On the Negative Pell Equation

$$y^2 = 54x^2 - 5$$

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***Abstract*—** The negative pell equation represented by the binary quadratic equation $y^2 = 54x^2 - 5$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented employing the solutions of the equation under consideration. The integer solutions for a few choices of hyperbola and parabola are obtained.

***Keywords*—** Binary Quadratic, Hyperbola, Parabola, Integral Solutions, Pell Equation

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1.Introduction

Diophantine equation of the form $y^2 = Dx^2 + 1$ where D is a given positive square free integer is known as pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over the world, since antiquity, J.L. Lagrange proved that the positive pell equation $y^2 = Dx^2 - 1$ has infinitely many distinct integer solutions where as the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a ceriterium for the solubility of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solutions, where as $y^2 = 65x^2 - 1, y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2-11]. More specifically one may refer "The on-line encyclopedia of integer sequences" (A031396, A130226, A031398) for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not.

In this communication, the negative pell equation given by $y^2 = 54x^2 - 5$ is considered and infinitively many integer solutions are obtained. A few interesting relations among the solutions are presented.

2.Methods For Analysis:

The negative pell equation representing hyperbola under consideration is

$$y^2 = 54x^2 - 5 \tag{1}$$

whose smallest positive integer solution is $x_0 = 1, y_0 = 7$ .To obtain the other solutions of (1), consider the pell equation

$$y^2 = 54x^2 + 1$$

whose general solution is given by

$$\tilde{y}_n = \frac{1}{2} f_n ; \tilde{x}_n = \frac{1}{2\sqrt{54}} g_n$$

Where
$$f_n = (485 + 66\sqrt{54})^{n+1} + (485 - 66\sqrt{54})^{n+1}$$
$$g_n = (485 + 66\sqrt{54})^{n+1} - (485 - 66\sqrt{54})^{n+1}, n = 0,1,2,3,\ldots\ldots..$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solutions of (1) are given by

$$108x_{n+1} = 54 f_n + 7\sqrt{54} g_n$$
$$2y_{n+1} = 7 f_n + \sqrt{54} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 970x_{n+2} + x_{n+1} = 0, x_0 = 1, x_1 = 947$$
$$y_{n+3} - 970y_{n+2} + y_{n+1} = 0, y_0 = 7, y_1 = 6959$$

Some numerical examples of x and y satisfying (1) are given in the following table1.

Table 1: Numerical Examples

n	x_n	y_n
0	1	7
1	947	6959
2	918589	6750223
3	891030383	6547709351

A few interesting properties between the solutions and special numbers are given below:

1. $\frac{6}{5}[108x_{2n+2} - 14y_{2n+2} + 10]$ is a nasty number
2. $\frac{1}{5}(108x_{3n+3} - 14y_{3n+3} + 3(108x_{n+1} - 14y_{n+1}))$ is a cubic integer
3. $\frac{1}{5}(108x_{2n+2} - 14y_{2n+2} + 10)$ is a perfect square
4. $44604\ x_{n+1} = 6372\ y_{n+1}$
5. $x_{n+3} = 485x_{n+1} + 66y_{n+1}$
6. $x_{n+3} = 470449x_{n+1} + 64020y_{n+1}$
7. $y_{n+2} = 3564x_{n+1} + 485y_{n+1}$
8. $Y_{n+3} = 3457080x_{n+1} + 470449y_{n+1}$
9. $108x_{n+2} = 52380x_{n+1} + 7128y_{n+1}$
10. $108x_{n+3} = 50808492x_{n+1} + 6914160y_{n+1}$

REMARKABLE OBSERVATIONS:

1. Let p,q be two non-zero distinct positive integers such that $p = x_n + 2y_n, q = x_n$ note that

$p > q > 0$.Treat p,q as the generators of the Pythagorean triangle T(α,β,γ) where α= 2pq, $\beta = p^2 - q^2$ and $\gamma = p^2 + q^2$.Let A,P represent the area and perimeter of

T (α,β,γ). Then the following interesting relations are observed:

1. $\alpha - 27\beta + 26\gamma = 5$
2. $\dfrac{2A}{P} = x_{n+1} y_{n+1}$
3. By considering linear combination among the solutions of (1), one may obtain solutions of different hyperbolas. A few examples are given in table2 below:

Table 2:Hyperbolas

S. No	X,Y	Hyperbola
1	$108x_{n+1} - 14y_{n+1}, 108y_{n+1} - 756x_{n+1}$	$54X^2 - Y^2 = 5400$
2	$6959x_{n+1} - 7x_{n+2}, 108y_{n+1} - 756x_{n+1}$	$54X^2 - 1089Y^2 = 5880600$
3	$6750223x_{n+1} - 7x_{n+3}, 108y_{n+1} - 756x_{n+1}$	$54X^2 - 1024640100Y^2 = 5.53305654 \times 10^{12}$
4	$108x_{n+1} - 14y_{n+1}, 54x_{n+2} - 51138x_{n+1}$	$58806X^2 - Y^2 = 5880600$
5	$108x_{n+1} - 14y_{n+1}, 54x_{n+3} - 49603806x_{n+1}$	$5.53305654 \times 10^{10} X^2 - Y^2$ $= 5.53305654 \times 10^{12}$
6	$6959x_{n+1} - 7x_{n+2}, 54x_{n+2} - 51138x_{n+1}$	$54X^2 - Y^2 = 5880600$
7	$6750223x_{n+1} - 7x_{n+3}, 54x_{n+3} - 49603806x_{n+1}$	$54X^2 - Y^2 = 5.53305654 \times 10^{12}$
8	$2y_{n+2} - 1894y_{n+1}, 108y_{n+1} - 756x_{n+1}$	$54X^2 - 4356Y^2 = 23522400$
9	$y_{n+3} - 918589y_{n+1}, 108y_{n+1} - 756x_{n+1}$	$25X^2 - 4743704167Y^2$ $= 2.56160025 \times 10^{12}$
10	$108x_{n+1} - 14y_{n+1}, 13918y_{n+1} - 14y_{n+2}$	$235224X^2 - Y^2 = 23522400$
11	$108x_{n+1} - 14y_{n+1}, 13500446y_{n+1} - 14y_{n+3}$	$2.213222616 \times 10^{11} X^2 - Y^2$ $= 2.213222616 \times 10^{13}$

12	$1894y_{n+3}-1837178y_{n+2}$, $13500446y_{n+2}-13918y_{n+3}$	$54X^2-Y^2=23522400$
13	$y_{n+3}-918589y_{n+1}, 13500446y_{n+1}-14y_{n+3}$	$216X^2-Y^2=2.21322261\ 6\times10^{13}$

3. By considering linear combination among the solutions of (1), one may obtain solutions of different parabolas. A few examples are given in table 3 below:

Table 3:Parabolas

S.No	X,Y	Parabola
1	$108x_{2n+2}-14y_{2n+2}+10,\ 108y_{n+1}-756x_{n+1}$	$Y^2=270X-5400$
2	$108x_{2n+2}-14y_{2n+2}+10,\ 54x_{n+2}-51138x_{n+1}$	$Y^2=294030X-5880600$
3	$108x_{2n+2}-14y_{2n+2}+10, 54x_{n+3}$ $-49603806x_{n+1}$	$Y^2=2.76652827\times10^{11}X-5.53305654\times10^{12}$
4	$108x_{2n+2}-14y_{2n+2}+10,\ 13918y_{n+1}-14y_{n+2}$	$Y^2=1176120X-23522400$
5	$108x_{2n+2}-14y_{2n+2}+10,$ $13500446y_{n+1}-14y_{n+3}$	$Y^2=1.106611308\times10^{12}X-2.213222616\times10^{13}$

3.CONCLUSIONS

In this paper, We have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2=54x^2-5$. As the binary quadratic Diophantine equation are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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