On the Cublic Equation with Four Unknowns

$$x^3+4z^3=y^3+4w^3+6(x-y)^3$$

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Abstract:

The sequences of integral solutions to the cubic equation with four variables are obtained. A few properties among the solutions are also presented.

Index Terms: Cubic equation having four unknowns with integral solutions.

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I. INTRODUCTION

The Diophantine equation offer an unlimited field for research due to their variety [1-2]. In particular, one may refer [3-14] for cubic equation with three unknowns. In [15-19] cubic equation with four unknowns are studied for its non-trivial integral solutions. This communication concerns with the problem of obtaining non-zer integral solution of cubic with four variables is given by $x^3+4z^3=y^3+4w^3+6(x-y)^3$.A few properties among the solutions and special numbers are presented.

II. NOTATIONS

$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$

$p_n^m=\frac{n(n+1)}{6}[n(m-2)+5-m)]$

$pr_n=n(n+1)$

$S_n=6n(n-1)+1$

$SO_n=n(2n^2-1)$

$j_n=2^n+(-1)^n$

$J_n=\frac{1}{3}[2^n+(-1)^n]$

III. METHOD OF ANALYSIS

The cubic Diophantine equation with four unknowns to be solved for getting non-zero integral solution is $\quad x^3+4z^3=y^3+4w^3+6(x-y)^3 \quad (1)$

on substituting the linear transformation

$$x = u + v, \quad y = u - v, \qquad w = s + v, \quad z = s - v \tag{2}$$

in (1) leads to

$$u^2 = 9v^2 + 4s^2 \tag{3}$$

We obtain different choices of integral solution to (1) through solving (3) which are illustrated as follows:

Choice 1 :

In equation (3), which is satisfied by

$$v = \frac{2}{3}pq$$

$$s = \frac{p^2 - q^2}{2}$$

$$u = p^2 + q^2$$

Put $p = 6P$ & $q = 6Q$ in v, s, u we get

$$v = 24PQ$$

$$s = 18(P^2 - Q^2)$$

$$u = 36(P^2 + Q^2)$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x(P,Q) = 36[P^2 + Q^2] + 24PQ$$

$$y(P,Q) = 36[P^2 + Q^2] - 24PQ$$

$$w(P,Q) = 18[P^2 - Q^2] + 24PQ$$

$$z(P,Q) = 18[P^2 - Q^2] - 24PQ$$

Properties:

1. $6\left[\frac{x(P,P+1) + y(P,P+1) - 72}{pr_P}\right]$ is a Nasty Number.
2. $x(P, t_{3,P}) - y(P, t_{3,P}) = 48p_P^5$
3. $\frac{w(P,P+1) + z(P,P+1)}{36} \equiv 1(md - 2)$

Choice 2:

Equation (3) can be re-written as

$$u^2 - (3v)^2 = (2s)^2$$

which is written in the form of ratio as,

$$\frac{u+3v}{2s} = \frac{2s}{u-3v} = \frac{\alpha}{\beta}, \quad \beta \neq 0 \tag{4}$$

which is equivalent to the system of equations,

$$u\beta + 3\beta v - 2\alpha s = 0$$

$$-u\alpha + 3\alpha v + 2\beta s = 0$$

applying the method of cross multiplication, we have

$$u = 6\alpha^2 + 6\beta^2$$
$$v = 2\alpha^2 - 2\beta^2$$
$$s = 6\alpha\beta$$

in view of (2) the non-zero integer solutions to (1) are given by

$$x(\alpha,\beta) = 8\alpha^2 + 4\beta^2$$

$$y(\alpha,\beta) = 4\alpha^2 + 8\beta^2$$

$$w(\alpha,\beta) = 2\alpha^2 - 2\beta^2 + 6\alpha\beta$$

$$z(\alpha,\beta) = 2\beta^2 - 2\alpha^2 + 6\alpha\beta$$

Properties:

1. $x(\alpha,\alpha-1) + y(\alpha,\alpha-1) + w(\alpha,\alpha-1) - t_{50.\alpha} - S_\alpha \equiv 0 \pmod 8$
2. $w(\alpha, t_{3,\alpha}) + z(\alpha, t_{3,\alpha}) \equiv 12p_\alpha^5$
3. $w(\alpha,\alpha-1) + z(\alpha,\alpha-1) + 1 = 2\,S_\alpha$

Choice 3:

Equation (4) can also be written as

$$\frac{u+3v}{4s} = \frac{s}{u-3v} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

which is equivalent to the system of equations,

$$u\beta + 3\beta v - \alpha s = 0$$

$$-u\alpha + 3\alpha v + 4\beta s = 0$$

applying the method of cross multiplication, we have

$$u = 12\beta^2 + 3\alpha^2$$
$$v = \alpha^2 - 4\beta^2$$
$$s = 6\alpha\beta$$

in view of (2) the non-zero integer solutions to (1) are given by

$$x(\alpha,\beta) = 4\alpha^2 + 8\beta^2$$

$$y(\alpha,\beta) = 2\alpha^2 + 16\beta^2$$

$$w(\alpha,\beta) = \alpha^2 - 4\beta^2 + 6\alpha\beta$$

$$z(\alpha,\beta) = 4\beta^2 - \alpha^2 + 6\alpha\beta$$

Properties:

1. $x(2^n,1) + y(2^n,1) - 30 = 72\, J_{2n}$
2. $w(\alpha,\beta) - z(\alpha,\beta) - y(\alpha,\beta) + 6pr_\beta \equiv 0 \ (mod\ 6)$
3. $x(2^n+1\,,1) - y(2^n+1\,,1) = 8Ky_n + 14$

Choice 4:

Equation (4) can also be written as

$$\frac{u+3v}{s} = \frac{4s}{u-3v} = \frac{\alpha}{\beta}\,, \quad \beta \neq 0$$

which is equivalent to the system of equations,

$$u\beta + 3\beta v - 4\alpha s = 0$$

$$-u\alpha + 3\alpha v + \beta s = 0$$

applying the method of cross multiplication, we have

$$u = 12\alpha^2 + 3\beta^2$$
$$v = 4\alpha^2 - \beta^2$$
$$s = 6\alpha\beta$$

in view of (2) the non-zero integer solutions to (1) are given by

$$x(\alpha,\beta) = 16\alpha^2 + 2\beta^2$$

$$y(\alpha,\beta) = 8\alpha^2 + 4\beta^2$$

$$w(\alpha,\beta) = 4\alpha^2 - \beta^2 + 6\alpha\beta$$

$$z(\alpha,\beta) = \beta^2 - 4\alpha^2 + 6\alpha\beta$$

Properties:

1. $w(\alpha, t_{3,\alpha}) + z(\alpha, t_{3,\alpha}) \equiv 12p_\alpha^5$
2. $x(n,n+1) + y(n,n+1) + w(n,n+1) + z(n,n+1) - t_{78,n} - 12pr_n \equiv -62 \pmod{86}$
3. $x(2^n,1) - y(2^n,1) - w(2^n,1) - z(2^n,1) - 3J_{2n} - j_{2n} = 8$

Choice 5:

Equation (3) can be re-written as

$$u^2 - (2s)^2 = (3v)^2$$

which is written in the form of ratio as,

$$\frac{u+2s}{3v} = \frac{3v}{u-2s} = \frac{\alpha}{\beta}, \quad \beta \neq 0 \tag{5}$$

which is equivalent to the system of equations,

$$u\beta - 3\alpha v + 2\beta s = 0$$

$$-u\alpha + 3\beta v + 2\alpha s = 0$$

applying the method of cross multiplication, we have

$$u = 6\alpha^2 + 6\beta^2$$
$$s = 3\alpha^2 - 3\beta^2$$

$$v = 4\alpha\beta$$

in view of (2) the non-zero integer solutions to (1) are given by

$$x(\alpha,\beta) = 6\alpha^2 + 6\beta^2 + 4\alpha\beta$$

$$y(\alpha,\beta) = 6\alpha^2 + 6\beta^2 - 4\alpha\beta$$

$$w(\alpha,\beta) = 3\alpha^2 - 3\beta^2 + 4\alpha\beta$$

$$z(\alpha,\beta) = 3\alpha^2 - 3\beta^2 - 4\alpha\beta$$

Properties:

1. $x(2^{2a-1},1) + y(2^{2a-1},1) = 12j_{4a-2}$
2. $x(2^n,1) + y(2^n,1) - w(2^n,1) + z(2^n,1) - 24 = 36\,J_{2n} - 12J_n - 4j_n$
3. $6p_\alpha^5\left[x(\alpha,t_{3,\alpha}) + w(\alpha,t_{3,\alpha}) - y(\alpha,t_{3,\alpha}) - z(\alpha,t_{3,\alpha})\right]$ is a Nasty Number.

Choice 6:

Equation (5) can also be written as

$$\frac{u+2s}{v} = \frac{9v}{u-2s} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

which is equivalent to the system of equations,

$$u\beta - \alpha v + 2\beta s = 0$$

$$-u\alpha + 9\beta v + 2\alpha s = 0$$

applying the method of cross multiplication, we have

$$u = 2\alpha^2 + 18\beta^2$$
$$s = \alpha^2 - 9\beta^2$$
$$v = 4\alpha\beta$$

in view of (2) the non-zero integer solutions to (1) are given by

$$x(\alpha,\beta) = 2\alpha^2 + 18\beta^2 + 4\alpha\beta$$

$$y(\alpha,\beta) = 2\alpha^2 + 18\beta^2 - 4\alpha\beta$$

$$w(\alpha,\beta) = \alpha^2 - 9\beta^2 + 4\alpha\beta$$

$$z(\alpha,\beta) = \alpha^2 - 9\beta^2 - 4\alpha\beta$$

Properties:

1. $x(\alpha,\beta) - y(\alpha,\beta) = w(\alpha,\beta) - z(\alpha,\beta)$
2. $48p_\alpha^5[w(\alpha,t_{3,\alpha}) - z(\alpha,t_{3,\alpha})]$ is a Nasty Number.
3. $w(2^n,1) + z(2^n,1) + 18 = 3\,J_{2n} + j_{2n}$.
4. $48pr_n[x(n,n+1) + y(n,n+1)]$ is a Nasty Number.

Choice 7:

Equation (5) can also be written as

$$\frac{u+2s}{9v} = \frac{v}{u-2s} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

which is equivalent to the system of equations,

$$u\beta - 9\alpha v + 2\beta s = 0$$

$$-u\alpha + \beta v + 2\alpha s = 0$$

applying the method of cross multiplication, we have

$$u = 2\beta^2 + 18\alpha^2$$
$$s = 9\alpha^2 - \beta^2$$
$$v = 4\alpha\beta$$

in view of (2) the non-zero integer solutions to (1) are given by

$$x(\alpha,\beta) = 18\alpha^2 + 2\beta^2 + 4\alpha\beta$$

$$y(\alpha,\beta) = 18\alpha^2 + 2\beta^2 + 4\alpha\beta$$

$$w(\alpha,\beta) = 9\alpha^2 - \beta^2 + 4\alpha\beta$$

$$z(\alpha,\beta) = 9\alpha^2 - \beta^2 - 4\alpha\beta$$

Properties:

1. $x(2^n,1) + y(2^n,1) + w(2^n,1) + z(2^n,1) - 56 = 162\,J_{2n}$.
2. $[x(2^n,1) + y(2^n,1)] - [w(2^n,1) + z(2^n,1)] - 12 = 6TK_{2n}$.

3. $6[y(t_{3,\alpha}, 1) - w(t_{3,\alpha}, 1) - 3 + 8t_{3,\alpha}]$ is a Nasty Number.

Choice 8 :

Equation (3) can be re-written as

$$u^2 - (3v)^2 = 4s^2$$

which is equivalent to the system of equations,

$$\left.\begin{aligned} u + 3v &= 4s \\ u - 3v &= s \end{aligned}\right\} \qquad (6)$$

Solving these two linear equation, we get

$$u = \frac{5s}{2}$$

$$v = \frac{s}{2}$$

put s=2k then we get the integer solution of u and v are as

$$u = 5k$$

$$v = k$$

Substituting the values of u, v, s in (2), we get the non-trivial integer solutions of equation (1) are given by

$$x = 6k$$

$$y = 4k$$

$$w = 3k$$

$$z = k$$

Choice 9:

The system of equation can be written as

$$\left.\begin{aligned} u + 3v &= s^2 \\ u - 3v &= 4 \end{aligned}\right\} \qquad (7)$$

Solving these two equation, we get

$$u = \frac{s^2+4}{2} \quad , \quad v = \frac{s^2-4}{2}$$

Taking $s = k^2 + k + 2$ in the above equations, we get

$$u = \frac{(k^2+k)^2}{2} + 2\,(k^2+k) + 4$$

$$v = \frac{(k^2+k)(k^2+k+2)}{6}$$

in view of (2) the non-zero integer solutions to (1) are given by

$$x = \frac{4k^4+8k^3+20k^2+16k+24}{6}$$

$$y = \frac{k^4+4k^3+10k^2+8k+24}{6}$$

$$w = \frac{k^4+2k^3+11k^2+10k+12}{6}$$

$$z = \frac{-k^4-2k^3+k^2+2k+12}{6}$$

Properties:

1. $z+w = 4t_{3,k}+4$,
2. $2z+2w-7 = ct_{8,k}$.
3. $2y-x=4$.
4. $2w-y=4t_{3,k}$.
5. $4z+2y-15 = ct_{8,k}$.
6. $y+z=w+4$.

CONCLUSION:

One may search for other Choices of solutions and their corresponding properties.

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