\in} (G, E)\}$ for a soft subset (G, E) of X.

Definition 13. Let $F: S(X, E) \to S(Y, K)$ be a soft multifunction.
The soft upper inverse image of (H, K) denoted by $F^+(H, K)$ and defined as $F^+(H, K) = \{E_e{}^x \tilde{\in} \tilde{X}: F(E_e{}^x) \tilde{\subseteq} (H, K)\}$.
The soft lower inverse image of (H, K) denoted by $F^-(H, K)$ and defined as $F^-(H, K) = \{E_e{}^x \tilde{\in} \tilde{X}: F(E_e{}^x) \tilde{\cap} (H, K) \neq \Phi\}$.

Example 1. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, $E = \{e_1, e_2, e_3\}$, $K = \{k_1, k_2\}$. Let $u: X \to Y$ be a multifunction defined as: $u(x_1) = \{y_1\}$, $u(x_2) = \{y_2\}$, $u(x_3) = \{y_1, y_3\}$. Let $p: E \to K$ be a mapping defined as: $p(e_1) = \{k_1\}$, $p(e_2) = k_2$, $p(e_3) = k_1$. Then $F: S(X, E) \to S(Y, K)$ is soft multifunction.

i) The image of every soft point of X as follows:

$$F((e_1, \{x_1\})) = (k_1, \{y_1\})$$
$$F((e_1, \{x_2\})) = (k_1, \{y_2\})$$
$$F((e_1, \{x_3\})) = (k_1, \{y_1, y_3\})$$
$$F((e_2, \{x_1\})) = (k_2, \{y_1\})$$
$$F((e_2, \{x_2\})) = (k_2, \{y_2\})$$
$$F((e_2, \{x_3\})) = (k_2, \{y_1, y_3\})$$
$$F((e_3, \{x_1\})) = (k_1, \{y_1\})$$
$$F((e_3, \{x_2\})) = (k_1, \{y_2\})$$

$$F((e_3,\{x_3\})) = (k_1,\{y_1,y_3\})$$

ii) For a soft subset $(G,E) = \{(e_1,\{x_2\}),(e_2,\{x_2\}),(e_3,\{x_1,x_3\})\}$ in (X,E), the soft image of (G,E) is denoted by $F(G,E)$ is $F(G,E) = \{(k_1,Y),(k_2,\{y_2\})\}$. Because;

$$F(G,E)(k_1) = u(G(e_1)) \cup u(G(e_3)) = u(x_1) \cup u(X) = \{y_1\} \cup Y = Y$$
$$F(G,E)(k_2) = u(G(e_2)) = u(x_1,x_2) = \{y_1,y_2\}.$$

iii) The upper inverse image and lower inverse image of every soft point of Y as follows:

$$F^+((k_1,\{y_1\})) = \{(e_1,\{x_1\}),(e_3,\{x_1\})\}$$
$$F^+((k_1,\{y_2\})) = \{(e_1,\{x_2\}),(e_3,\{x_2\})\}$$
$$F^+((k_1,\{y_3\})) = \{(e_1,\{x_3\}),(e_3,\{x_3\})\}$$
$$F^+((k_2,\{y_1\})) = \{(e_2,\{x_1\})\}$$
$$F^+((k_2,\{y_2\})) = \{(e_2,\{x_2\})\}$$
$$F^+((k_2,\{y_3\})) = \{(e_2,\{x_3\})\}$$
$$F^-((k_1,\{y_1\})) = \{(e_1,\{x_1,x_3\}),(e_3,\{x_1,x_3\})\}$$
$$F^-((k_1,\{y_2\})) = \{(e_1,\{x_2\}),(e_3,\{x_2\})\}$$
$$F^-((k_1,\{y_3\})) = \{(e_1,\{x_3\}),(e_3,\{x_3\})\}$$
$$F^-((k_2,\{y_1\})) = \{(e_2,\{x_1\})\}$$
$$F^-((k_2,\{y_2\})) = \{(e_2,\{x_2\})\}$$
$$F^-((k_2,\{y_3\})) = \{(e_2,\{x_3\})\}$$

iv) For a soft subset $(H,K) = \{(k_1,\{y_2\}),(k_2,\{y_1,y_3\})\}$ in (Y,K), the upper inverse image and lower inverse image of a soft set (H,K) is denoted by $F^+(H,K)$ and $F^-(H,K)$ respectively is

$$F^+(H,K) = \{(e_1,\{x_2\}),(e_2,\{x_1\}),(e_3,\{x_2\})\}$$
$$F^-(H,K) = \{(e_1,\{x_2\}),(e_2,\{x_1,x_3\}),(e_3,\{x_2\})\}.$$

Definition 14. Let $F: S(X,E) \to S(Y,K)$ and $G: S(X,E) \to S(Y,K)$ be two soft multifunctions. Then, F equal to G if $F({E_e}^x) = G({E_e}^x)$, for each ${E_e}^x \in X$.

Definition 15. The soft multifunction $F: S(X,E) \to S(Y,K)$ is called surjective if p and u are surjective.

Theorem 1. Let $F: S(X,E) \to S(Y,K)$ be a soft multifunction. Then, for soft sets (F,E), (G,E) and for a family of soft sets $(G_i,E)_{i\in I}$ in the soft class $S(X,E)$ the following are hold:

(a) $F(\Phi) = \Phi$
(b) $F(\tilde{X}) \tilde{\subseteq} \tilde{Y}$
(c) $F((G,A)\,\tilde{\cup}\,(H,B)) = F(G,A)\,\tilde{\cup}\,F(H,B)$ in general $F(\tilde{\cup}_i\,(G_i,E)) = \tilde{\cup}_i\,F(G_i,E)$
(d) $F((G,A)\,\tilde{\cap}\,(H,B)) \tilde{\subseteq} F(G,A)\,\tilde{\cap}\,F(H,B)$ in general $F(\cap_i\,(G_i,E)) \tilde{\subseteq} \tilde{\cap}_i\,F(G_i,E)$
(e) If $(G,E) \tilde{\subseteq} (H,E)$, then $F(G,E) \tilde{\subseteq} F(H,E)$.

Proof. (a) and (b) obvious.

(c) For $k \in K$, if $p^{-1}(k) \cap (A \cup B) = \emptyset$, then the proof is obvious. Suppose that $p^{-1}(k) \cap (A \cup B) \neq \emptyset$. Then, there exists three cases.

Case 1. If $e \in p^{-1}(k) \cap (A - B)$ then $F((G,A)\,\tilde{\cup}\,(H,B))(k) = \cup u(G(e)) = F(G,A)(k) = F(G,A)(k) \cup \emptyset = F(G,A)(k) \cup F(H,B)(k)$.

Case 2. If $e \in p^{-1}(k) \cap (B - A)$ then $F((G,A)\,\tilde{\cup}\,(H,B))(k) = \cup u(H(e)) = F(H,B)(k) = \emptyset \cup F(H,B)(k) = F(G,A)(k) \cup F(H,B)(k)$.

Case 3. If $e \in p^{-1}(k) \cap (A \cap B)$ then $F((G,A)\,\tilde{\cup}\,(H,B))(k) = \cup(u(G(e)) \cup u(H(e))) = (\cup u(G(e))) \cup (\cup u(H(e)))$ $= F(G,A)(k) \cup F(H,B)(k)$. Therefore we have, $F((G,A)\,\tilde{\cup}\,(H,B)) = F(G,A)\,\tilde{\cup}\,F(H,B)$.

(d) For $k \in K$, if $p^{-1}(k) \cap (A \cap B) = \emptyset$, then the proof is obvious. Suppose that $p^{-1}(k) \cap (A \cap B) \neq \emptyset$. Then, $\big(F(G,A)(k)\big) \cap \big(F(H,B)(k)\big) = \Big(\cup u\big(G(e)\big)\Big) \cap \Big(\cup u\big(H(e)\big)\Big) \subset \cup u\big(G(e) \cap H(e)\big) = F\big((G,A)\,\tilde{\cap}\,(H,B)\big)(k)$. Therefore we have, $F(G,A)\,\tilde{\cap}\,F(H,B) \tilde{\subseteq} F((G,A)\,\tilde{\cap}\,(H,B))$.

(e) For $k \in K$, if $p^{-1}(k) \cap E = \emptyset$, then the proof is obvious. Suppose that $p^{-1}(k) \cap E \neq \emptyset$. Then $F(G,E)(k) = \cup u(G(e)) \subset \cup u(H(e)) = F(H,E)(k)$. Thus we have $F(G,E) \tilde{\subseteq} F(H,E)$.

In the following examples is shown that the statements of a) and b) is not to be equal, respectively.

Example 2. Let the soft multifunction $F: S(X,E) \to S(Y,K)$ defined as follows:
$X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, $E = \{e_1, e_2\}$, $K = \{k_1, k_2\}$ $u(x_1) = \{y_1\}$, $u(x_2) = \{y_1\}$, $p(e_1) = k_1$, $p(e_2) = k_2$. Then $F(\tilde{X}) \neq \tilde{Y}$.

Example 3. Let the soft multifunction $F: S(X,E) \to S(Y,K)$ defined as follows:
$X = \{x_1, x\_\{2,\} x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $E = \{e_1, e_2, e_3\}$, $K = \{k_1, k_2, k_3\}$. $u(x_1) = \{y_1, y_2\}$, $u(x_2) = \{y_2, y_3\}$, $u(x_3) = \{y_4\}$, $u(x_4) = \{y_1, y_3, y_4\}$, $p(e_1) = k_2$, $p(e_2) = k_3$, $p(e_3) = k_1$.
Then, $F((G,A) \tilde{\cap} (H,A)) \neq F(G,A) \tilde{\cap} F(H,A)$ for a soft subsets $(G,A) = \{(e_1, \{x_1\}), (e_2, \{x_3\}), (e_3, \{x_1, x_2\})\}$ and $(H,A) = \{(e_1, \{x_2\}), (e_2, \{x_3\}), (e_3, \{x_3\})\}$ in $S(X,E)$.

Theorem 2. Let $F: S(X,E) \to S(Y,K)$ be a soft multifunction. Then the follows are true:
(a) $F^-(\Phi) = \Phi$ and $F^+(\Phi) = \Phi$
(b) $F^-(\tilde{Y}) = \tilde{X}$ and $F^+(\tilde{Y}) = \tilde{X}$
(c) $F^-((G,K) \tilde{\cup} (H,K)) = F^-(G,K) \tilde{\cup} F^-(H,K)$
(d) $F^+(G,K) \tilde{\cup} F^+(H,K) \tilde{\subseteq} F^+((G,K) \tilde{\cup} (H,K))$
(e) $F^-((G,K) \tilde{\cap} (H,K)) \tilde{\subseteq} F^-(G,K) \tilde{\cap} F^-(H,K)$
(f) $F^+(G,K) \tilde{\cap} F^+(H,K) = F^+((G,K) \tilde{\cap} (H,K))$
(g) If $(G,K) \tilde{\subseteq} (H,K)$, then $F^-(G,K) \tilde{\subseteq} F^-(H,K)$ and $F^+(G,K) \tilde{\subseteq} F^+(H,K)$.

Proof. (a) and (b) obvious.

(c) Let ${E_e}^x \tilde{\in} F^-((G,K) \tilde{\cup} (H,K))$, then $F({E_e}^x) \tilde{\cap} ((G,K) \tilde{\cup} (H,K)) \neq \Phi$. Then, $(F({E_e}^x) \tilde{\cap} (G,K)) \tilde{\cup} (F({E_e}^x) \tilde{\cap} (H,K)) \neq \Phi$. Then, $F({E_e}^x) \tilde{\cap} (G,K) \neq \Phi$ or $F({E_e}^x) \tilde{\cap} (H,K) \neq \Phi$. Thus, ${E_e}^x \tilde{\in} F^-(G,K)$ or ${E_e}^x \tilde{\in} F^-(H,K)$. Then, ${E_e}^x \tilde{\in} F^-(G,K) \tilde{\cup} F^-(H,K)$. Thus we have, $F^-((G,K) \tilde{\cup} (H,K)) \tilde{\subseteq} F^-(G,K) \tilde{\cup} F^-(H,K)$.
Conversely, let ${E_e}^x \tilde{\in} F^-(G,K) \tilde{\cup} F^-(H,K)$. Then, ${E_e}^x \tilde{\in} F^-(G,K)$ or ${E_e}^x \tilde{\in} F^-(H,K)$. Then, $F({E_e}^x) \tilde{\cap} (G,K) \neq \Phi$ or $F({E_e}^x) \tilde{\cap} (H,K) \neq \Phi$. Then $(F({E_e}^x) \tilde{\cap} (G,K)) \tilde{\cup} (F({E_e}^x) \tilde{\cap} (H,K)) \neq \Phi$. Then, $F({E_e}^x) \tilde{\cap} ((G,K) \tilde{\cup} (H,K)) \neq \Phi$. Then, ${E_e}^x \tilde{\in} F^-((G,K) \tilde{\cup} (H,K))$. Thus we have $F^-(G,K) \tilde{\cup} F^-(H,K) \tilde{\subseteq} F^-((G,K) \tilde{\cup} (H,K))$.

(d) Let ${E_e}^x \tilde{\in} F^+(G,K) \tilde{\cup} F^+(H,K)$, then ${E_e}^x \tilde{\in} F^+(G,K)$ or ${E_e}^x \tilde{\in} F^+(H,K)$. Thus $F({E_e}^x) \tilde{\subseteq} (G,K)$ or $F({E_e}^x) \tilde{\subseteq} (H,K)$. Thus $F({E_e}^x) \tilde{\subseteq} (G,K) \tilde{\cup} (H,K)$. Therefore, ${E_e}^x \tilde{\in} F^+((G,K) \tilde{\cup} (H,K))$.

(e) Let ${E_e}^x \tilde{\in} F^-((G,K) \tilde{\cap} (H,K))$ then $F({E_e}^x) \tilde{\cap} ((G,K) \tilde{\cap} (H,K)) \neq \Phi$. Then $(F({E_e}^x) \tilde{\cap} (G,K)) \tilde{\cap} (F({E_e}^x) \tilde{\cap} (H,K)) \neq \Phi$. Then $F({E_e}^x) \tilde{\cap} (G,K) \neq \Phi$ and $F({E_e}^x) \tilde{\cap} (H,K) \neq \Phi$. Thus ${E_e}^x \tilde{\in} F^-(G,K)$ and ${E_e}^x \tilde{\in} F^-(H,K)$. Therefore ${E_e}^x \tilde{\in} F^-(G,K) \tilde{\cap} F^-(H,K)$.
Conversely, let ${E_e}^x \tilde{\in} F^-(G,K) \tilde{\cap} F^-(H,K)$ then, ${E_e}^x \tilde{\in} F^-(G,K)$ and ${E_e}^x \tilde{\in} F^-(H,K)$. Then, $F({E_e}^x) \tilde{\cap} (G,K) \neq \Phi$ and $F({E_e}^x) \tilde{\cap} (H,K) \neq \Phi$. Thus, $F({E_e}^x) \tilde{\cap} ((G,K) \tilde{\cap} (H,K)) \neq \Phi$. Therefore, ${E_e}^x \tilde{\in} F^-((G,K) \tilde{\cap} (H,K))$.

(f) Let ${E_e}^x \tilde{\in} F^+((G,K) \tilde{\cap} (H,K))$ then, $F({E_e}^x) \tilde{\subseteq} (G,K) \tilde{\cap} (H,K)$. Then, $F({E_e}^x) \tilde{\subseteq} (G,K)$ and $F({E_e}^x) \tilde{\subseteq} (H,K)$. Thus, ${E_e}^x \tilde{\in} F^+(G,K)$ and ${E_e}^x \tilde{\in} F^+(H,K)$. Therefore, ${E_e}^x \tilde{\in} F^+(G,K) \tilde{\cap} F^+(H,K)$.
Conversely, let ${E_e}^x \tilde{\in} F^+(G,K) \tilde{\cap} F^+(H,K)$. Then, ${E_e}^x \tilde{\in} F^+(G,K)$ and ${E_e}^x \tilde{\in} F^+(H,K)$. Then, $F({E_e}^x) \tilde{\subseteq} (G,K)$ and $F({E_e}^x) \tilde{\subseteq} (H,K)$ hence, $F({E_e}^x) \tilde{\subseteq} ((G,K) \tilde{\cap} (H,K))$. Therefore, ${E_e}^x \tilde{\in} F^+((G,K) \tilde{\cap} (H,K))$.

(g) Let ${E_e}^x \tilde{\in} F^-(G,K)$ then, $F({E_e}^x) \tilde{\cap} (G,K) \neq \Phi$. Since $(G,K) \tilde{\subseteq} (H,K)$ then, $F({E_e}^x) \tilde{\cap} (H,K) \neq \Phi$. Thus we have ${E_e}^x \tilde{\in} F^-(H,K)$.
Let ${E_e}^x \tilde{\in} F^+(G,K)$ then, $F({E_e}^x) \tilde{\subseteq} (G,K)$. Since $(G,K) \tilde{\subseteq} (H,K)$ then, $F({E_e}^x) \tilde{\subseteq} (H,K)$. Thus we have ${E_e}^x \tilde{\in} F^+(H,K)$.

In the following examples is shown that the statements of d) and e) is not to be equal, respectively.

Example 4. Let the soft multifunction $F: S(X,E) \to S(Y,K)$ defined as follows:
$X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $E = \{e_1, e_2, e_3\}$, $K = \{k_1, k_2, k_3\}$.
$u(x_1) = \{y_1, y_2\}$, $u(x_2) = \{y_2, y_3\}$, $u(x_3) = \{y_4\}$, $u(x_4) = \{y_1, y_3, y_4\}$
$p(e_1) = k_2$, $p(e_2) = k_3$, $p(e_3) = k_1$.
For a soft subsets $(G,K) = \{(k_1, \{y_1\}), (k_2, \{y_1, y_2\}), (k_3, \{y_4\})\}$ and $(H,K) = \{(k_1, \{y_2\}), (k_2, \{y_2, y_3\}), (k_3, \{y_1, y_2\})\}$ in $S(X,E)$.

i) $F^{-}((G,K)\tilde{\cap}(H,K))\neq F^{-}(G,K)\tilde{\cap}F^{-}(H,K)$
ii) $F^{+}((G,K)\tilde{\cup}(H,K))\neq F^{+}(G,K)\tilde{\cup}F^{+}(H,K)$.

Proposition 3. Let (G_i,K) be soft sets over Y for each $i\in I$. Then the follows are true for a soft multifunction $F: S(X,E)\to S(Y,K)$:
(a) $F^{-}(\tilde{\cup}(G_i,K)) = \tilde{\cup}(F^{-}(G_i,K))$.
(b) $\tilde{\cap}(F^{+}(G_i,K)) = F^{+}(\tilde{\cap}(G_i,K))$.
(c) $\tilde{\cup}F^{+}(G_i,K)\tilde{\subseteq}F^{+}(\tilde{\cup}(G_i,K))$.
(d) $F^{-}(\tilde{\cap}(G_i,K))\tilde{\subseteq}\tilde{\cap}(F^{-}(G_i,K))$.

Proof. (a) For every ${E_e}^x\tilde{\in}F^{-}(\tilde{\cup}(G_i,K))$, $F({E_e}^x)\tilde{\cap}(\tilde{\cup}(G_i,K))\neq\Phi$. Then, there exists $i\in I$ such that $F({E_e}^x)\tilde{\cap}(G_i,K)\neq\Phi$. For the same $i\in I$, ${E_e}^x\tilde{\in}F^{-}(G_i,K)$. Therefore ${E_e}^x\tilde{\in}\tilde{\cup}(F^{-}(G_i,K))$. Thus $F^{-}(\tilde{\cup}(G_i,K))\tilde{\subseteq}\tilde{\cup}(F^{-}(G_i,K))$.
Conversely, for every ${E_e}^x\tilde{\in}\tilde{\cup}(F^{-}(G_i,K))$, there exists $i\in I$ such that ${E_e}^x\tilde{\in}F^{-}(G_i,K)$. For the same $i\in I$, $F({E_e}^x)\tilde{\cap}(G_i,K)\neq\Phi$. Therefore, $F({E_e}^x)\tilde{\cap}(\tilde{\cup}(G_i,K))\neq\Phi$ and ${E_e}^x\tilde{\in}F^{-}(\tilde{\cup}(G_i,K))$. Thus $\tilde{\cup}(F^{-}(G_i,K))\tilde{\subseteq}F^{-}(\tilde{\cup}(G_i,K))$.

(b) For every ${E_e}^x\tilde{\in}\tilde{\cup}(F^{+}(G_i,K))$, there exists $i\in I$ such that ${E_e}^x\tilde{\in}F^{+}(G_i,K)$. For the same $i\in I$, $F({E_e}^x)\tilde{\subseteq}(G_i,K)$. Therefore, $F({E_e}^x)\tilde{\subseteq}(\cup(G_i,K))$ and ${E_e}^x\tilde{\in}F^{+}\big(\tilde{\cup}(G_i,K)\big)$. Thus $\tilde{\cup}(F^{+}(G_i,K))\tilde{\subseteq}F^{+}(\tilde{\cup}(G_i,K))$.
Conversely, for every ${E_e}^x\tilde{\in}F^{+}(\tilde{\cup}(G_i,K))$, $F({E_e}^x)\tilde{\subseteq}(\tilde{\cup}(G_i,K))$. Then, there exists $i\in I$ such that $F({E_e}^x)\tilde{\subseteq}(G_i,K)$. For the same $i\in I$, ${E_e}^x\tilde{\in}F^{+}(G_i,K)$. Thus, ${E_e}^x\tilde{\in}\tilde{\cup}(F^{+}(G_i,K))$.
The proof (c) and (d) similar to proof (a) and (b).

Proposition 4. Let $F: S(X,E)\to S(Y,K)$ be a soft multifunction. Then the follows are true:
(a) $(G,A)\tilde{\subseteq}F^{+}(F(G,A))\tilde{\subseteq}F^{-}(F(G,A))$ for a soft subset (G,A) in X. If F is surjectice then $(G,A)=F^{+}(F(G,A))=F^{-}(F(G,A))$
(b) $F(F^{+}(H,B))\tilde{\subseteq}(H,B)\tilde{\subseteq}F(F^{-}(H,B))$ for a soft subset (H,B) in Y.
(c) For two soft subsets (H,B) and (U,C) in Y such that $(H,B)\tilde{\cap}(U,C)=\Phi$ then $F^{+}(H,B)\tilde{\cap}F^{-}(U,C)=\Phi$.

Proof. (a) Let ${E_e}^x\tilde{\in}(G,A)$, then $F({E_e}^x)\tilde{\subseteq}(G,A)$. By definition of upper inverse of F, ${E_e}^x\tilde{\in}F^{+}(F(G,A))$.
Let ${E_e}^x\tilde{\in}F^{+}(F(G,A))$, then $F({E_e}^x)\tilde{\subseteq}(G,A)$. Thus we have, $F({E_e}^x)\cap(G,A)\neq\Phi$. By definition of lower inverse of F, ${E_e}^x\tilde{\in}F^{-}(F(G,A))$.

(b) The proof is similar to (a).

(c) Suppose that ${E_e}^x\tilde{\in}F^{+}(H,B)\cap F^{-}(U,C)$, then ${E_e}^x\tilde{\in}F^{+}(H,B)$ and ${E_e}^x\tilde{\in}F^{-}(U,C)$. Thus $F({E_e}^x)\tilde{\subseteq}(H,B)$ and $F({E_e}^x)\tilde{\cap}(U,C)\neq\Phi$. Hence, we have $(H,B)\tilde{\cap}(U,C)\neq\Phi$. This is a contradiction. Therefore we have $F^{+}(H,B)\tilde{\cap}F^{-}(U,C)=\Phi$.

Proposition 5. Let $F:(X,\tau,E)\to(Y,\sigma,K)$ and $G:(Y,\sigma,K)\to(Z,\eta,L)$ be two soft multifunction. Then the follows are true:
(a) $(F^{-})^{-}=F$
(b) For a soft subset (T,C) in Z, $(GoF)^{-}(T,C)=F^{-}(G^{-}(T,C))$ and $(GoF)^{+}(T,C)=F^{+}(G^{+}(T,C))$.

Proof. Straightforward.

Proposition 6. Let (G,E) be a soft set over Y. Then the followings are true for a soft multifunction $F:(X,\tau,E)\to(Y,\sigma,K)$:
(a) $F^{+}(\tilde{Y}-(G,K))=\tilde{X}-F^{-}(G,K)$
(b) $F^{-}(\tilde{Y}-(G,K))=\tilde{X}-F^{+}(G,K)$.

Proof. (a) If ${E_e}^x\tilde{\in}\tilde{X}-F^{-}(G,K)$ then ${E_e}^x\tilde{\notin}F^{-}(G,K)$ which implies $F({E_e}^x)\tilde{\cap}(G,K)=\Phi$ and therefore we have $F({E_e}^x)\tilde{\subseteq}\tilde{Y}-(G,K)$. Thus ${E_e}^x\tilde{\in}F^{+}(\tilde{Y}-(G,K))$. Hence $\tilde{X}-F^{-}(G,K)\tilde{\subseteq}F^{+}(\tilde{Y}-(G,K))$.
Conversely, if ${E_e}^x\tilde{\in}F^{+}(\tilde{Y}-(G,K))$ then $F({E_e}^x)\tilde{\subseteq}\tilde{Y}-(G,K)$ which implies $F({E_e}^x)\tilde{\cap}(G,K)=\Phi$ and therefore, ${E_e}^x\tilde{\notin}F^{-}(G,K)$. Thus ${E_e}^x\tilde{\in}\tilde{X}-F^{-}(G,K)$. Hence $F^{+}(\tilde{Y}-(G,K))\tilde{\subseteq}\tilde{X}-F^{-}(G,K)$.

(b) Let ${E_e}^x\tilde{\in}\tilde{X}-F^{+}(G,K)$ then, ${E_e}^x\tilde{\notin}F^{+}(G,K)$ which implies $F({E_e}^x)\tilde{\nsubseteq}(G,K)$ and therefore $F({E_e}^x)\tilde{\cap}(\tilde{Y}-(G,K))\neq\Phi$. Thus ${E_e}^x\tilde{\in}F^{-}(\tilde{Y}-(G,K))$ and $\tilde{X}-F^{+}(G,K)\tilde{\subseteq}F^{-}(\tilde{Y}-(G,K))$.
Conversely, let ${E_e}^x\tilde{\in}F^{-}(\tilde{Y}-(G,K))$ then $F({E_e}^x)\tilde{\cap}(\tilde{Y}-(G,K))\neq\Phi$ which implies $F({E_e}^x)\tilde{\nsubseteq}(G,K)$ and therefore ${E_e}^x\tilde{\notin}F^{+}(G,K)$. Thus ${E_e}^x\tilde{\in}\tilde{X}-F^{+}(G,K)$ and $F^{-}(\tilde{Y}-(G,K))\tilde{\subseteq}\tilde{X}-F^{+}(G,K)$.

III. UPPER AND LOWER CONTINUITY OF SOFT MULTIFUNCTION

Definition 16. Let (X,τ,E) and (Y,σ,E) be two soft topological spaces. Then a soft multifunction $F:(X,\tau,E) \to (Y,\sigma,K)$ is said to be;

(a) soft upper continuous at a soft point $E_e{}^x \tilde{\in} \tilde{X}$ if for each soft open (G,K) such that $F(E_e{}^x) \tilde{\subseteq} (G,K)$, there exists an open neighborhood $P(E_e{}^x)$ of $E_e{}^x$ such that $F(E_e{}^z) \tilde{\subseteq} (G,K)$ for all $E_e{}^z \tilde{\in} P(E_e{}^x)$.

(b) soft lower continuous at a soft point $E_e{}^x$ if for each soft open (G,K) such that $F(E_e{}^x) \tilde{\cap} (G,K) \neq \Phi$, there exists an open neighbourhood $P(E_e{}^x)$ of $E_e{}^x$such that $F(E_e{}^z) \tilde{\cap} (G,K) \neq \Phi$ for all $E_e{}^z \tilde{\in} P(E_e{}^x)$.

(c) soft upper(lower) continuous if F has this property at every soft point of X.

Proposition 7. A soft multifunction $F:(X,\tau,E) \to (Y,\sigma,K)$ is soft upper continuous if and only if $F^{+}(G,K)$ is soft open in X, for every soft open set (G,K).

Proof. First suppose that F is soft upper continuous. Let (G,K) be any soft open set over Y and $E_e{}^x \tilde{\in} F^{+}(G,K)$. Then, by from Definition 16, we know that there exists a soft open neighbourhood $P(E_e{}^x)$ of $E_e{}^x$ such that $F(E_e{}^z) \tilde{\subseteq} (G,K)$ for all $E_e{}^z \tilde{\in} P(E_e{}^x)$. This means that $F^{+}(G,K)$ is soft open as claimed.
The other direction is just the definition of soft upper continuity of F.

Proposition 8. $F:(X,\tau,E) \to (Y,\sigma,K)$ is soft lower continuous multifunction if and only if $F^{-}(G,K)$ is soft open set in X, for every soft set (G,K).

Proof. First assume that F is soft lower continuous. Let (G,K) be any soft open set over Y and $E_e{}^x \tilde{\in} F^{-}(G,K)$. Then there exists an soft open neighbourhood $P(E_e{}^x)$ of $E_e{}^x$ such that $F(E_e{}^z) \tilde{\cap} (G,K) \neq \Phi$ for all $E_e{}^z \tilde{\in} P(E_e{}^x)$. So, $P(E_e{}^x) \tilde{\subseteq} F^{-}(G,K)$ which implies that $F^{-}(G,K)$ is soft open in X.

Now suppose that $F^{-}(G,K)$ is soft open. Let $E_e{}^x \tilde{\in} F^{-}(G,K)$. Then $F^{-}(G,K)$ is an soft open neighbourhood of $E_e{}^x$ and for all $E_e{}^z \tilde{\in} F^{-}(G,K)$ we have $F(E_e{}^z) \tilde{\cap} (G,K) \neq \Phi$. So, F is soft lower continuous.

Example 5. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, $E = \{e_1, e_2\}$, $K = \{k_1, k_2\}$. $\tau = \{\Phi, \tilde{X}, (G,E)\}$, where $(G,E) = \{(e_2, \{x_1\})\}$ and $\sigma = \{\Phi, \tilde{Y}, (H,K)\}$, where $(H,K) = \{(k_1, \{y_1\}), (k_2, \{y_2\})\}$.

Let $u: X \to Y$ be multifunction defined as $u(x_1) = \{y_1\}$, $u(x_2) = \{y_1, y_2\}$ and $p: E \to K$ be mapping defined as $p(e_1) = k_2$, $p(e_2) = k_1$.

Then the soft multifunction $F:(X,\tau,E) \to (Y,\sigma,K)$ is soft upper continuous. Because for a soft open set (H,K) in Y, $F^{+}(H,K) = \{(e_2, \{x_1\})\} = (G,E)$ is soft open set in X.

Example 6. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, $E = \{e_1, e_2\}$, $K = \{k_1, k_2\}$, $\tau = \{\Phi, \tilde{X}, (G,E)$, where $(G,E) = \{(e_1, \{x_2\}), (e_2, X)\}$ and $\sigma = \{\Phi, \tilde{Y}, (H,K)\}$, where $(H,K) = \{(k_1, \{y_1\}), (k_2, \{y_2\})\}$. Let $u: X \to Y$ be multifunction defined as $u(x_1) = \{y_1\}$, $u(x_2) = \{y_1, y_2\}$ and $p: E \to K$ be mapping defined as $p(e_1) = k_2$, $p(e_2) = k_1$. Then the soft multifunction $F:(X,\tau,E) \to (Y,\sigma,K)$ is soft lower continuous. Because for a soft open set (H,K) in Y, $F^{-}(H,K) = \{(e_1, \{x_2\}), (e_2, X)\} = (G,E)$ is soft open set in X.

Theorem 3. The followings are equivalent for a soft multifunction $F:(X,\tau,E) \to (Y,\sigma,K)$:

(a) F is soft upper continuous

(b) $F^{-}(G,K)$ is soft closed in X, for each soft closed set (G,K) over Y.

(c) $cl(F^{-}(G,K)) \tilde{\subseteq} F^{-}(cl(G,K))$, for each soft set (G,K) over Y.

(d) $F^{+}(Int(G,K)) \tilde{\subseteq} Int(F^{+}(G,K))$, for each soft set (G,K) over Y.

Proof. (a)⇒(b) Let (G,K) be a soft closed set over Y. Then $\tilde{Y} - (G,K)$ is soft open set. Since F is soft upper continuous, by Proposition 7, $F^{+}(\tilde{Y} - (G,K))$ is soft open set. Also, since $F^{+}(\tilde{Y} - (G,K)) = \tilde{X} - F^{-}(G,K)$, then $F^{-}(G,K)$ is soft closed.

(b)⇒(c) Let (G,K) be any soft set over Y. Then $cl(G,K)$ is soft closed set. By (b) $F^{-}(cl(G,K))$ is soft closed in X. Hence, $cl(F^{-}(G,K)) \tilde{\subseteq} F^{-}(cl(G,K))$ and since $F^{-}(G,K) \tilde{\subseteq} F^{-}(cl(G,K))$, then $cl(F^{-}(G,K)) \tilde{\subseteq} F^{-}(cl(G,K))$.

(c)⇒(d) Let (G,K) be any soft set over Y. By (c), $cl(F^{-}(\tilde{Y} - (G,K))) \tilde{\subseteq} F^{-}(cl(\tilde{Y} - (G,K)))$, then $\tilde{X} - F^{-}((int(\tilde{Y} - (G,K))) \tilde{\subseteq} int(\tilde{X} - F^{-}((int(\tilde{Y} - (G,K))))$ and $\tilde{X} - (\tilde{X} - F^{+}(int(G,K))) \subset int(F^{+}(G,K)$.

(d)⇒(a) Let (G,K) be any soft set over Y. By (d), $F^{+}(Int(G,K)) = F^{+}(G,K) \tilde{\subseteq} Int(F^{+}(G,K))$ and so $F^{+}(G,K)$ is soft open in X. Then by proposition 7, F is soft upper continuous.

Theorem 4. The following are equivalent for a soft multifunction $F: (X,\tau,E) \to (Y,\sigma,K)$:
(a) F is soft lower continuous.
(b) $F^{+}(G,K)$ is soft closed in X, for each soft closed set (G,K) over Y .
(c) $cl(F^{+}(G,K)) \tilde{\subseteq} F^{+}(cl(G,K))$, for each soft set (G,K) over Y .
(d) $F^{-}(Int(G,K)) \tilde{\subseteq} int(F^{-}(G,K))$, for each soft set (G,K) over Y.

Proof. The proof is similar to previous theorem.

IV. An Application For Soft Multifunctions

An important task of a medical expert system is to transform a patient's complaints into a set of possible causes. A patient's case may easily be encoded into a soft set.

Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$
$E = \{e_1, e_2, e_3, e_4\}$
$Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}$
$K = \{k_1, k_2, k_3, k_4\}$, where
$x_1 = egzema,$
$x_2 = migraine,$
$x_3 = herpes,$
$x_4 = anxiety,$
$x_5 = backpone\ pain,$
$x_6 = joint\ pain,$
$x_7 = sleepnessless,$
$x_8 = headache,$
$x_9 = burning\ in\ stomach$
$y_1 = lack\ of\ appetite,$
$y_2 = allergy,$
$y_3 = obesity,$
$y_4 = mood\ disorder,$
$y_5 = depression,$
$y_6 = wrong\ posture,$
$y_7 = fatigue,$
$y_8 = blood\ pressure,$
$y_9 = acidity$
$e_1 = high\ importance,$
$e_2 = medium\ importance,$
$e_3 = low\ importance,$
$e_4 = very\ low\ importance,$
$e_5 = unimportance$
$k_1 = frequent\ high\ potency,$
$k_2 = infrequent\ high\ potency,$
$k_3 = frequent\ low\ potency,$
$k_4 = infrequent\ low\ potency,$
$k_5 = impotency.$

Thus we have two soft classes (X,E) and (Y,K). Where (X,E) is the soft class of symptoms and their importance for the patient, (Y,K) represents causes and medical preference for treatment. The soft set of patient may be given as:

$(G,E) = \{(e_1,\{x_2,x_4,x_8\}),(e_2,\{x_5,x_6\}),(e_3,\{x_1,x_7,x_9\}),(e_4,\{x_3\}),(e_5,\emptyset)\}$
$= \{(high\ importance,\{migraine, anxiety, headache\}),(medium\ importance,\{backpone\ pain, joint\ pain\}),$
$(low\ importance,\{egzema, sleepnessless, burning\ in\ stomach\}),(very\ low\ importance,\{herpes\}),(unimportance,\emptyset)\}$
.

As a first of the medical exper system, stored medical knowledge is to be applied on the given case.

Mapping $u: X \to Y$ and $p: E \to K$ defined as follows:

$u(x_1) = \{y_2, y_8\}$,
$u(x_2) = \{y_4, y_5, y_9\}$,
$u(x_3) = \{y_7\}$,
$u(x_4) = \{y_4, y_5\}$,
$u(x_5) = \{y_6, y_7\}$,
$u(x_6) = \{y_7\}$,
$u(x_7) = \{y_3, y_5\}$,
$u(x_8) = \{y_1, y_2\}$,
$u(x_9) = \{y_8, y_9\}$
$p(e_1) = k_1$,
$p(e_2) = k_2$,
$p(e_3) = k_3$,
$p(e_4) = k_4$,
$p(e_5) = k_5$.

The calculations give us the represents causes and medical preference for treatment as follows:

$$F(G,E) = \{(k_1,\{y_1,y_2,y_4,y_5,y_9\}),(k_2,\{y_6,y_7\}),(k_3,\{y_2,y_3,y_5,y_8,y_9\}),(k_4,\{y_7\}),(k_5,\emptyset)\}$$
$$= \{(frequent\ high\ potency,\{lack\ of\ appetite, allergy, mood\ disorder, depression, acidity\}),$$
$$(infrequent\ high\ potency,\{wrong\ posture, fatigue\}),$$
$$(frequent\ low\ potency,\{allergy, obesity, depression, blood\ pressure, acidity\}),$$
$$(infrequent\ low\ potency,\{fatigue\}),(impotency,\emptyset)\}.$$

V. CONCLUSIONS

In this paper, we define the concept of multifunction between the soft classes. We give the notion upper and lower inverse image of these multifunctions. Then using these notions, we defined the upper and lower continuity of these multifunctions. Also, by examples and counter examples several properties of these multifunctions have been given. Finally we have been applied these multifunctions to the medical diagnosis. Since the soft sets is very useful in information systems, we hoped that these multifunctions will be useful for the many sciences, such as medical, engineering, economy, social science, etc.

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