Some Inequalities for the Polar Derivative of a Polynomial

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***Abstract*— In this paper we establish L^p inequalities for polar derivatives of polynomials not vanishing in $|z|<1$. Also we obtain inequalities for polar derivatives of polynomials satisfying $p(z)=z^n p\left(\frac{1}{z}\right)$. Our results generalize some well-known results in this direction.**

***Keywords*— Inequalities, Polar Derivative, Polynomial.**

I. Introduction

If $p(z)$ is a polynomial of degree at most n, then according to the famous result known as Bernstein's inequality,

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{1.1}$$

Here equality holds for $p(z)=\lambda z^n$, λ being a complex number.

For the class of polynomials not vanishing in $|z|<1$, (1.1) can be sharpened. In this connection we have the following inequality, which was conjectured by Erdös and later verified by Lax [9].

$$\max_{|z|=1}|p'(z)|\le \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.2}$$

Equality in (1.2) holds for $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$.

The L^p analogue of (1.2) was obtained by de-Bruijn [4]. He proved that

Theorem A. If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, then for each $r\ge 1$,

$$\left(\int_0^{2\pi}|p'(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}\le nc_r\left(\int_0^{2\pi}|p(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}, \tag{1.3}$$

where $c_r=\left(\frac{2\pi}{\int_0^{2\pi}|1+e^{i\theta}|^r\,d\theta}\right)^{\frac{1}{r}}$.

This inequality is sharp and equality holds for $p(z)=\lambda+\mu z^n$ where $|\lambda|=|\mu|$.

Let $p(z)$ be a polynomial of degree n and α be any real or complex number. Then the polar derivative of $p(z)$, denoted by $D_\alpha p(z)$, is defined as

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z).$$

The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ of $p(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p'(z)$$

The polynomial $D_\alpha p(z)$ is called by Laguerre [8] the "émanant" of $p(z)$, by Pólya and Szegö [11] the "derivative of $p(z)$ with respect to the point α" and by Marden [10] simply "the polar derivative of $p(z)$".

In this paper we firstly obtain integral inequalities for the polar derivative of a polynomial. In this direction, we prove the following generalizations of de-Bruijn's Theorem [4] to the polar derivative.

Theorem 1. If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, then for every real or complex number α with $|\alpha|\geq 1$ and for every given complex number β with $|\beta|\leq 1$, we have for $r\geq 1$,

$$\left(\int_0^{2\pi}|D_\alpha\{p(e^{i\theta})\}+\alpha nm\beta e^{i(n-1)\theta}|^r\,d\theta\right)^{\frac{1}{r}}\leq n(|\alpha|+1)E_r\left(\int_0^{2\pi}|p(e^{i\theta})+m\beta e^{in\theta}|^r\,d\theta\right)^{\frac{1}{r}},\tag{1.4}$$

where $E_r=\left(\frac{2\pi}{\int_0^{2\pi}|1+e^{i\theta}|^r\,d\theta}\right)^{\frac{1}{r}}$ and $m=\min_{|z|=1}|p(z)|$.

Dividing both sides of (1.4) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get the following generalization of Theorem A due to Aziz [3].

Corollary 1. If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, then for every given complex number β with $|\beta|\leq 1$, we have for $r\geq 1$,

$$\left(\int_0^{2\pi}|p'(e^{i\theta})+mn\beta e^{i(n-1)\theta}|^r\,d\theta\right)^{\frac{1}{r}}\leq nE_r\left(\int_0^{2\pi}|p(e^{i\theta})+m\beta e^{in\theta}|^r\,d\theta\right)^{\frac{1}{r}}\tag{1.5}$$

where E_r is as in Theorem 1.

Letting $r\to\infty$ in (1.4) and choosing the argument of β, with $|\beta|=1$, suitably we get the following inequality for polynomials not vanishing in $|z|<1$, which improves upon a result due to Aziz [2]

$$\max_{|z|=1}|D_\alpha p(z)|+|\alpha|nm\leq\frac{n}{2}(|\alpha|+1)[\max_{|z|=1}|p(z)|+m].$$

Equivalently,

$$\max_{|z|=1}|D_\alpha p(z)|\leq\frac{n}{2}[(|\alpha|+1)\max_{|z|=1}|p(z)|-(|\alpha|-1)m].$$

The above inequality is best possible for the polynomials $p(z)=\lambda+\mu z^n$ where $|\lambda|=|\mu|$.

Theorem 2. If $p(z)$ is a polynomial of degree n, having all its zeros in $|z|\geq 1$ and s -fold zeros at $z=0$, then for every real or complex number α with $|\alpha|\geq 1$ and for $r\geq 1$,

$$\left(\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}\leq\{|\alpha|s+(n-s)(|\alpha|+1)F_r\}\left(\int_0^{2\pi}|p(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}\tag{1.6}$$

where $F_r=\left(\frac{2\pi}{\int_0^{2\pi}|1+e^{i\theta}|^r\,d\theta}\right)^{\frac{1}{r}}$.

Letting $r\to\infty$ in (1.6), we get

Corollary 2. If a polynomial $p(z)$ has all zeros in $|z|\geq 1$ and a zero of order s at $z=0$, then for every real or complex number α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha p(z)|\leq\left[|\alpha|s+\frac{(n-s)(|\alpha|+1)}{2}\right]\max_{|z|=1}|p(z)|.\tag{1.7}$$

(1.7) is best possible. Equality holds for $p(z)=z^s(z+1)^{n-s}$.

For $s=0$, Corollary 2 reduces to a result due to Aziz [2].

Dividing both sides of (1.6) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get

Corollary 3. If $p(z)$ is a polynomial of degree n, having all its zeros in $|z|\geq 1$, with s-fold zeros at origin, then for $r\geq 1$,

$$\left(\int_0^{2\pi}|p'(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}\leq G_r\left(\int_0^{2\pi}|p(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}, \tag{1.8}$$

where $G_r=s+\frac{n-s}{\left\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\theta}|^r\,d\theta\right\}^{\frac{1}{r}}}$.

For $s=0$, Corollary 3 reduces to Theorem A due to de-bruijn [4].

Finally, we consider the class of self-reciprocal polynomials i.e. the polynomials satisfying $p(z)\equiv z^n p(1/z)$ and prove the following result for the polar derivatives of polynomials.

Theorem 3. If $p(z)$ is a polynomial of degree n satisfying $p(z)=z^n p(1/z)$ with all its coefficients lying in a sector of opening at most γ where $0\leq\gamma\leq 2\pi/3$, then for every real or complex number α,

$$\max_{|z|=1}|D_\alpha p(z)|\leq\frac{n(|\alpha|+1)}{2\cos\frac{\gamma}{2}}\max_{|z|=1}|p(z)|. \tag{1.9}$$

For $\alpha=0$ the above result is best possible for $0\leq\gamma\leq\pi/2$ with equality holding for the polynomial $p(z)=z^n+2e^{i\gamma}z^{n/2}+1$, n being even.

Dividing both sides of (1.9) by $|\alpha|$ and Lemma 2 making $|\alpha|\to\infty$, Theorem 3 reduces to a result due to Govil and Vetterlein [6]. This result can also be obtained directly from (1.9) by putting $\alpha=0$.

For $\gamma=\frac{\pi}{2}$, Theorem 3 extends a result of Jain [7] to the polar derivative.

II. LEMMAS

For the proofs of the theorems we required following lemmas.

Lemma 1. If $p(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then for $r\geq 1$ and for every real or complex number α with $|\alpha|\geq 1$,

$$\left(\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}\leq n(|\alpha|+1)F_r\left(\int_0^{2\pi}|p(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}, \tag{2.1}$$

where $F_r=\left(\frac{2\pi}{\int_0^{2\pi}|1+e^{i\theta}|^r\,d\theta}\right)^{\frac{1}{r}}$.

In the limiting case, when $r\to\infty$, the above inequality is sharp and equality holds for the polynomial $p(z)=\lambda+\mu z^n$, $|\lambda|=|\mu|$.

The above lemma was proved by Govil, Nyuydinkong and Tameru [5].

Lemma 2. If $p(z)$ is a polynomial satisfying $p(z)=z^n p(1/z)$, with all its coefficients lying in a sector of opening at most γ where $0\leq\gamma\leq 2\pi/3$, then

$$\max_{|z|=1}|p'(z)|\le \frac{n}{2\cos\frac{\gamma}{2}}\max_{|z|=1}|p(z)|. \qquad (2.2)$$

This lemma is due to Govil and Vetterlein [6].

III. Proofs of the Theorems

Proof of Theorem 1. By hypothesis, the polynomial $p(z)$ does not vanish in $|z|<1$ and $m=\min_{|z|=1}|p(z)|$, therefore, $m\le|p(z)|$ for $|z|\le 1$. We first show that for any given complex number β with $|\beta|\le 1$, the polynomial $G(z)=p(z)+m\beta z^n$ has no zeros in $|z|<1$. If $m=0$, then this is obvious. Therefore, for $m>0$ let us assume that $G(z)$ has a zero in $|z|<1$, say at $z=z_0$ with $|z_0|<1$, then we have $p(z_0)+m\beta z_0^n=G(z_0)=0$. This gives $|p(z_0)|=|m\beta z_0^n|\le m|z_0|^n<m$, which is a contradiction. Hence $G(z)$ has no zeros in $|z|<1$ for every β with $|\beta|\le 1$. Applying Lemma 1 to the polynomial $G(z)$, we get

$$\left(\int_0^{2\pi}|D_\alpha\ p(e^{i\theta})+m\beta e^{in\theta}\ |^r\,d\theta\right)^{\frac{1}{r}}\le n(|\alpha|+1)E_r\left(\int_0^{2\pi}|p(e^{i\theta})+m\beta e^{in\theta}|^r\,d\theta\right)^{\frac{1}{r}}$$

where $E_r=\left(\frac{2\pi}{\int_0^{2\pi}|1+e^{i\theta}|^r\,d\theta}\right)^{\frac{1}{r}}$

or

$$\left(\int_0^{2\pi}|n\ \ p(e^{i\theta})+m\beta e^{in\theta}\ \ +(\alpha-e^{i\theta})\{p'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\}|^r\,d\theta\right)^{\frac{1}{r}}\le n(|\alpha|+1)E_r\left(\int_0^{2\pi}|p(e^{i\theta})+m\beta e^{in\theta}|^r\,d\theta\right)^{\frac{1}{r}}$$

or

$$\left(\int_0^{2\pi}|D_\alpha\ \ p(e^{i\theta})\ \ +\alpha nm\beta e^{i(n-1)\theta}\ |^r\,d\theta\right)^{\frac{1}{r}}\le n(|\alpha|+1)E_r\left(\int_0^{2\pi}|p(e^{i\theta})+m\beta e^{in\theta}|^r\,d\theta\right)^{\frac{1}{r}}$$

which is (1.4). Hence the Theorem is proved.

Proof of Theorem 2. By hypothesis, the polynomial $p(z)$ has s zeros at the origin and $n-s$ zeros in $|z|\ge 1$, therefore we can take

$$p(z)=z^s h(z), \qquad (3.1)$$

where $h(z)$ is a polynomial of degree $n-s$, having all zeros in $|z|\ge 1$. Now

$$\begin{aligned}D_\alpha p(z)&=np(z)+(\alpha-z)p'(z)=n[z^s h(z)]+(\alpha-z)[sz^{s-1}h(z)+z^s h'(z)]\\&=z^s\ [(n-s)h(z)+(\alpha-z)h'(z)]+\alpha s\,z^{s-1}h(z),\end{aligned}$$

which implies for $0\le\theta<2\pi$

$$\begin{aligned}|D_\alpha p(e^{i\theta})|&=|e^{is\theta}[(n-s)h(e^{i\theta})+(\alpha-e^{i\theta})h'(e^{i\theta})]+\alpha s e^{i(s-1)\theta}h(e^{i\theta})|\\&=|e^{i\theta}D_\alpha\{h(e^{i\theta})\}+\alpha s h(e^{i\theta})|.\end{aligned}$$

This gives by using Minkowski's inequality for every $r\ge 1$,

$$\begin{aligned}\left(\int_0^{2\pi}|D_\alpha\ \ p(e^{i\theta})\ |^r\,d\theta\right)^{\frac{1}{r}}&=\left(\int_0^{2\pi}|\alpha s h(e^{i\theta})+e^{i\theta}D_\alpha\ \ h(e^{i\theta})\ |^r\,d\theta\right)^{\frac{1}{r}}\\&\le\left(\int_0^{2\pi}s^r\,|\alpha|^r|h(e^{i\theta})|^r\,d\theta\right)^{\frac{1}{r}}+\left(\int_0^{2\pi}|D_\alpha\ \ h(e^{i\theta})\ |^r\,d\theta\right)^{\frac{1}{r}}.\end{aligned} \qquad (3.2)$$

Now since $h(z)$ is a polynomial of degree $(n-s)$, having all zeros in $|z|\ge 1$, therefore, applying Lemma 1 to the polynomial $h(z)$, we get

$$\left(\int_0^{2\pi} |D_\alpha \; h(e^{i\theta}) \;|^r \, d\theta\right)^{\frac{1}{r}} \le (n-s)(|\alpha|+1)F_r\left(\int_0^{2\pi} | \; h(e^{i\theta}) \;|^r \, d\theta\right)^{\frac{1}{r}} \quad (3.3)$$

where $F_r = \left(\frac{2\pi}{\int_0^{2\pi}|1+e^{i\theta}|^r\, d\theta}\right)^{\frac{1}{r}}$.

Combining (3.2) and (3.3) and noting that

$|h(e^{i\theta})| = |e^{is\theta}h(e^{i\theta})| = |p(e^{i\theta})|$ for $|z|=1$, we get for every $r \ge 1$,

$$\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^r \, d\theta\right)^{\frac{1}{r}} \le \{|\alpha|s + (n-s)(|\alpha|+1)F_r\}\left(\int_0^{2\pi} |p(e^{i\theta})|^r \, d\theta\right)^{\frac{1}{r}}.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Since $p(z)$ is a self-reciprocal polynomial of degree n, we have

$p(z) = z^n p(1/z)$ for all $z \in \mathbb{C}$.

This implies

$$z^{n-1}p'(1/z) = np(z) - zp'(z),$$

which in particular gives

$$\max_{|z|=1} |p'(z)| = \max_{|z|=1} |z^{n-1}p'(1/z)| = \max_{|z|=1} |np(z) - zp'(z)|.$$

Now for $|z|=1$

$$|D_\alpha p(z)| = |np(z) + (\alpha - z)p'(z)| \le |\alpha||p'(z)| + |np(z) - zp'(z)|.$$

Equivalently,

$$\max_{|z|=1} |D_\alpha p(z)| \le |\alpha| \max_{|z|=1} |p'(z)| + \max_{|z|=1} |np(z) - zp'(z)| = (|\alpha|+1)\max_{|z|=1} |p'(z)|.$$

Combining above inequality with (2.2) of Lemma 2, we conclude that'

$$\max_{|z|=1} |D_\alpha p(z)| \le \frac{n(|\alpha|+1)}{2\cos\frac{\gamma}{2}} \max_{|z|=1} |p(z)|.$$

This proves the desired result.

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