Closed form continued fraction expansions for the powers of Generalized Lucas golden proportion

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***Abstract*— Generalized Lucas sequence $\left\{L_n^{(a,b)}\right\}$ is defined by the recurrence relation $L_n^{(a,b)} = aL_{n-1}^{(a,b)} + bL_{n-2}^{(a,b)}$ for $n \geq 2$ with initial conditions $L_0^{(a,b)} = 2, L_1^{(a,b)} = a$ and a, b are positive integers. We define generalized golden proportion for this sequence by $\phi_{a,b} = \lim_{n\to\infty} \frac{L_n^{(a,b)}}{L_{n-1}^{(a,b)}}$. In this paper by using elementary properties of the recurrence relation and the continued fractions we find the closed form continued fraction expansion for $\phi_{a,1}^k$ and $\phi_{a,-1}^k$, for any positive integer k.**

***Keywords*— Lucas sequence, Generalized Lucas sequence, Continued fraction, Golden proportion.**

I. Introduction

Continued fractions provide deep insight into mathematical problems; particularly into the nature of numbers. Continued fractions have found applications in various areas of Physics such as Fabry-Perot interferometry, quasi-amorphous states of matter and chaos. It encodes much useful information about the algebraic structure of a number and frequently arises in approximation theory and dynamical systems. Van der Poorten [1- 4] wrote that the elementary nature and simplicity of the theory of continued fractions is mostly buried in the literature. Our work is an outgrowth of [5, 6, 9]. We refer the readers to these papers for some basic background information on continued fractions, and to the books [7, 8, 10, 11] for more details.

It is known that every real number α has a continued fraction expansion

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}},$$

Where each a_i is an integer (and a positive integer unless $i = 0$). For brevity we write $\alpha = [a_0, a_1, a_2, \dots]$. Clearly, α is rational if and only if its continued fraction is finite, and a beautiful theorem of Lagrange asserts that α is a quadratic irrational if and only if the continued fraction expansion is periodic.

The continued fraction expansion consisting of the number 1 repeated indefinitely represents the '*golden mean*'. This satisfies the quadratic equation $x^2 = x + 1$. The convergents of this continued fraction are obtained as the ratio of the successive terms of the Fibonacci sequence [9, 10]. Recall that the Fibonacci sequence $\{F_n\}$, named after Leonardo Pisano Fibonacci (1170–1250), is defined as $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2};\ n \geq 2$, which gives the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 … .The Lucas numbers $\{L_n\}$, named after François Lucas (1842–1891), are defined by $L_0 = 2, L_1 = 1$ and the recurrence relation $L_n = L_{n-1} + L_{n-2}; n \geq 2$. First few terms of the sequence are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123,...

We define the generalized Fibonacci sequence $\left\{F_n^{(a,\,b)}\right\}$ by the general difference equation of the form $F_n^{(a,\,b)} = a\,F_{n-1}^{(a,\,b)} + b\,F_{n-2}^{(a,\,b)}; n \geq 2$ with the initial conditions $F_0^{(a,\,b)} = F_1^{(a,\,b)} = 1$.

It is known [2] that extended Binet formula for this sequence is of the form

$$F_n^{(a,\,b)} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{;where } \alpha = \frac{a + \sqrt{a^2+4b}}{2} \text{ and } \beta = \frac{a - \sqrt{a^2+4b}}{2}. \qquad (1)$$

We also define generalized Lucas sequence $\left\{L_n^{(a,\,b)}\right\}$ by the recurrence relation $L_n^{(a,\,b)} = aL_{n-1}^{(a,\,b)} + bL_{n-2}^{(a,\,b)}; n \geq 2$ with $L_0^{(a,\,b)} = 2, L_1^{(a,\,b)} = a$.

II. Theorems and Properties of Generalized Lucas Sequences

We first derive the extended Binet's formula for $L_n^{(a,\,b)}$.

Theorem 2.1: Extended Binet's formula for the terms of generalized Lucas sequence $\left\{L_n^{(a,\,b)}\right\}$ is given by $L_n^{(a,\,b)} = \left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n + \left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n = \alpha^n + \beta^n$.

***Proof*:** We prove the result by induction.

We have $L_1^{(a,\,b)} = \frac{a+\sqrt{a^2+4b}}{2} + \frac{a-\sqrt{a^2+4b}}{2} = a$, proving the result for $n = 1$. Assume that it holds for all integers up to *k*. i.e. $L_k^{(a,\,b)} = \alpha^k + \beta^k$ and $L_{k-1}^{(a,\,b)} = \alpha^{k-1} + \beta^{k-1}$ holds. This gives

$$L_{k+1}^{(a,\,b)} = a\,L_k^{(a,\,b)} + b\,L_{k-1}^{(a,\,b)} = a\,[\alpha^k + \beta^k] + b\,[\alpha^{k-1} + \beta^{k-1}]$$
$$= \alpha^{k-1}(a\alpha + b) + \beta^{k-1}(a\beta + b)$$

We note that $a\alpha + b = a\left(\frac{a+\sqrt{a^2+4b}}{2}\right) + b = \left(\frac{a+\sqrt{a^2+4b}}{2}\right)^2 = \alpha^2$ and $a\beta + b = \beta^2$.

Thus $L_{k+1}^{(a,\,b)} = (\alpha)^{k-1}\alpha^2 + (\beta)^{k-2}\beta^2 = \alpha^{k+1} + \beta^{k+1}$.

This proves the result for $k + 1$, which proves the required result.

Here we note that $\alpha\beta = -b$.

Corollary 2.2: $\lim_{n\to\infty} \frac{L_n^{(a,\,b)}}{L_{n-1}^{(a,\,b)}} = \frac{a \pm \sqrt{a^2+4b}}{2}$.

***Proof*:** Assume that $\lim_{n\to\infty} \frac{L_n^{(a,\,b)}}{L_{n-1}^{(a,\,b)}} = x$. This gives $x = \lim_{n\to\infty} \frac{aL_{n-1}^{(a,\,b)} + bL_{n-2}^{(a,\,b)}}{L_{n-1}^{(a,\,b)}}$

$x = a + \frac{b}{x} \Rightarrow x^2 - ax - b = 0$. This gives $x = \frac{a \pm \sqrt{a^2+4b}}{2}$, as required.

We now derive a result which connects generalized Fibonacci numbers and Generalized Lucas numbers.

Lemma 2.3: $L_n^{(a,\,b)} = F_{k+1}^{(a,\,b)} L_{n-k}^{(a,\,b)} + b\,F_k^{(a,\,b)} L_{n-k-1}^{(a,\,b)}$.

Proof: First we express $L_n^{(a,\,b)} = \alpha^n + \beta^n$ in terms of any two previous consecutive terms, viz. $L_{n-k}^{(a,\,b)} = \alpha^{n-k} + \beta^{n-k}$ and $L_{n-k-1}^{(a,\,b)} = \alpha^{n-k-1} + \beta^{n-k-1}$. This can be written as

$$\begin{pmatrix} L_{n-k}^{(a,\,b)} \\ L_{n-k-1}^{(a,\,b)} \end{pmatrix} = \begin{pmatrix} \alpha^{n-k} & \beta^{n-k} \\ \alpha^{n-k-1} & \beta^{n-k-1} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix};$$

where $C_1 = C_2 = 1$. This gives

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \alpha^{n-k} & \beta^{n-k} \\ \alpha^{n-k-1} & \beta^{n-k-1} \end{pmatrix}^{-1} \begin{pmatrix} L_{n-k}^{(a,\,b)} \\ L_{n-k-1}^{(a,\,b)} \end{pmatrix}$$

$$= \frac{1}{\alpha^{n-k}\beta^{n-k-1} - \alpha^{n-k-1}\beta^{n-k}} \begin{pmatrix} \beta^{n-k-1} & -\beta^{n-k} \\ -\alpha^{n-k-1} & \alpha^{n-k} \end{pmatrix} \begin{pmatrix} L_{n-k}^{(a,\,b)} \\ L_{n-k-1}^{(a,\,b)} \end{pmatrix}$$

$$= \frac{1}{(\alpha\beta)^{n-k-1}(\alpha-\beta)} \begin{pmatrix} L_{n-k}^{(a,\,b)}\beta^{n-k-1} - L_{n-k-1}^{(a,\,b)}\beta^{n-k} \\ -L_{n-k}^{(a,\,b)}\alpha^{n-k-1} + L_{n-k-1}^{(a,\,b)}\alpha^{n-k} \end{pmatrix}$$

Thus

$$C_1 = \frac{L_{n-k}^{(a,\,b)}\beta^{n-k-1} - L_{n-k-1}^{(a,\,b)}\beta^{n-k}}{(\alpha\beta)^{n-k-1}(\alpha-\beta)},\; C_2 = \frac{-L_{n-k}^{(a,\,b)}\alpha^{n-k-1} + L_{n-k-1}^{(a,\,b)}\alpha^{n-k}}{(\alpha\beta)^{n-k-1}(\alpha-\beta)}.$$

This gives

$$C_1 = \frac{L_{n-k}^{(a,\,b)} - L_{n-k-1}^{(a,\,b)}\beta}{\alpha^{n-k-1}(\alpha-\beta)},\; C_2 = \frac{-L_{n-k}^{(a,\,b)} + L_{n-k-1}^{(a,\,b)}\alpha}{\beta^{n-k-1}(\alpha-\beta)}.$$

Using these values in $L_n^{(a,\,b)} = C_1\alpha^n + C_2\beta^n$, we get

$$L_n^{(a,\,b)} = \frac{(L_{n-k}^{(a,\,b)} - L_{n-k-1}^{(a,\,b)}\beta)\alpha^n}{\alpha^{n-k-1}(\alpha-\beta)} + \frac{(-L_{n-k}^{(a,\,b)} + L_{n-k-1}^{(a,\,b)}\alpha)\beta^n}{\beta^{n-k-1}(\alpha-\beta)}$$

$$= \frac{\alpha^{k+1}(L_{n-k}^{(a,b)} - \beta L_{n-k-1}^{(a,b)})}{(\alpha-\beta)} + \frac{\beta^{k+1}(-L_{n-k}^{(a,\,b)} + \alpha\, L_{n-k-1}^{(a,\,b)})}{(\alpha-\beta)}$$

$$= L_{n-k}^{(a,b)}\left(\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha-\beta}\right) - \alpha\beta\, L_{n-k-1}^{(a,b)}\left(\frac{\alpha^k - \beta^k}{\alpha-\beta}\right).$$

Since $\alpha\beta = -b$, we finally get

$$L_n^{(a,\,b)} = F_{k+1}^{(a,\,b)} L_{n-k}^{(a,\,b)} + b\,F_k^{(a,\,b)} L_{n-k-1}^{(a,\,b)}.$$

Lemma 2.4: $F_{n-2}^{(a,b)} F_n^{(a,b)} - \left(F_{n-1}^{(a,b)}\right)^2 = (-1)^{n-1}b^{n-2}$.

Proof: We use (1) to write

$$F_{n-2}^{(a,b)} F_n^{(a,b)} - \left(F_{n-1}^{(a,b)}\right)^2 = (C_1\alpha^{n-2} + C_2\beta^{n-2})(C_1\alpha^n + C_2\beta^n) - (C_1\alpha^{n-1} + C_2\beta^{n-1})^2,$$

Where $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$, $\beta = \frac{a-\sqrt{a^2+4b}}{2}$ and $C_1 = \frac{1}{\sqrt{a^2+4b}}$, $C_1 = -C_2$.

Then LHS$= C_1^{\,2}[(\alpha^{n-2} - \beta^{n-2})(\alpha^n - \beta^n) - (\alpha^{n-1} - \beta^{n-1})^2]$

$= C_1^{\,2}[\alpha^{2n-2} - \alpha^{n-2}\beta^n - \alpha^n\beta^{n-2} + \beta^{2n-2} - (\alpha^{2n-2} - 2\alpha^{n-1}\beta^{k-1} + \beta^{2n-2})]$

$$= C_1{}^2[-\alpha^{n-2}\beta^{n-2}(\alpha^2+\beta^2)+2\alpha^{n-1}\beta^{n-1}]$$
$$= C_1{}^2[-(\alpha\beta)^{n-2}(\alpha^2+\beta^2)+2(\alpha\beta)^{n-1}]$$

Now, $\alpha\beta = \left(\frac{a+\sqrt{a^2+4b}}{2}\right)\left(\frac{a-\sqrt{a^2+4b}}{2}\right) = \frac{a^2-(a^2+4b)}{4} = -b.$

This gives LHS$= C_1{}^2[-(-b)^{n-2}(\alpha^2+\beta^2)+2(-b)^{n-1}]$

$$= (-1)^{n-1}(-b)^{n-2}C_1{}^2[(\alpha^2+\beta^2)+2b]$$

Substituting back in for C_1, α, β yields:

$$\text{LHS} = (-1)^{n-1}(-b)^{n-2}\left(\frac{1}{a^2+4b}\right)\left(\frac{a^2+\sqrt{a^2+4b}+2b}{2}+\frac{a^2-\sqrt{a^2+4b}+2b}{2}+2b\right)$$
$$= (-1)^{n-1}(-b)^{n-2}\left(\frac{1}{a^2+4b}\right)\left(\frac{2a^2+4b}{2}+2b\right)$$
$$= (-1)^{n-1}(-b)^{n-2}\left(\frac{1}{a^2+4b}\right)(a^2+4b)$$
$$= (-1)^{n-1}(-b)^{n-2}\text{, as required.}$$

III. Continued fractions of Powers of Golden mean

Define $\phi_{a,b} = \lim_{n\to\infty}\frac{L_n^{(a,b)}}{L_{n-1}^{(a,b)}} = \frac{a\pm\sqrt{a^2+4b}}{2}$. We find closed form expressions for the continued fractions of the $\phi_{a,b}^k$ for any integer k and for $b = \pm 1$.As the continued fraction of α is trivially related to that of $\frac{1}{\alpha}$, it suffices to study the case $k > 0$.

Theorem 3.1: $\phi_{a,1}^k = \begin{cases} \left[L_k^{(a,b)}; \overline{L_k^{(a,b)}}\right];\ \textit{if k is odd} \\ \left[L_k^{(a,b)}-1, \overline{1\ , L_k^{(a,b)}-2}\right];\ \textit{if k is even}\end{cases}$

Proof: By lemma 2.3 we have

$$L_n^{(a,b)} = F_{k+1}^{(a,b)}L_{n-k}^{(a,b)} + b\,F_k^{(a,b)}L_{n-k-1}^{(a,b)}.$$

$$\therefore \frac{L_n^{(a,b)}}{L_{n-k}^{(a,b)}} = \frac{F_{k+1}^{(a,b)}L_{n-k}^{(a,b)} + b\,F_k^{(a,b)}L_{n-k-1}^{(a,b)}}{L_{n-k}^{(a,b)}}$$
$$= F_{k+1}^{(a,b)} + b\,F_k^{(a,b)}\frac{L_{n-k-1}^{(a,b)}}{L_{n-k}^{(a,b)}}$$
$$= F_{k+1}^{(a,b)} + \frac{b\left(a\,F_{k-1}^{(a,b)}L_{n-k-1}^{(a,b)} + b\,F_{k-2}^{(a,b)}L_{n-k-1}^{(a,b)}\right)}{L_{n-k}^{(a,b)}}$$
$$= F_{k+1}^{(a,b)} + b\begin{bmatrix}\left(a\,F_{k-1}^{(a,b)}L_{n-k-1}^{(a,b)} + b\,F_{k-1}^{(a,b)}L_{n-k-2}^{(a,b)}\right)\\ +\left(b\,F_{k-2}^{(a,b)}L_{n-k-1}^{(a,b)} - b\,F_{k-1}^{(a,b)}L_{n-k-2}^{(a,b)}\right)\end{bmatrix}\times\left(L_{n-k}^{(a,b)}\right)^{-1}$$

$$= F_{k+1}^{(a,b)} + \frac{b\,F_{k-1}^{(a,b)} L_{n-k}^{(a,b)}}{L_{n-k}^{(a,b)}} + \frac{b\left(b\,F_{k-2}^{(a,b)} L_{n-k-1}^{(a,b)} - b\,F_{k-1}^{(a,b)} L_{n-k-2}^{(a,b)}\right)}{L_{n-k}^{(a,b)}}$$

Since$L_{n-k-1}^{(a,b)} = F_k^{(a,b)} L_{n-k-k}^{(a,b)} + b\,F_{k-1}^{(a,b)} L_{n-k-k-1}^{(a,b)}$ and

$L_{n-k-2}^{(a,b)} = F_{k-1}^{(a,b)} L_{n-k-k-1}^{(a,b)} + b\,F_{k-2}^{(a,b)} L_{n-k-k-2}^{(a,b)}$, the above reduces to

$$\frac{L_n^{(a,b)}}{L_{n-k}^{(a,b)}} = F_{k+1}^{(a,b)} + b\,F_{k-1}^{(a,b)} + b^2 \begin{bmatrix} F_{k-2}^{(a,b)}\left(F_k^{(a,b)} L_{n-2k}^{(a,b)} + b\,F_{k-1}^{(a,b)} L_{n-2k-1}^{(a,b)}\right) \\ - F_{k-1}^{(a,b)}\left(F_{k-1}^{(a,b)} L_{n-2k}^{(a,b)} + b\,F_{k-2}^{(a,b)} L_{n-2k-1}^{(a,b)}\right) \end{bmatrix} \times \left(L_{n-k}^{(a,b)}\right)^{-1}$$

$$= \left(F_{k+1}^{(a,b)} + b\,F_{k-1}^{(a,b)}\right) + \frac{b^2\left(F_{k-2}^{(a,b)} F_k^{(a,b)} L_{n-2k}^{(a,b)} - \left(F_{k-1}^{(a,b)}\right)^2 L_{n-2k}^{(a,b)}\right)}{L_{n-k}^{(a,b)}}$$

But by lemma 2.4 we know that

$$F_{k-2}^{(a,b)} F_k^{(a,b)} - (F_{k-1}^{(a,b)})^2 = (-1)^{\mathrm{k-1}} \mathrm{b}^{\mathrm{k-2}}.$$

And thus,

$$\frac{L_n^{(a,b)}}{L_{n-k}^{(a,b)}} = \left(F_{k+1}^{(a,b)} + bF_{k-1}^{(a,b)}\right) + \frac{L_{n-2k}^{(a,b)} \mathrm{b}^{\mathrm{k}} (-1)^{\mathrm{k-1}}}{L_{n-k}^{(a,b)}}. \qquad (2)$$

Now when we consider$b = 1$, we have$L_n^{(a,b)} = aL_{n-1}^{(a,b)} + L_{n-2}^{(a,b)}$. In this case (2) becomes

$$\frac{L_n^{(a,b)}}{L_{n-k}^{(a,b)}} = L_k^{(a,b)} + \frac{L_{n-2k}^{(a,b)} (-1)^{\mathrm{k-1}}}{L_{n-k}^{(a,b)}}.$$

Now if *k* is odd then we get

$$\frac{L_n^{(a,b)}}{L_{n-k}^{(a,b)}} = L_k^{(a,b)} + \frac{1}{L_{n-k}^{(a,b)}/L_{n-2k}^{(a,b)}}.$$

Also if *k* is even then we get

$$\frac{L_n^{(a,b)}}{L_{n-k}^{(a,b)}} = L_k^{(a,b)} + \frac{-1}{L_{n-k}^{(a,b)}/L_{n-2k}^{(a,b)}}.$$

In this case we manipulate further. We write it as

$$\frac{L_n^{(a,b)}}{L_{n-k}^{(a,b)}} = L_k^{(a,b)} - 1 + 1 - \frac{1}{L_{n-k}^{(a,b)}/L_{n-2k}^{(a,b)}}.$$

Now,

$$1 - \frac{1}{L_{n-k}^{(a,b)}/L_{n-2k}^{(a,b)}} = \frac{L_{n-k}^{(a,b)} - L_{n-2k}^{(a,b)}}{L_{n-k}^{(a,b)}}$$

$$= \frac{1}{L_{n-k}^{(a,b)}/(L_{n-k}^{(a,b)} - L_{n-2k}^{(a,b)})}$$

$$= \frac{1}{\left(\left(L_{n-k}^{(a,b)} - L_{n-2k}^{(a,b)}\right) + L_{n-2k}^{(a,b)}\right) / \left(L_{n-k}^{(a,b)} - L_{n-2k}^{(a,b)}\right)}$$

$$= \frac{1}{1 + \left(L_{n-2k}^{(a,b)} / \left(L_{n-k}^{(a,b)} - L_{n-2k}^{(a,b)}\right)\right)}$$

$$= \frac{1}{1 + \left(1 / \left(\left(L_{n-k}^{(a,b)} - L_{n-2k}^{(a,b)}\right) / L_{n-2k}^{(a,b)}\right)\right)}$$

$$= \frac{1}{1 + \left(1 / \left(\left(L_{n-k}^{(a,b)} / L_{n-2k}^{(a,b)}\right) - 1\right)\right)}$$

Taking the limit as $n \to \infty$, we get

$$\phi_{a,1}^{k} = L_k^{(a,b)} - 1 + \frac{1}{1 + (1/(\phi_{a,1}^{k} - 1))}.$$

Thus,

$$\phi_{a,1}^{k} = \begin{cases} \left[L_k^{(a,b)}; \overline{L_k^{(a,b)}}\right]; \ if\ k\ is\ odd \\ \left[L_k^{(a,b)} - 1, \overline{1\ , L_k^{(a,b)} - 2}\right]; \ if\ k\ is\ even \end{cases}.$$

Finally the continued fraction of $\phi_{a,1}^{k}$ follows easily.

Theorem 3.2: $\phi_{a,-1}^{k} = \left[F_{k+1}^{(a,b)} - F_{k-1}^{(a,b)} - 1, \overline{1, (F_{k+1}^{(a,b)} - F_{k-1}^{(a,b)} - 2)}\right]$

Proof: When $b = -1$ we have $L_n^{(a,b)} = aL_{n-1}^{(a,b)} - L_{n-2}^{(a,b)}$ .In this case, (2) becomes

$$\frac{L_n^{(a,b)}}{L_{n-k}^{(a,b)}} = \left(F_{k+1}^{(a,b)} - F_{k-1}^{(a,b)}\right) + (-1)^{k}(-1)^{k-1} \frac{L_{n-2k}^{(a,b)}}{L_{n-k}^{(a,b)}}$$

$$= \left(F_{k+1}^{(a,b)} - F_{k-1}^{(a,b)}\right) - \frac{L_{n-2k}^{(a,b)}}{L_{n-k}^{(a,b)}}$$

Arguing according to last theorem, and using the fact that $\phi_{a,-1} = \frac{a+\sqrt{a^2-4}}{2}$, we get

$$\phi_{a,-1}^{k} = \left(F_{k+1}^{(a,b)} - F_{k-1}^{(a,b)} - 1\right) + \frac{1}{1 + (1/(\phi_{a,-1}^{k} - 1))}.$$

This clearly determines a continued fraction with repeating block of length 2 of the following form:

$$\phi_{a,-1}^{k} = \left[F_{k+1}^{(a,b)} - F_{k-1}^{(a,b)} - 1, \overline{1, (F_{k+1}^{(a,b)} - F_{k-1}^{(a,b)} - 2)}\right].$$

This proves the result.

IV. Conclusions

The techniques developed in this paper have allowed us to determine closed form expressions for the continued fraction expansions of some special quadratic numbers. However, since our method applies only to a relatively small class of numbers, it does not allow us to abandon the algorithm. We have been able to prove the structure of the continued fraction of a sizeable class of numbers. Although it was pretty clear at the outset that there was a nice structure to this class, we have successfully proven it, and can now use these results to possibly derive similar results for other classes.

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