An extension of a fixed point results with Φ_P operator in Cone Banach Space

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***Abstract*— In this paper we study a class of self-mappings on a Cone Banach Space which have at least one fixed point. More precisely for a closed and convex subset C of a cone Banach space with a generalized norm that satisfy a special condition. We are proposing some extensions of the results of Karapinar and idea Multu and Yolku using operator Φ_P**

Keywords— **fixed point, contraction, Cone Banach Space, Φ_P operator**

I. INTRODUCTION AND PRELIMINARIES

Rzepecki [1] in 1980 was the first that introduced the concept of a generalized metric d_E on a set X in a way that $d_E : X \times X \to S$, where E is Banach space and S is a normal cone in E with partial order $\leq$. In that paper, the author generalized the fixed point theorems of Maia type [2].
Seven years later Lin [3] considered the notion of K-metric space by replacing real numbers with cone K in metric function, that is $d : X \times X \to K$. In 2007, Huang and Zhang [4] announced the notion of cone metric spaces (CMS) by replacing real numbers with an ordering Banach space. In that paper, they also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces.

Recently, many results on fixed point theorems have been extended to cone metric spaces.(see example [3-8],). Karapinar [5] proved some fixed point theorems for self mappings satisfying some contractive condition in these spaces. Mutlu and Yolcu (2013)[6] used the idea of Karapinar to obtain some results through Φ_p operator. We try to use the two ideas to profit some propositions.

Throughout this paper $E := E, \|\cdot\|$ stands for real Banach space. Let $P := P_E$ always be a closed nonempty subset of E. P is called cone if $ax + by \in P$ for all for all $x, y \in P$ and nonnegative number a, b where $P \cap -P = 0$ and $P \neq 0$.

For a given cone P, one can define a partial ordering (denoted by $\leq$ or $\leq_P$)with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$ while $x \ll y$ will show $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of P. From now on, it is assumed that $intP \neq \phi$

The cone P is called

N *normal* if there is a number $K \geq 1$ such that for all $x, y \in E$:

$$x \leq y \Rightarrow \|x\| \leq K\|y\| \quad (1.2)$$

R *Regular* if every increasing sequence which is bounded from above is convergent.

That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$

In N the least positive integer K, satisfying (1.2), is called the normal constant of P.

Lemma 1.1 (see [7, 11])
(i) *Every regular cone is normal.*
(ii) *For each* $k > 1$, *there is a normal cone with normal constant* $K > k$.
(iii) *The cone* P *is regular if every decreasing sequence which is bounded from below is convergent.*
Proofs of (i) and (ii) are given in [7] and the last one follows from definition.

Definition 1.2 (see [4]). Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies

$(M1)$ $0 \leq d(x, y)$ for all $x, y \in X$,

$(M2)$ $d(x, y) = 0$ if and only if $x = y$

$(M3)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$,

$(M4)$ $d(x, y) = d(y, x)$ for all $x, y \in X$,

then d is called cone metric on X, and the pair (X, d) is called a cone metric space (CMS).

*Example 1.3.*Let $E = \mathbb{R}^3$, $P = \{(x, y, z) \in E : x, y, z \geq 0\}$, and $X = \mathbb{R}$. Define $d: X \times X \to E$ by $d(x, \tilde{x}) = \left(\frac{1}{2}|x^3 - \tilde{x}^3|, \frac{1}{3}|x^3 - \tilde{x}^3|, \frac{1}{4}|x^3 - \tilde{x}^3|\right)$. Then (X, d) is aCMS. Note that the cone P is normal with the normal constant $K = 1$.

It is quite natural to consider Cone Normed Spaces (CNS).

Definition 1.4 (see[10], [11]). Let X be a vector space over $\mathbb{R}$. Suppose the mapping $\|\cdot\|_c : X \to E$ satisfies:

$(N1)$ $\|x\|_C > 0$ for all $x \in X$

$(N2)$ $\|x\|_C = 0$ if and only if $x = 0$

$(N3)$ $\|x + y\|_C \leq \|x\|_C + \|y\|_C$ $x, y \in X$

$(N4)$ $\|kx\|_C = |k| \cdot \|x\|_C$ for all $k \in \mathbb{R}$

then $\|.\|_p$is called cone norm on X, and the pair $(X, \|\cdot\|_c)$ is called a cone normed space (CNS).

Note that each CNS is CMS. Indeed, $d(x, y) = \|x - y\|_C$

Definition 1.5. Let (X, $\|.\|_p$) be a CNS, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ *converges to* x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x\|_C \leq c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$

(ii) $\{x_n\}_{n\geq 1}$ is a *Cauchy sequence* whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x\|_P \ll c$ for all $n, m \geq N$.

(iii) $(X, \|\cdot\|_c)$ is a complete cone normed space if every Cauchy sequence is convergent.

Complete cone normed spaces will be called cone Banach spaces.

Lemma 1.6.*Let* (X, $\|.\|_C$) *be a CNS, P a normal cone with normal constant K, and* $\{x_n\}$ *a sequence in X. Then,*

(i) *the sequence* $\{x_n\}$ *converges to* x *if and only if* $\|x_n - x\|_C \to 0$, *as* $n \to \infty$;

(ii) *the sequence* $\{x_n\}$ *is Cauchy if and only if* $\|x_n - x_m\|_C \to 0$ *as* $n, m \to \infty$

(iii) *the sequence* $\{x_n\}$ *converges to* x *and the sequence* $\{y_n\}$ *converges to* y *then* $\|x_n - y_n\|_C \to \|x - y\|_C$

The proof is direct by applying [4] to the cone metric space (X, d)

where $d(x, y) \to \|x - y\|_C$, for all $x, y \in X$.

Lemma 1.7 [8]. *Let* (X, $\|.\|_C$) *be a CNS over a cone P in E. Then:*

(i) $\operatorname{int}(P) + \operatorname{int}(P) \subseteq \operatorname{int}(P)$ and $\lambda \operatorname{int}(P) \subseteq \operatorname{int}(P), \lambda > 0$.

(ii) *If* $c \gg 0$ *then there exists* $\delta > 0$ *such that* $\|b\| < \delta$ *implies* $b \ll c$

(ii) *For any given* $c \gg 0$ *and* $c_0 \gg 0$, *there exists* $n_0 \in N$ *such that* $c_0 / n_0 \ll c$.

(iv) *If* a_n, b_n *are sequences in E such that* $a_n \to a, b_n \to b$, *and* $a_n \leq b_n$, *for all* n *then* $a \leq b$

The proofs of the first two parts followed from the definition of $\operatorname{int}(P)$. The third part

is obtained by the second part. Namely, if $c \gg 0$ is given then find $\delta > 0$ such that $\|b\| < \delta$

implies $b \ll c$. Then find n_0 such that $1/n_0 < \delta / \|c_0\|$ and hence $c_0 / n_0 \ll c$ Since P is closed,

the proof of fourth part is achieved.

II. MAIN RESULTS

From now on, $X = (X, \|\cdot\|_c)$ will be a cone Banach space, P will be a normal cone with a normal constant K and T will be self-mapping operator defined on a subset C of X. Let Φ_P be an increasing, positive and self-mapping operator defined on E, where E is Banach algebra. In this paper we represent a generalization of Theorem 2.4 in [5] as well as theorem 6 in Multu [6].

Definition 2.1. Let E be Banach algebra and $E = (E, \|\cdot\|)$ be a Banach space. $\Phi_P : E \to E$ is an increasing and positive mapping, i. e. $\Phi_p(x) = \|x\|^{p-2} x$, where for a $q > 0$ holds $\frac{1}{p} + \frac{1}{q} = 1$.

If E = ₹ , then Φ_p: ₹ →₹ is a p- Laplacian operator, i.e. $\Phi_p(x) = \|x\|^{p-2} x$ for some p > 1.

By using definition, we can show that the operator $\Phi_P : E \to E$ holds the following properties:

(1) if $x \leq y$, then $\Phi_p(x) \leq \Phi_p(y)$ for all $x, y \in E$

(2) Φ_P is continuous bijection and its inverse mapping is also continuous,

(3) $\Phi_p(x \cdot y) \leq \Phi_p(x) \cdot \Phi_p(y)$ for all $x, y \in E$

(4) $\Phi_p(x + y) \leq \Phi_p(x) + \Phi_p(y)$ for all $x, y \in E$

Theorem 2.2. Let C be a closed and convex subset of a cone Banach space X with norm $\|x\|_C = d(x, 0)$ Let E be a Banach algebra and $\Phi_P : E \to E$ and $T : C \to C$ be mappings and T satisfy the following condition

$$\Phi_p(d(x, Tx)) + \Phi_p(d(y, Ty)) \leq k \cdot \Phi_p(d(x, y)) \qquad (2.1)$$

for all $x, y \in C$ and $\lambda > 1$, where $\lambda^{p-1} < k < (2\lambda)^{p-1}$, Then T has at least one fixed point.

Proof. Let $x_0 \in C$ be an arbitrary point, Define a sequence $\{x_n\}$ as follows:

$$x_{n+1} := \frac{(\lambda - 1)x_n + T(x_n)}{\lambda} \quad n = 0, 1, 2, \ldots \qquad (2.2)$$

Notice that

$$x_n - Tx_n = \lambda(x_n - (\frac{(\lambda - 1)x_n + Tx_n}{\lambda}) = \lambda(x_n - x_{n+1}) \qquad (2.3)$$

which can write

$$d(x_n,Tx_n)=\|x_n-Tx_n\|=\lambda\|x_n-x_{n+1}\|=\lambda d(x_n,x_{n+1}) \qquad (2.4)$$

for $n = 0,1\ 2,\ \ldots$. If we substitute $x = x_{n-1}$ *and* $y = x_n$ then from (2.1) we have

$$\Phi_p(d(x_{n-1},Tx_{n-1}))+\Phi_p(d(x_n,Tx_n))\le k\Phi_p(d(x_{n-1},x_n))$$

and from(2.4) we obtain

$$\Phi_p(\lambda d(x_{n-1},x_n))+\Phi_p(\lambda d(x_n,x_{n+1}))\le k\Phi_p(d(x_{n-1},x_n)) \qquad (2.5)$$

or

$$\Phi_p(\lambda[d(x_{n-1},x_n))+\Phi_p(d(x_n,x_{n+1})]) \le k\Phi_p(d(x_{n-1},x_n))$$

regarding property (3) and invers property of the operator, on obtain

$$d(x_n,x_{n+1})\le\left(\frac{\Phi_q\ \ k}{\lambda}-1\right)d(x_{n-1},x_n)$$

iterative process gives

$$d(x_n,x_{n+1})\le\left(\frac{\Phi_q\ \ k}{\lambda}-1\right)^n d(x_0,x_1) \qquad (2.6)$$

Let $m>n$ then from (2.6) we have

$$d(x_m,x_n)\le 1\frac{(\frac{\Phi_q(k)}{\lambda}-1)^n}{2-\frac{\Phi_q(k)}{\lambda}}d(x_0,x_1)$$

Since $\lambda^{p-1}<k<(2\lambda)^{p-1}$. Hence, the sequence x_n is a Cauchy sequence in C. As the C is closed and convex subset of a cone Banach space then it converges to the point $z\in C$. Regarding the inequality

$$d(z,Tx_n)\le d(z,x_n)+d(x_n,Tx_n)=d(z,x_n)+\lambda d(x_n,x_{n+1})$$

and the Lemma (1.6) we have $Tx_n\to z$.

If we substitute $x=z$ and $y = x_n$ then inequality (2.1) and (2.3) imply

$$\Phi_p(d(z,Tz))+\Phi_p(d(x_n,\mathrm{T}\,x_n))\le k\Phi_p(d(z,x_n))$$

or

$$\Phi_p(d(z,Tz))+\lambda(d(x_n,\mathrm{x}_{n+1}))\le k\Phi_p(d(z,x_n))$$

By using (2,4)

$$d(z,Tz)+\lambda d(x_n,x_{n+1})\le\Phi_p(k)d(z,x_n)$$

when $n\to\infty$, $d(z,Tz)=0$. Then $Tz=z$.

Corollary 2.3. In condition of the theorem 2.2. if $\lambda=2$ we obtain the result of the theorem (6) in [6]

Theorem 2.4. Let C be a closed and convex subset of a cone Banach space X with norm $\|x\|_C=d\quad x,0$ Let E be a Banach algebra and $\Phi_p:E\to E$ and $T\ :\ C\to C$ be mappings and T satisfy the following condition

$$\Phi_p(d\quad Tx,Ty\ \ )+\Phi_p(d\quad x,Tx\ \ )+\Phi_p(d\quad y,Ty\ \ )\le r\Phi_p(\ d\quad x,y\ \ ) \quad (2.7)$$

for all $x,y\in C$ and $\lambda>1$, where $1<r<(2\lambda+1)^{p-1}$. Then, T has at least one fixed point.

Proof. Construct the sequence x_n in same way as in proof of the theorem (2.2) and equalities (2.3) and (2.4) hold. Let we see also

$$x_n-Tx_{n-1}=\frac{(\lambda-1)x_{n-1}+Tx_{n-1}}{\lambda}-Tx_{n-1}=\frac{\lambda-1}{\lambda}(x_{n-1}-Tx_{n-1}) \quad (2.8)$$

from that

$$d(x_n,Tx_{n-1})=\|x_n-Tx_{n-1}\|_p=\frac{\lambda-1}{\lambda}\|x_{n-1}-Tx_{n-1}\|_p=\frac{\lambda-1}{\lambda}d(x_{n-1},Tx_{n-1}) \qquad (2.9)$$

Triangle inequality for the points x_n, Tx_n, Tx_{n-1} and property (4) of the operator Φ_p implies

$$\Phi_p(d(x_n,Tx_n)) - \Phi_p(d(x_n,Tx_{n-1})) \le \Phi_p(d(Tx_{n-1},Tx_n)) \quad (2.10)$$

then by (2.4), (2.8) and (2.9) we obtain

$$\Phi_p(\lambda d(x_n,x_{n+1})) - \Phi_p(\frac{\lambda-1}{\lambda} d(x_{n-1},Tx_{n-1})) \le \Phi_p(d(Tx_{n-1},Tx_{n)}) \quad (2.11).$$

Replacing $x = x_{n-1}$ and $y = x_n$ in (2.7) we have

$$\Phi_p(d(Tx_{n-1},Tx_n)) + \Phi_p(d(x_{n-1},Tx_{n-1})) + \Phi_p(d(x_n,Tx_n)) \le r\Phi_p \cdot d(x_{n-1},x_n) \quad (2.12)$$

regarding (2.9), (2.10) and (2.8) one can obtain

$$\Phi_p(\lambda d(x_n,x_{n+1})) - \Phi_p(\frac{\lambda-1}{\lambda} d(x_{n-1},Tx_{n-1})) + \Phi_p(\mathrm{d}(\mathrm{x}_{n-1},Tx_{n-1})) + \Phi_p(d(x_n,Tx_n)) \le r\Phi_p(d(x_{n-1},x_n)) \quad (2.12)$$

or

$$\Phi_p(2\lambda d(x_n,x_{n+1}) + \mathrm{d}(\mathrm{x}_{n-1},x_n)) \le r\Phi_p(d(x_{n-1},x_n))$$

by (2.8)

$$d(x_n,x_{n+1}) \le \frac{\Phi_q(r)-1}{2\lambda} d(x_{n-1},x_n) \quad (2.13)$$

if $0 < \frac{r^{q-1}-1}{2\lambda} < 1$ or $1 < r < (2\lambda+1)^{p-1}$.

For these values, the sequence x_n is a Cauchy sequence that converges to any $z \in C$. Since the sequence Tx_n also converges to z as in proof of the theorem (2.2) under the assumption that $x = z$ and $y = x_n$ and by help of lemma (1.6) which yields $d(Tz,z) + d\ z,Tz\ \le 0$ which is equivalent with $z = Tz$.

Theorem 2.5. Let C be a closed and convex subset of a cone Banach space X with norm $\|x\|_C = d\ x,0$ Let E be a Banach algebra and $\Phi_p : E \to E$ and $T\ :\ C \to C$ be mappings and T satisfy the following condition

$$\Phi_p(d(Tx,Ty)) \le k\Phi_p(d(x,y)) \quad (2.14)$$

where $0 < k < 1$. Then T has one fixed point.

Proof. Let we regard again three points x_n, Tx_n, Tx_{n-1}. The triangle property one can write

$$d\ Tx_n,Tx_{n-1}\ \ge d\ Tx_n,x_n\ - d\ x_n,Tx_{n-1}$$

or

$$\Phi_p(d(\mathrm{T}x_n,Tx_{n-1}) \ge \Phi_p(d(Tx_n,x_n) - \Phi_p(d(x_n,Tx_{n-1})$$

replacing two terms of the right side with (2.5) and (2.9) we have

$$\Phi_p(d(Tx_n,Tx_{n-1})) \ge \Phi_p(\lambda d(x_n,x_{n+1}) - \ \lambda-1\ d(x_{n-1},x_n)$$

If we substitute the right side to the inequality (2.14) on can obtain

$$\Phi_p(\lambda d(x_n,x_{n+1})) - \ \lambda-1\ d(x_{n-1},x_n) \le k \cdot \Phi_p(d(x_n,x_{n-1}))$$

or

$$d(x_n,x_{n+1}) \le \frac{\Phi_q(k)+(\lambda-1)}{\lambda} d(x_n,x_{n-1}) \quad (2.15).$$

The sequence x_n is a Cauchy sequence and also convergent if is satisfied the condition $\frac{k^{q-1}+(\lambda-1)}{\lambda} < 1$ or $k < 1$.

Notice that we get another proof for well known Banach Theorem in cone Banach space.

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