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Classification of All Ideals of the Group Algebra of Some Lie Groups .

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Abstract

Let $G = \mathbb{R}^n \underset{\rho}{\rtimes} \mathbb{R}^m$ be the Lie group, which is the semi-direct product of the real vector group $\mathbb{R}^n$ and $\mathbb{R}^m$, $1 \leq m \leq n$. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra $\underline{g}$ of G. The purpose of this paper is to give a characterization of the all ideals of the group algebra $L^1(G)$ of G. Besides, we prove some existence theorems for $\mathcal{U}$.

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1 Preliminaries and Results

1.1. If G is a Lie group, we denote by $C^\infty(G)$, $\mathcal{D}(G)$, $\mathcal{D}'(G)$, $\mathcal{E}'(G)$ be the space of C^∞- functions, C^∞ with compact support, distributions and distributions with compact support on G respectively.. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra $\underline{g}$ of G; which is canonically isomorphic to the algebra of all distributions on G supported by $\{0\}$, where 0 is the identity element of G. For any $u \in \mathcal{U}$ one can define a differential operator P_u on G as follows:

$$P_u f(x) = u * f(x) = \int_K f(y^{-1}x)u(y)dy \tag{1}$$

for any $f \in C^\infty(G)$, where dy is the right Haar measure on G , $y \in G$, $x \in G$ and $*$ denotes the convolution product on G. The mapping $u \to P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on G. For more details see [3, 9]. We denote by $L^1(G)$ the Banach algebra that consists of all complex valued functions on the group G, which are integrable with respect to the Haar measure of G and multiplication is defined by convolution on G as

$$\phi * f(g) = \int_G f(h^{-1}g)\phi(h)dh \tag{2}$$

for every ϕ and f belong to $L^1(G)$, and we denote by $L^2(G)$ its Hilberst spac

1.2. Let B be the vector group of G , which is the vector of space of the Lie algebra of G . We denote also by $\mathcal{U}$ the complexified enveloping algebra of the real Lie algebra $\underline{b}$ of B. For every $u \in \mathcal{U}$, we can associate a differential operator Q_u on B as follows

$$Q_u f(x) = u *_c f(x) = \int_B f((x-y)u(y)dy \tag{3}$$

for any $f \in C^\infty(B)$, $x \in B, y \in B$, where $*_c$ signify the convolution product on the real vector group B and dy is the Lebesgue measure on B. The mapping $u \mapsto Q_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on B, which are nothing but the algebra of differential operator with constant coefficients on B. Also, We denote by $L^1(B)$ the Banach algebra that consists of all complex valued functions on the group B, which are integrable with respect to the Haar measure of B and multiplication is defined by convolution on B as

$$\phi *_c f(g) = \int_G f(g-h)\phi(h)dh \tag{4}$$

for every ϕ and f belong to $L^1(B)$, and we denote by $L^2(B)$ its Hilberst space

2 Left Ideals of Group Algebra of G

2.1. For any $k \in \mathbb{N}$, we denot by $\mathbb{R}^k$ the $k-$dimensional real vector group. Let $G = \mathbb{R}^n \rtimes_\rho \mathbb{R}^m$ be the Lie group of the semidi-direct of the two real vector groups $\mathbb{R}^n$and $\mathbb{R}^m$, via the group homomorphism $\rho : \mathbb{R}^m \to Aut(\mathbb{R}^n)$, where $Aut(\mathbb{R}^n)$ is the group of all automorphisms of $\mathbb{R}^n$. The multiplication of two elements $X = (x, y)$ and $Y = (x', y')$ in G is given by :

$$X \cdot Y = (x, y)(x', y') = (x + \rho(y)(x'), y + y') \tag{5}$$

The inverse X^{-1} of an element X in G is :

$$X^{-1} = (x, y)^{-1} = (\rho(-y)(-x'), -y)$$

for any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^m$. In the next, we write yx in the place of $\rho(y)(x)$

Let $L = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ be the group with law:

$$(x, t, r)(y, s, q) = (x + \rho(r)y, t + s, r + q)$$

for all $(x, t, r) \in L$ and $(y, s, q) \in L$. In this case the group G can be identified with the closed subgroup $\mathbb{R}^n \times \{0\} \times_\rho \mathbb{R}^m$ of L and B with the subgroup $\mathbb{R}^n \times \mathbb{R}^m \times \{0\}$of L.

Definition 2.1. *For every $f \in C^\infty(G)$, one can define a function $\widetilde{f} \in C^\infty(L)$ as follows:*

$$\widetilde{f}(x, t, r) = f(\rho(t)x, r + t) \tag{6}$$

for all $(x, t, r) \in L$. So every function $\psi(x, r)$ on G extends uniquely as an invariant function $\widetilde{\psi}(x, t, r)$ on L.

Remark 2.1 *The function $\widetilde{f}$ is invariant in the following sense:*

$$\widetilde{f}(\rho(s)x, t - s, r + s) = \widetilde{f}(x, t, r) \tag{7}$$

for any $(x, t, r) \in L$ and $s \in \mathbb{R}^m$.

Lemma 2.1 *For every function $F \in C^\infty(L)$ invariant in sense (7) and for every $u \in \mathcal{U}$, we have*

$$u \ * F(x, t, r) = u \ *_c F(x, t, r) \tag{8}$$

for every $(x, t, r) \in L$, where $*$ signifies the convolution product on G with respect the variables (x, r) and $*_c$signifies the commutative convolution product on B with respect the variables (x, t).

Proof: In fact we have

$$\begin{aligned} & P_uF(x,t,r) = u * F(x,t,r) \\ = & \int_G F(y,s)^{-1}(x,t,r)u(y,s)dyds \\ = & \int_G F\left[(\rho(-s)(-y),-s)(x,t,r)\right]u(y,s)dyds \\ = & \int_G F\left[\rho(-s)(x-y),t,r-s\right]u(y,s)dyds \\ = & \int_G F\left[x-y,t-s,r\right]u(y,s)dyds \\ = & u *_c F(x,t,r) = Q_uF(x,t,r) \end{aligned} \tag{9}$$

where P_u and Q_u are the invariant differential operators on G and B respectively.

Definition 2.1. *If $u \in L^1(G)$, then one can define two convolutions product on the group L by:*

$$\begin{aligned} (i)\ u * F(x,y,z) &= \int_G F\left[(t,s)^{-1}(x,y,z)\right]u(t,s)dtds \\ &= \int_G F\left[\rho(-s)(x-t)y,z-s)\right]u(t,s)dtds \end{aligned} \tag{10}$$

$$(ii)\ u *_c F(x,y,z) = \int_B F\left[x-t,y-s,z)\right]u(t,s)dtds \tag{11}$$

for any $F \in L^1(L), (x,y,z) \in L$ and $(t,s) \in \mathbb{R}^{2m}$, where $dtds$ is the left Haar measure on G, $*$ is the convolution product on G and $*_c$ is the convolution product on B. It results

$$u * \widetilde{F}(x,y,z) = u *_c \widetilde{F}(x,y,z) \tag{12}$$

for each $F \in L^1(G)$

Proposition 2.1. *The mapping γ from $L^1(G)|_B$ to $L^1(G)|_G$ defined by*

$$\widetilde{F}|_B\ (x,y,0) \to \gamma(\widetilde{F}|_B\ )(x,0,y) = \widetilde{F}|_G\ (x,0,y) \tag{13}$$

is a topological isomorphism, and

$$\gamma(u \ *_c \widetilde{F}|_B)(x,0,y) = u \ * \widetilde{F}|_G(x,0,y) \tag{14}$$

where

$$(u *_c \widetilde{F}|_B)(x,y,0) = \int_B \widetilde{F}\left[x-t, y-s, 0)\right] u(t,s)dtds, \ F \in L^1(G) \tag{15}$$

$$\begin{aligned}\gamma(u \ * \widetilde{F}|_G)(x,0,y) &= \int_B \widetilde{F}\left[\rho(-s)(x-t), 0, y-s)\right] u(t,s)dtds \\ &= u \ * F(x,y), \ F \in L^1(G)\end{aligned} \tag{16}$$

Proof: It easy to see that $\gamma : L^1(G)|_B \to L^1(G)|_G$ is a topological isomorphism and

$$\begin{aligned}&\gamma(u \ *_c \widetilde{F}|_B)(x,0,y) \\ = &\int_G \widetilde{F}\left[x-t, -s, y\right] u(t,s)dtds \\ = &\int_G \widetilde{F}\left[\rho(-s)(x-t), 0, y-s\right] u(t,s)dtds \\ = &\ u \ * \widetilde{F}|_G(x,0,y)\end{aligned} \tag{17}$$

for every $F \in L^1(G)$. The fact that

$$\gamma^{-1} : \widetilde{L^1(G)}|_G \ \to \widetilde{L^1(G)}|_B \tag{18}$$

is a topological isomorphismm, we get

$$\begin{aligned}\widetilde{F}|_G \ (x,0,y) &\to \gamma^{-1}(\widetilde{F}|_G \ )(x,y,0) \\ &= \widetilde{F}|_B \ (x,y,0)\end{aligned} \tag{19}$$

Hence the proposition

If I is a subspace of $L^1(G)$, we denote by $\widetilde{I}$ its image by the mapping $\sim$. Let $J = \widetilde{I} \ |_B$. Our main result is:

Theorem 2.1. *Let I be a subspace of $L^1(G)$, then the following conditions are equivalents.*

(i) $J = \widetilde{I}\ |_B$ *is an ideal in the Banach algebra* $L^1(B)$.
(ii) I *is a left ideal in the Banach algebra* $L^1(G)$.

Proof: (i) implies (ii) Let I be a subspace of the space $L^1(B)$ such that $J = \widetilde{I}|_B$ is an ideal in $L^1(B)$, then we have:

$$u *_c \widetilde{I}\ |_B(x, y, 0) \subseteq \widetilde{I}\ |_B(x, y, 0) \tag{20}$$

for any $u \in L^1(B)$ and $(x, y) \in B$, where

$$u *_c \widetilde{I}\ |_B(x, y, 0) = \left\{ \int_B \widetilde{f}|_B\ [x - t, y - s, 0)]\, u(t, s)dtds,\ f \in\ I \right\} \tag{21}$$

It shows that

$$u *_c \widetilde{f}\ |_B(x, y, 0) \in \widetilde{I}\ |_B(x, y, 0) \tag{22}$$

for any $\widetilde{f} \in \widetilde{I}$. According to equation(14), we get

$$\begin{aligned} & \gamma(u *_c \widetilde{f}|_B)(x, 0, y) \\ = \ & u\ * \widetilde{f}(x, 0, y) \in \gamma(\widetilde{I}\ |_B(x, 0, y) \\ = \ & \widetilde{I}\ |_G(x, 0, y) = I \end{aligned} \tag{23}$$

(ii) implies (i) If I is an ideal in $L^1(G)$, then we get

$$\begin{aligned} & u * \widetilde{I}\ |_G(x, 0, y) \\ = \ & u * I\ (x, y) \subseteq \widetilde{I}\ |_G(x, 0, y) = I\ (x, y) \end{aligned} \tag{24}$$

where

$$u*\widetilde{I}\ |_G(x, 0, y) = \left\{ \int_B \widetilde{f}|_G\ [\rho(-s)(x - t), 0, y - s]\, u(t, s)dtds,\ f \in\ I \right\} \tag{25}$$

Apply now equation (19),we obtain

$$\begin{aligned} & \gamma^{-1}(u * \widetilde{f}\ |_G)(x, 0, y) \\ = \ & u *_c \widetilde{f}|_B(x, y, 0) \in \gamma^{-1}(u * \widetilde{I}\ |_G)(x, y, 0) \\ = \ & u * \widetilde{I}\ |_B(x, y, 0) \end{aligned} \tag{26}$$

Corollary 2.1. *Let I be a subspace of the space $L^1(G)$ and $\widetilde{I}$ its image by the mapping $\sim$ such that $J = \widetilde{I}|_B$ is an ideal in $L^1(B)$, then the following conditions are verified.*

(i) J is a closed ideal in the algebra $L^1(B)$ if and only if I is a left closed ideal in the algebra $L^1(G)$.

(ii)J is a prime ideal in the algebra $L^1(B)$ if and only if I is a left prime ideal in the algebra $L^1(G)$

(iii)J is a maximal ideal in the algebra $L^1(B)$ if and only if I is a left maximal ideal in the algebra $L^1(G)$

(iv) J is a dense ideal in the algebra $L^1(B)$ if and only if I is a left dense ideal in the algebra $L^1(G)$.

The proof of this corollary results immediately from theorem **2.1.**

3 Fourier Transform and Existence theorems

In this paragraph, we will prove the solvability of any element of $\mathcal{U}$. Let $G = \mathbb{R}^n \underset{\rho}{\rtimes} \mathbb{R}^m$ be the Lie group which is the semi-direct product of the two real vector groups $\mathbb{R}^n$ and $\mathbb{R}^m$. Therefor we define the Fourier transform on G in view of its vector group, in order to obtain the Plancherel formula. Besides we prove the existence theorem for the algebra of all invariant differential operators. As in [3], we will define the Fourier transform on G. Therefor let $\mathcal{S}(G)$ be the Schwartz space of G which can be considered as the Schwartz space of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, and let $\mathcal{S}'(G)$ be the space of all tempered distributions on G. The action ρ of the group $\mathbb{R}^m$ on $\mathbb{R}^n$ defines a natural action ρ of the dual group $(\mathbb{R}^m)^*$of the group $\mathbb{R}^m$ $((\mathbb{R}^m)^* \simeq \mathbb{R}^m)$ on $(\mathbb{R}^n)^*$, which is given by :

$$\langle \rho(t)(\xi), x \rangle = \langle \xi, \rho(t)(x) \rangle \tag{27}$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $t = (t_1, t_2, ..., t_m) \in \mathbb{R}^m$ and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$

Definition 3.1. *If $f \in \mathcal{S}(G)$, one can define its Fourier transform $\mathcal{F}f$ by :*

$$\mathcal{F}f \quad (\xi, \lambda) = \int_G f(x,t)\ e^{-\ i\ (\langle \xi, x \rangle + \langle \lambda, t \rangle)}\ dxdt \tag{28}$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m$ and $t = (t_1, t_2, ..., t_m) \in \mathbb{R}^m$, *where* $\langle \xi, x\rangle = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$ *and* $\langle \lambda, t\rangle = \lambda_1 t_1 + \lambda_2 t_2 + ... + \lambda_m t_m$. *It is clear that* $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^{n+m})$ *and the mapping* $f \rightarrow \mathcal{F}f$ *is isomorphism of the topological vector space* $\mathcal{S}(G)$ *onto* $\mathcal{S}(\mathbb{R}^{n+m})$.

Definition 3.2. *If* $f \in \mathcal{S}(G)$, *we define the Fourier transform of its invariant* $\widetilde{f}$ *as follows*

$$\mathcal{F}(\widetilde{f})(\xi, \lambda, 0) = \int\limits_{G\times\mathbb{R}^m} \widetilde{f}(x, t, s) e^{-\, i\, (\langle \xi, x\rangle + \langle \lambda, t\rangle)}\, e^{-\, i\, \langle \mu, s\rangle}\, dx dt ds d\mu \tag{29}$$

where $(\mu, s) \in \mathbb{R}^{n+m}$ *and* $\langle \mu, s\rangle = \mu_1 s_1 + \mu_2 s_2 + ... + \mu_m s_m$

Corollary 3.1. *For every* $u \in \mathcal{S}(G)$, *and* $f \in \mathcal{S}(G)$, *we have*

$$\int\limits_{\mathbb{R}^m} \mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi, \lambda, \mu) d\mu = \int\limits_{\mathbb{R}^m} \mathcal{F}(\widetilde{f})(\xi, \lambda, \mu) \mathcal{F}(\overset{\vee}{u})(\xi, \lambda) d\mu = \mathcal{F}(\widetilde{f})(\xi, \lambda, 0)\overline{\mathcal{F}(u)(\xi, \lambda)} \tag{30}$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m$ *and* $\mu = (\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$, *where* $\overset{\vee}{u}(x, t) = \overline{u(x, t)^{-1}}$

Proof: By **Lemma 2.1**, we have

$$\overset{\vee}{u} * \widetilde{f}(x, t, r) = \overset{\vee}{u} *_c \widetilde{f}(x, t, r) \tag{31}$$

and

$$\int\limits_{\mathbb{R}^m} \mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi, \lambda, \mu) d\mu = \mathcal{F}(\overset{\vee}{u} *_c \widetilde{f})(\xi, \lambda, 0) = \mathcal{F}(\widetilde{f})(\xi, \lambda, 0)\overline{\mathcal{F}(u)(\xi, \lambda)} \tag{32}$$

Theorem 3.1.(*Plancherel's formula*). *For any* $f \in L^1(G) \cap L^2(G)$, *we get*

$$\int\limits_G |f(x, t)|^2\, dx dt = \int\limits_{\mathbb{R}^{n+m}} |\mathcal{F}f(\xi, \lambda)|^2\, d\xi d\lambda \tag{33}$$

Proof: First, let $\overset{\widetilde{\vee}}{f}$ be the function defined by

$$\overset{\widetilde{\vee}}{f}(x,t,r) = \overline{f((tx, r+t)^{-1})} \tag{34}$$

then we have

$$f \; * \overset{\widetilde{\vee}}{f}(0,0,0) = \int\limits_G \overset{\widetilde{\vee}}{f}\left[(x,t)^{-1}(0,0,0)\right] f(x,t)dxdt$$

$$= \int\limits_G \overset{\widetilde{\vee}}{f}\left[\rho(-t)((-x)+(0)), 0, 0-t\right] f(x,t)dxdt$$

$$= \int\limits_G \overset{\widetilde{\vee}}{f}\left[\rho(-t)(-x), 0, -t\right] f(x,t)dxdt = \int\limits_G \overset{\vee}{f}\left[\rho(-t)(-x), -t\right] f(x,t)dxdt$$

$$= \int\limits_G \overline{f(x,t)} f(x,t)dxdt = \int\limits_G |f(x,t)|^2 \, dxdt \tag{35}$$

Second by (31), we obtain

$$f \; * \overset{\widetilde{\vee}}{f}(0,0,0)$$

$$= \int\limits_{\mathbb{R}^{n+2m}} \mathcal{F}(f * \overset{\widetilde{\vee}}{f})(\xi,\lambda,\mu)d\xi d\lambda d\mu = \int\limits_{\mathbb{R}^{n+2m}} \mathcal{F}(f *_c \overset{\widetilde{\vee}}{f})(\xi,\lambda,\mu)d\xi d\lambda d\mu$$

$$= \int\limits_{\mathbb{R}^{n+m}} \mathcal{F}(\overset{\widetilde{\vee}}{f})(\xi,\lambda,0)\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda = \int\limits_{\mathbb{R}^{n+m}} \widetilde{\overset{\vee}{\overbrace{(\mathcal{F}f)}}}(\xi,\lambda,0)\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda$$

$$= \int\limits_{\mathbb{R}^{n+m}} \overset{\vee}{\overbrace{(\mathcal{F}f)}}(\lambda\xi,\lambda)\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda = \int\limits_{\mathbb{R}^{n+m}} \overline{\mathcal{F}f[(\lambda\xi,\lambda)^{-1}]}\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda$$

$$= \int\limits_{\mathbb{R}^{n+m}} \overline{\mathcal{F}(f)}\,(\xi,\lambda)\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda$$

$$= \int\limits_{\mathbb{R}^{n+m}} |\mathcal{F}(f)(\xi,\lambda)|^2 \, d\xi d\lambda = \int\limits_G |f(x,t)|^2 \, dxdt \tag{36}$$

Which is the Plancherel's formula on G. So the Fourier transform can be extended to an isometry of $L^2(G)$ onto $L^2(\mathbb{R}^{n+m})$.

Corollary 3.2. *In equation* (36), *if we replace the first f by g, we obtain the Parseval formula on G*

$$\int_G \overline{f(x,t)}g(x,t)dxdt = \int_{\mathbb{R}^{n+m}} \overline{\mathcal{F}(f)(\xi,\lambda}\mathcal{F}g(\xi,\lambda)d\xi d\lambda \tag{37}$$

In the following, we introduce some existence theorems, the first one is:

Theorem 3.2. *Every invariant differential operator on G which is not identically* 0 *has a tempered fundamental solution.*

Proof: For each complex number s with positive real part, we can define a distribution T^s on $G \times \mathbb{R}^m$ by:

$$\langle T^s, f\rangle = \int_G \int_{\mathbb{R}^m} \left[|\mathcal{F}(u)(\xi,\lambda)|^2\right]^s \mathcal{F}(f)(\xi,\lambda)\ d\xi d\lambda$$

for each $f \in \mathcal{S}(G \times \mathbb{R}^m)$. By Atiyah-Bernstein theorems [1], the function $s \mapsto T^s$ has a meromorphic continuation in the whole complex plan, which is analytic at $s = 0$ and its value at this point is the Dirac measure on the group $L = G \times \mathbb{R}^m$. Now we can define another distribution, $\widehat{T^s}$, as follows.

$$\left\langle \widehat{T^s}, f\right\rangle = \left\langle T^s, \widehat{f}\right\rangle = \int_G \int_{\mathbb{R}^m} \left[|\mathcal{F}(u)(\xi,\lambda)|^2\right]^s \mathcal{F}(\widehat{f})(\xi,\lambda,v)\ d\xi d\lambda dv$$

for any $f \in \mathcal{S}(G \times \mathbb{R}^n)$ and $s \in \mathbb{C}$, with $\text{Re}(s) \geq 0$.

Note that the distribution $\widehat{T^s}$ is invariant in sense (7) and we have

$$\begin{aligned}\left\langle \widehat{u * \check{u} *_c T^s}, f\right\rangle &= \left\langle u * \widetilde{u} *_c T^s, \widehat{f}\right\rangle = \left\langle T^s, u *_c \check{u} * \widehat{f}\right\rangle \\ &= \int_G \int_{\mathbb{R}^n} \left[\left|\mathcal{F}(\check{u})(\xi,\lambda)\right|^2\right]^s \mathcal{F}(\widetilde{u} * c\check{u} * \widehat{f})(\xi,\lambda,v)\ d\xi d\lambda dv\end{aligned}$$

here

$$u *_c f = \int_G f((z-a, y-b, x-c)u((a,b,c)dadbdc$$

is the commutative convolution product on G. By proposition **2.1** , we get:

$$\left\langle \widehat{u * \overset{\vee}{u} *_c T^s}, f \right\rangle = \int_G \int_{\mathbb{R}^n} \left[|\mathcal{F}(u)(\xi, \lambda)|^2 \right]^{s+1} \mathcal{F}(\widehat{f})(\xi, \lambda, v) d\xi d\lambda dv$$

Hence

$$\widehat{u * \overset{\vee}{u} *_c T^s} = \widehat{T^{s+1}}$$

In view of invariance (7), the restriction of the distributions $\widehat{u * \overset{\vee}{u} *_c T^s} = \widehat{T^{s+1}}$ on the sub-group $\mathbb{R}^n \times \{0\} \underset{\rho}{\propto} \mathbb{R}^m \simeq G$ are nothing but the distributions

$$u * \overset{\vee}{u} *_c T^s = T^{s+1}.$$

The distribution T^s can be expanded a round $s = -1$ in the form

$$T^s = \sum_{j=-(n+2m)}^{\infty} \alpha_j (s+1)^j \tag{38}$$

where each α_j is a distribution on G. But $u * \widetilde{u} * c T^s = T^{s+1}$ can not have a pole at $s = -1$ (since $T^0 = \delta_G$) and so we must have:

$$\begin{aligned} u * \overset{\vee}{u} *_c \alpha_j &= 0 \quad \text{for} \quad j < 0 \\ u * \overset{\vee}{u} *_c \alpha_0 &= \delta_G \end{aligned} \tag{39}$$

Hence the theorem.

The second is

If we consider the group G as a subgroup of L, then $\widetilde{f}(x,s,t) \in \mathcal{S}(G)$ for s is fixed, and if we consider B as a subgroup of L, then $\widetilde{f}(x,s,t) \in \mathcal{S}(B)$ for

t fixed. This being so; denote by $\mathcal{S}_E(L)$ the space of all functions $\phi(x,s,t) \in C^\infty(L)$ such that $\phi(x,s,t) \in \mathcal{S}(G)$ for s is fixed, and $\phi(x,s,t) \in \mathcal{S}(B)$ for t is fixed. We equip $\mathcal{S}_E(L)$ with the natural topology defined by the seminomas:.

$$\phi \to \sup_{(x,t)\in G} |Q(x,t)\ P(D)\phi(x,s,t)| \qquad s\ fixed. \tag{40}$$

$$\phi \to \sup_{(x,s)\in B} |R(x,s)H(D)\phi(x,s,t)| \qquad t\ fixed. \tag{41}$$

where P, Q, R and H run over the family of all complex polynomial in $n+m$ variables. Let $\mathcal{S}_E^I(L)$ be the subspace of all functions $F \in \mathcal{S}_E(L)$, which are invariant in sense (7), then we have the following result.

lemma 3.1. *Let $u \in \mathcal{U}$ and Q_u be the invariant differential operator on the group B*, which is associated to u, then we have:

(i) The mapping $f \mapsto \widetilde{f}$ is a topological isomorphism of $\mathcal{S}(G)$ onto $\mathcal{S}_E^I(L)$

.

(ii) The mapping $F \mapsto Q_uF$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image, where Q_u acts on the variables $(x,s) \in B$.

*Proof***:** (i) In fact $\sim$ is continuous and the restriction mapping $F \mapsto RF$ on G is continuous from $\mathcal{S}_E^I(L)$ into $\mathcal{S}(G)$ that satisfies $R\circ \sim = Id_{\mathcal{S}(G)}$ and $\sim \circ R = Id_{\mathcal{S}_E^I(L)}$, where $Id_{\mathcal{S}(G)}$ $(resp.\ Id_{\mathcal{S}_E^I(L)})$ is the identity mapping of $\mathcal{S}(G)$ $(resp.\ \mathcal{S}_E^I(L))$ and G is considered as a subgroup of L. To prove(ii) we refer to[14, $P.313-315$] and his famous result that is:

"*Any invariant differential operator on B, is a topological isomorphism of $S(B)$ onto its image*" From this result, we obtain:

$$Q_u : \mathcal{S}_E(L) \to \mathcal{S}_E(L) \tag{42}$$

is a topological isomorphism and its restriction on $\mathcal{S}_E^I(L)$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image. Hence the lemma is proved.

In the following we will prove that every invariant differential operator on $G = \mathbb{R}^n \times \{0\} \times_\rho \mathbb{R}^m$ has a tempered fundamental solution. As in the introduction, we will consider the two invariant differential operators P_u and Q_u, the first on the group $G = \mathbb{R}^n \times \{0\} \times \rho\ \mathbb{R}^m$, and the second on the commutative group $B = \mathbb{R}^n \times \mathbb{R}^m \times \{0\}$. Our main result is:

Theorem 3.3. *Every nonzero invariant differential operator P_u on G associated to $\mathcal{U}$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image.*

Proof **:** By equation (31) we have for every $u \in \mathcal{U}$ and $F \in \mathcal{S}_E^I(L)$

$$P_uF(x,s,t) = u * F(x,s,t)$$
$$= \quad u *_c (x,s,t) = Q_uF(x,s,t) \qquad (43)$$

This shows that:

$$P_uF(x,s,t) = Q_uF(x,s,t) \qquad (44)$$

for all $(x,s,t) \in L$, where $\star$ is the convolution product on $G = \mathbb{R}^n \times \{0\} \times \mathbb{R}^m$ and $\star_c$ is the convolution product on the group $B = \mathbb{R}^n \times \mathbb{R}^m \times \{0\}$. By lemma 3.1 the mapping $F \mapsto Q_uF$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image, then the mapping $F \mapsto P_uF$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image. Since

$$R(P_uF)(x,s,t) = P_u(RF)(x,s,t) \qquad (45)$$

so the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}_E^I(L) & \overset{P_u}{\longrightarrow} & P_u\mathcal{S}_E^I(L) \\ \sim\uparrow\downarrow R & & \downarrow R \\ \mathcal{S}(G) & \overset{P_u}{\longrightarrow} & P_u\mathcal{S}(G) \end{array}$$

Hence the mapping $F \mapsto P_uF$ is a topological isomorphism of $\mathcal{S}(G)$ onto its image.

Corollary 3.3. *Every nonzero invariant differential operator on G has a tempered fundamental solution.*

Proof **:** The transpose ${}^tP_u$ of P_u is a continuous mapping of $\mathcal{S}'(G)$ onto $\mathcal{S}'(G)$. This means that for every tempered distribution$T$ on G there is a tempered distribution E on G such that

$$P_uE = T \qquad (46)$$

Indeed the Dirac measure δ belongs to $\mathcal{S}'(G)$.

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