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The (A,D) - Ascending Subgraph Decomposition of Cartesian Product of some Simple Graphs

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Abstract - Alavi et al[1] defined Ascending Subgraph Decomposition(ASD) as decomposition of G with size $\binom{n+1}{2}$ into n subgraphs $G_1,G_2,G_3, \ldots ,G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1\leq i \leq n-1$ and $|E(G_i)| = i$ for $1\leq i \leq n$. Let G be a graph of size $\frac{n}{2}(2a + (n-1)d)$ where a, n, d are positive integers. Then G is said to have (a,d) - Ascending Subgraph Decomposition ((a,d) -ASD) into n parts if the edge set of G can be partitioned into n non-empty sets generating subgraphs $G_1,G_2, \ldots,G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1\leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$ for $1 \leq i \leq n$. The cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is defined to be the graph whose vertex set is $V_1 \times V_2$ and two vertices $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in $V = V_1 \times V_2$ are adjacent in $G_1 \times G_2$ if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1. In this paper, I investigate the (a,d) - Ascending Subgraph Decomposition of $P_{n+1} \times K_2$.

Keywords: Ascending Subgraph Decomposition , cartesian product.

1. INTRODUCTION

By a graph we mean a finite undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary[3].

Definition 1.1. Let G = (V,E) be a simple graph of order p and size q. If $G_1,G_2, \ldots ,G_n$ are edge disjoint subgraphs of G such that $E(G) = E(G_1)\cup E(G_2)\cup \ldots \cup E(G_n)$ then $\{G_1,G_2, \ldots,G_n\}$ is said to be a decomposition of G.

Definition 1.2. Alavi et al[1] defined Ascending Subgraph Decomposition(ASD) as decomposition of G with size $\binom{n+1}{2}$ into n-subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le n-1$ and $|E(G_i)| = i$ for $1 \le i \le n$.

Definition 1.3. Let G be a graph of size $\frac{n}{2}(2a + (n-1)d)$, where a, n, d are positive integers. Then G is said to have (a,d) - Ascending Subgraph Decomposition ((a,d) – ASD) into n parts if the edge set of G can be partitioned into n non-empty sets generating subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le n-1$ and $|E(G_i)| = a + (i-1)d$ for $1 \le i \le n$.

Definition 1.4. The cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is defined to be the graph whose vertex set is $V_1 \times V_2$ and two vertices $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in $V = V_1 \times V_2$ are adjacent in $G_1 \times G_2$ if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.

2. The (a,d) - ASD of $P_{n+1} \times K_2$.

Here, I investigate under what conditions $P_{n+1} \times K_2$ admits (a,d) - ASD.

Theorem 2.1. If $k \equiv 0,3 \pmod 6$, then $P_{n+1} \times K_2$ does not admit (a,d) - ASD into k parts.

Proof. Suppose $P_{n+1} \times K_2$ admits (a,d) - ASD into k parts then we have

$$a + (a + d) + (a + 2d) + \ldots + (a + (k-1)d) = q.$$

Since $q = 3n + 1$, $\frac{k}{2}(2a + (k-1)d) = 3n + 1$. ----------- (1)

Case (i) : Suppose $k \equiv 0 \pmod 6$.

Let $k = 6r$, $r \in Z^+$.

From (1) we have,

$$\frac{6r}{2}(2a + (6r-1)d) = 3n + 1$$

$$3r(2a + (6r-1)d = 3n + 1.$$

This is not possible. Hence, $P_{n+1} \times K_2$ does not admit (a,d) - ASD into k parts.

Case (ii) : Suppose $k \equiv 3 \pmod 6$.

Let $k = 6r + 3$, $r \in \{0\} \cup Z^+$.

Using (1) we have,

$$\frac{6r+3}{2}(2a + (6r + 2)d) = 3n + 1$$

$$(6r + 3)(a + (3r + 1)d) = 3n + 1$$

$$3(2r + 1)(a + (3r + 1)d) = 3n + 1.$$

This is also not possible. Hence $P_{n+1} \times K_2$ does not admit (a,d) - ASD into k parts. □

Theorem 2.2. If $P_{n+1} \times K_2$ admits (a,d) - ASD into k parts, then

(a) For $k \equiv 1 (\text{mod } 6)$,

(i) $3n + 1 \equiv 0 (\text{mod } k)$ (ii) $a \equiv 1 (\text{mod } 3)$ and (iii) $n \geq \frac{k(k+1)-2}{6}$.

(b) For $k \equiv 2 (\text{mod } 6)$,

(i) $3n + 1 \equiv 0 (\text{mod } \frac{k}{2})$ (ii) $a \equiv 0 (\text{mod } 3)$ and $d \equiv 1 (\text{mod } 3)$; $a \equiv 1 (\text{mod } 3)$ and $d \equiv 2 (\text{mod } 3)$; $a \equiv 2 (\text{mod } 3)$ and $d \equiv 0 (\text{mod } 3)$ and (iii) $n \geq \frac{k(k+5)-2}{6}$.

(c) For $k \equiv 4 (\text{mod } 6)$,

(i) $3n + 1 \equiv 0 (\text{mod } \frac{k}{2})$ (ii) $a \equiv 1 (\text{mod } 3)$ and

(iii) $n \geq \frac{k(k+1)-2}{6}$ except $n = \frac{k(k+1)}{6} + \frac{k}{2} l$ where $l = 1,3,5,\ldots,2r - 1$.

(d) For $k \equiv 5 (\text{mod } 6)$,

(i) $3n + 1 \equiv 0 (\text{mod } k)$

(ii) $a \equiv 0 (\text{mod } 3)$ and $d \equiv 1 (\text{mod } 3)$; $a \equiv 1 (\text{mod } 3)$ and $d \equiv 2 (\text{mod } 3)$; $a \equiv 2 (\text{mod } 3)$ and $d \equiv 0 (\text{mod } 3)$ and

(iii) $n \geq \frac{k(k+5)-2}{6}$.

***Proof* .** Suppose $P_{n+1} \times K_2$ admits (a,d) - ASD into k parts, then we have

$$a + (a + d) + (a + 2d) + \ldots + (a + (k - 1)d) = q$$

Since $q = 3n + 1$, $\frac{k}{2}(2a + (k - 1)d) = 3n + 1$ ----------- (1)

Case (a) : Suppose $k \equiv 1 (\text{mod } 6)$.

Let $k = 6r + 1, r \in Z^+$.

Using (1) we have,

$$\frac{(6r + 1)}{2}\,(2a + 6rd) = 3n + 1$$

$$(6r + 1)\,(a + 3rd) = 3n + 1$$

$$\text{That is, } k\,(a + 3rd) = 3n + 1. \qquad \text{-----------}\ (2)$$

$$\text{Therefore, } 3n + 1 \equiv 0 (\text{mod } k).$$

Also from (2), $a \equiv 1(\text{mod } 3)$.

If a, d = 1 then using (1) we get,

$$\frac{k}{2}\,(2 + (k - 1)) = 3n + 1$$

$$k(k + 1) = 6n + 2$$

$$\frac{k(k+1)-2}{6} = n.$$

Since $a \geq 1$, $d \geq 1$ using (1), we get

$$n \geq \frac{k(k+1)-2}{6}.$$

Case (b) : Suppose $k \equiv 2(\text{mod } 6)$.

Let $k = 6r + 2$, $r \in Z^{+}$.

Using (1) we get,

$$\frac{(6r+2)}{2}\,(2a + (6r + 1)d) = 3n + 1$$

$$(3r + 1)\,(2a + (6r + 1)d) = 3n + 1$$

$$\frac{k}{2}\,(2a + (6r + 1)d) = 3n + 1. \qquad \text{-------------}(3)$$

$$\text{Therefore, } 3n + 1 \equiv 0(\text{mod } \tfrac{k}{2}).$$

Also, from (3) we have

$a \equiv 0(\text{mod } 3)$ and $d \equiv 1(\text{mod } 3)$;

$a \equiv 1(\text{mod } 3)$ and $d \equiv 2(\text{mod } 3)$; and

$a \equiv 2(\text{mod } 3)$ and $d \equiv 0(\text{mod } 3)$.

Since $a \geq 3$, $d \geq 1$ and using (1), we get $n \geq \dfrac{k(k+5)-2}{6}$.

Case (c) : Suppose $k \equiv 4(\text{mod } 6)$.

Let $k = 6r + 4$, $r \in \{0\} \cup Z^{+}$.

Using (1) we have,

$$\frac{(6r+4)}{2}(2a + (6r + 3)d) = 3n + 1$$

$$(3r + 2)(2a + (6r + 3)d) = 3n + 1$$

$$\frac{k}{2}(2a + (6r + 3)d) = 3n + 1. \qquad \text{-------------(4)}$$

Therefore, $3n + 1 \equiv 0(\text{mod } \frac{k}{2})$.

Also, from (4) we have $a \equiv 1(\text{mod } 3)$. -------------(5)

Since $a \geq 1$, $d \geq 1$ and using (1), we get

$$n \geq \frac{k(k+1)-2}{6}$$

$$6n + 2 \geq k(k + 1)$$

$$3n + 1 \geq \frac{k(k+1)}{2}$$

Since $3n + 1 \equiv 0(\text{mod } \frac{k}{2})$,

$$3n + 1 - \frac{k(k+1)}{2} = \frac{k}{2}l,\ l \in Z^{+}.$$

$$3n + 1 = \frac{k(k+1)}{2} + \frac{k}{2}l,\ l \in Z^{+}.$$

Using (1), we get $\frac{k}{2}(2a + (k-1)d) = \frac{k(k+1)}{2} + \frac{k}{2}l,\ l \in Z^{+}$.

That is, $2a + (k-1)d = (k + 1) + l,\ l \in Z^{+}$.

Suppose $l = 6s - 3$ where s = 1, 2, . . . , r.

$$2a + (6r + 3)d = 6r + 5 + 6s - 3$$

$$2a + (6r + 3)d = 6r + 6s + 2.$$

Suppose a = 1, d = 1, then $s = \frac{1}{2}$.

Using (1), a and d should be $a \geq 4$, $d \geq 1$.

Suppose a = 4, d = 1, then

$$11 + 6r = 6r + 6s + 2 \text{ and } s = \frac{3}{2}.$$

Suppose $a \geq 4$ and $d \geq 2$, then

$$14 + 12r \leq 2a + (6r + 3)d = 6r + 6s + 2.$$

$14 + 12r \leq 12r + 2$ as $s \leq r$.

From the above arguments, we get a contradiction when $l = 3, 9, 15, \ldots, 6r - 3$.

Case(d) : Suppose $k \equiv 5 (\text{mod } 6)$.

Let $k = 6r + 5$, $r \in \{0\} \cup Z^+$.

Using (1) we have,

$$\frac{(6r+5)}{2}(2a + (6r + 4)d) = 3n + 1$$

$$(6r + 5)(a + (3r + 2)d) = 3n + 1$$

$$k(a + (3r + 2)d) = 3n + 1. \quad \text{------------(6)}$$

Therefore, $3n + 1 \equiv 0(\text{mod } k)$

Also, from (6), we have

$a \equiv 0(\text{mod } 3)$ and $d \equiv 1(\text{mod } 3)$;

$a \equiv 1(\text{mod } 3)$ and $d \equiv 2(\text{mod } 3)$; and

$a \equiv 2(\text{mod } 3)$ and $d \equiv 0(\text{mod } 3)$.

Since $a \geq 3$, $d \geq 1$ and using (1), we obtain

$$n \geq \frac{k(k+5)-2}{6}.$$

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