Analytical discourse on Linguistics in Classical and Fuzzy Mathematics

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Abstract**—** In this paper we emphasize, the importance of Linguistics in **Classical Logic** and its extended and generalized form "a branch of Mathematics Fuzzy Logic". Both are the basic building blocks of **Mathematics**. The **Classical logic** deals with propositions which can be either true or false but not both (vague or inexact concepts). Whereas **Fuzzy Logic** deals with propositions which can be true or false, as well as they can take all the values between true and false. We propose the basic aspects of propositional logic (PL). The PL has advantage of being capable of interpreting in English Language. Just as in English Language grammar is used to generate sentences, in PL too grammar is used to generate well –formed propositions which are analogues of sentences. We construct the **well – formed propositions** and develop reasoning theory utilizing basic **logical connectives**. The study of PL consists of grammar, dictionary meaning, inference rules and valuations.

Keywords**—** Classical logical connectives, Fuzzy logical connectives and Linguistics quantifiers.

1. Introduction

Mathematics is an exact science. Every proposition or statement in mathematics must be precisely written with proper English grammar and Logical reasoning. Also every proposition in Mathematics cannot be without proof and each proof needs proper logical reasoning. Therefore a mathematician must be grammarian and logician. The fact is illustrated by the following incidence.

A layman, Businessman and Mathematician were visiting to Mumbai for the first time. As they were approaching Mumbai, they saw a dog just like pink colored crossing the road. With a great eager the layman said, "Look dogs in Mumbai are pink". Immediately businessman corrected him and said "**some dogs** in Mumbai are Pink". To which the Mathematician responded and made a proposition or statement with proper logical reasoning and English grammar as, "In all ($\forall$) I can say is that, there exists at least one dog in Mumbai, who has a noticeable pink spot on one side". Here we notice that the first two statements are more or less incorrect, ambiguous and confusing. Clearly it happened due to lack of proper English grammar and Logical reasoning. On the contrary the third statement pervades very sensible massage to the society.

As a matter of fact, all the Sciences and Social Sciences progress only with the aid of Mathematical Logic. So it is aptly remarked, **Mathematics is a Science of all Sciences and art of all arts.**

Logic is the study of methods and principles of reasoning. More precisely, it is codification of first approximation to human reasoning. Logic goes by many names: classical logic, Boolean logic, two valued logic, Symbolic logic etc..

Here we focus mainly on propositional logic (PL) in the classical sense, as by name it deals with statement or propositions.

2 . The language and grammar of propositional Logic:

Many of the sentences of advanced Mathematics have a complicated structure which can become easier to understand, if one knows a few basic terms of mathematical grammar and Logical reasoning. We explain the most important basic of mathematical grammar namely "parts of speech", some of which are similar to those of natural languages and other quite different. The main reason for the importance of mathematical grammar is that the statement of mathematics are supposed to be precise, and it is not possible to achieve a high level of precision unless the language one uses is free of many of the vagueness and ambiguities of ordinary speech. To illustrate the sort of clarity and simplicity that is needed in mathematical discourse. Let us consider the famous mathematical sentence.

Two plus two equals four.

As a sentence of English rather than of Mathematics, try to analyze it grammatically. It contains a verb "equal" and a conjunction "plus". However looking more carefully we may begin to notice some oddities. Such as the word "plus" resembles the word "and" the paradigm example of conjunction, it does not behave in quite the same way, as shown in the sentence.

Mary and Peter Love Paris.

The verb in this sentence "love" is plural, whereas the verb in the previous sentence "equal" was singular. So the word "plus" seems to take two objects and produce out of them a new, single object, while "and" conjoins "Mary" and "Peter" in a looser way, living them as distinct people.

A word, which famously has three quite distinct meanings, is "is". The three meanings are illustrated in the following three sentences.

1) 5is a square root of 25.
2) 5 is less than 10.
3) 5 is a prime number.

In the first of these sentences, "is" could be replaced by "equals". It says that two objects 5 and the square root of 25 is in fact one and the same object, just as in the English sentence "London is the capital of the united Kingdom". In the second sentence, "is" plays a completely different role. The words in the adjectival phrase "less than 10", specifying a property that "numbers may or may not have", and "is" in this sentence is like "is" in the English sentence "grass is green." As for the third sentence, the word "is there means "is an example of" as it does in English sentence "Mercury is a planet."

Sets. Sets are very important in development of Mathematics. They are not only important in proving mathematical object but also in the process of mathematical reasoning itself. Hence sets are called building blocks of Mathematics. Sets allow one to reduce greatly the number of parts of speech that one needs, turning almost all of them in to nouns. For example, with the help of the membership symbol " $\in$ " one can do without adjectives, as the translation of "The proposition p: 5 is a prime number" (where prime function as an adjective) in to "$5 \in p$." The symbol "$\in$" is usually read, as an element of". So the "is" in sentence (3) is more like $\in$ than $=$.

Functions. : Functions seem to be more like process. So far as the term function belongs to the English Grammar it will act as nouns. Moreover, when they appear in mathematical sentences they do not behave like nouns. They are more like propositions. E.g. The set of even no. is expressed as $f(x) = 2x$, x belongs to set of positive integers.

2.1 Grammar of propositional Logic: The study of propositional logic consists of grammar (syntax), meaning (semantics), inference rules and derivation (valuation). Propositional logic can be viewed as a language and this language is based on alphabets i.e. symbols. The alphabets or primitive symbols of PL consist of:

1. Propositional variables, denoted by p, q, r, s, t,.
2. Constants, denoted by T & C and
3. Connectives denoted by $\sim$, $\wedge$, $\vee$, N, $\rightarrow$, $\leftrightarrow$.

Note that the five connectives: NOT ($\sim$), OR ($\vee$), AND ($\wedge$), IF . . . THEN ($\rightarrow$) and IF AND ONLY IF ($\leftrightarrow$) are called propositional connectives or sentential connectives or logical connectives or simply connectives in logic.We shall now discuss the sentential connectives in detail.

- **NOT ($\sim$): The negation**

Even though it is referred to as a 'connective' in logic, it is not used for combining two statements but it is used to modify a given statement. Hence it is called **unary connectives**. Note that, all other connectives combine two statements and thus they are **binary connectives**.

If p is a statement then negation of p i.e. NOT p is denoted by $\sim$p or $\bar{p}$. For example if p is "I know English" then $\sim$p is "I do not know English".

The truth table for NOT is given by

p	$\sim$p
T	F
F	T

Note that True is denoted by T or 1 and false is denoted by F or 0.

- **OR ($\vee$): Disjunction**

A logical connective "or" a word that has a more restrictive meaning to mathematician it does to normal speakers of English language. The mathematical use is illustrated by the joke of responding "Yes Please" to question such as, "would you like your coffee with or without sugar? Thus the statement p or q ($p \vee q$) is true if and only if p is true or q is true or both are true. The disjunction of two statements p and q is the composite statement p or q denoted by $p \vee q$. One of the most important points to be noted here is that in symbolic language and in Mathematics OR is always taken in the **inclusive sense** unless other stated. Therefore, by p or q, we mean **either P or q or both.**

For example:

p: There are 26 alphabets in English,

Q: Mathematics is an interesting subject, then

$p \vee q$: There are 26 alphabets in English or Mathematics is an interesting subject.

Now there are 26 alphabets in English is always true, therefore $p \vee q$ is true irrespective of q. Hence $p \vee q$ is true if at least one of p or q is true. We shall now write the truth table for OR ($\vee$).

P	q	$p \vee q.$
T	T	T
T	F	T
F	T	T
F	F	F

- **AND ($\wedge$) : Conjunction**

The conjunction of two statements p and q is the composite statement p and q denoted by $p \wedge q.$ We observe that p and q is true only when p and q are both true and in all other cases it is false.

For example:

P: I know English,

q: I know Mathematics, then

$p \wedge q$: I know English and I know Mathematics.

We shall now write the truth table for OR ($\vee$).

P	q	$p\wedge q.$
T	T	T
T	F	F
F	T	F
F	F	F

- **If p then q ($\rightarrow$) : Conditional**

The statement $p \rightarrow q$ means roughly speaking that q is a consequence of p, and it is read as “If p then q”, but as with “or” this does not mean quite what it would in English. To get a feel for difference, consider an even worse piece of mathematical pedantry, contained in the following exchange during a mathematics lecture.

Lecturer: Would you please raise your hand if you have solved the difficult problem. I mentioned in the last lecture?

Lecturer: [To a student in the first row] Ah, well done. And perhaps you would like to tell us the solution.

Student: But I have not solved it.

Lecturer: In that case why did you put your hand up?

Student: You asked to us to raise your hands if you had solved the problem. You gave no instructions about what to do if you had not.

The sentence $p \rightarrow q$ is considered to be true, under all circumstances except one: it is not true if p is true and q is false. This is the definition of “implies”. It can be confusing because in English the word “implies” suggest some sort of connection between p and q, that p is some way cause of q or at least relevant to it. If p causes q then certainly p cannot be true without q being true, but all mathematician care about is this logical consequence and not whether there is any reason for it. Thus, if you want to prove that $p \rightarrow q$ all you to do are rule out the possibility that p could be true and q false at the same time. To give an example: if n is positive integer, then the statement, “n is perfect square with final digit 7” implies the statement, “n is prime number” not because there is any connection between the two but no perfect square ends in a 7. Of course we avoid such implication, being confused by some of the ambiguities and subtle nuances of ordinary language.

If p and q are two statements, then, “If p then q” means that “the statement p implies the statement q”

Or “the statement q is implied by the statement p”. It is called a conditional statement and is denoted by

$p \rightarrow q$ or $p \Rightarrow q.$

For example:

p: I go mad.

q: I bite you, then

$p \rightarrow q$: If I go mad, then I bite you.

Let us analyze the example as follows:

For the implication "$p \rightarrow q$" to be true,

(i) If I go mad, then I must bite you, otherwise the implication is false. I.e. if p is true then, for the implication '$p \rightarrow q$' to be true q must be true.

(ii) If I do not go mad but still I bite you, then the outcome q has anyway occurred therefore implication $p \rightarrow q$ is true. I.e. when p is false but is true, $p \rightarrow q$ is true.

(iii) Suppose I do not go mad and I do not bite you then $p \rightarrow q$ is true. I.e. If p is false and q is false then $p \rightarrow q$ is true.

The truth table for $p \to q$ is as follows:

P	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The important point is to be noted here is that, if we write any false statement as p, then for every statement q, the implication $p \to q$ is true.

The following terminology is used for conditional statement.

Whenever $p \to q$ is a conditional statement then

(i) $q \to p$ is called its converse

(ii) $\sim q \to \sim p$ is called contrapositive and

(iii) $\sim p \to \sim q$ is called its inverse.

Finally we discuss we discuss the last connectives, namely IF AND ONLY IF (↔)

- **IF AND ONLY IF (↔): Biconditional**

Well-formed propositions (WFFs): Just as in English language, grammar or syntax is used to generate sentences in PL too grammar is used to generate well formed formulae which are analogous to sentences. For examples:

$p, p \vee q, p \to q$, $[p \wedge (q \to r)] \leftrightarrow [(p \wedge q) \to r)]$ etc. are well- formed propositions.

Meaning (semantics) of Propositional logic: Here we define meaning of propositional logic. Just as there is a dictionary for words, phrases and sentences of natural languages giving their meanings, analogous, we talk of meanings of WFFs in PL**.** The dictionary of PL is concise and compact as every primitive WFFs can have only one of the two meanings True (T) or False (F).

Valuation V is a procedure for assigning one of the two values - True or false – to each WFF

Subject to the following conditions:

- $V(NF)$ = True, if V(F) = False.

  = False, if V (F) = True
- $V(F \vee G)$ = False, if V(F) = False = V(G)

  = True, otherwise.
- $V(F \wedge G)$ = True, if V(F) = True = V(G)

  = False, otherwise.
- $V(F \to G)$ = False, if V(F) = True and V(G) = False

  = True, otherwise.
- $V(F \leftrightarrow G)$ = True, if V(F) = True = V(G) or V(F) = False = V(G)

  = False, otherwise.

Example. Consider the WFF

P: $[p \wedge (q \to r)] \to [(p \wedge q) \to r)]$.

Valuation or interpretation is can be given very easily.

Quantifiers:

Yet another ambiguity in the English language is exploited by the following old joke that suggests that our priorities need radically rethinking.

1) Nothing is better than lifelong happiness.
2) But a cheese sandwich is better than nothing.
3) Therefore, a cheese sandwich is better than nothing lifelong happiness.

Let us try to be precise about how this play on words works. It hinges on the word “nothing” which is used in two different ways. The first sentence means, “Some things are available there but no single thing that is better than lifelong happiness”, whereas the second means, “It is better to have a cheese sandwich than to have nothing at all.” In other words, in the second sentence, “nothing” stands for what one might call the null option, whereas in the first it does not (to have nothing is not better than to have lifelong happiness).

Words like “all”, “some”, “every” are called quantifiers, and in English language they are highly prone to this kind of ambiguity. Mathematics therefore makes do with just two quantifiers, and the rules for their use are much stricter. They tend to come at the beginning of sentences, and can be read as “for all” (or “for every”), it symbolized by “∀" and “there exist” (or “for some”) it symbolized by "∃". A rewriting of sentence 1) that renders it unambiguous is

1) for all x, lifelong happiness is better than x.

The second sentence cannot be rewritten in these terms because the word “nothing” is not playing the role of a quantifier. It’s nearest mathematical equivalent is something like the empty set.

Armed with “for all (∀)” and “there exists (∃)" we can be clear about the difference between the beginnings of the following sentences.

4) Everybody likes at least one drink, and that drink is water.
5) Everybody likes at least one drink; I myself go for red wine.

The symbols “for all (∀)” and “there exists (∃)" allow us to write quite complicated mathematical sentences in a highly symbolic form if we want to. For example suppose we let proposition p be the “set of all primes”. Then the following symbols make the claim that, there are infinitely many primes.

6) ∀ n, ∃ m : m > n, ∀ m, n ∈ p. In words, this means that, for every n, we can find m that is bigger than n and is a prime. This of course means there must be infinitely many primes.

Remark: It fells that the quantifiers are free standing, but actually they are always associated with a set.

Birth of Fuzzy Mathematics:

Fuzzy Mathematics was introduced by Lotfi A. Zadeh, who felt that the classical set theory is not adequate enough to provide the required mathematical framework. So he proposed the use of the fuzzy set theories for providing such a framework for varioua fuzzy logics.

The concept of a fuzzy set theory is a generalization as well as extension of the concept of a crisp set theory. Thus the basic theme and ideas of crisp set theory will be reflected in fuzzy set theory. Just as a crisp set on a universal set U is defined by a function,
$A: U \to \{0,1\}$.
Whereas a fuzzy set on U is meant a function,
$A: U \to [0,1]$.

We also write, $A = \{(x, A(x)) : x \in U\}$, where A is called the membership function, $A(x)$ is called the membership grade of x, 0 means no membership and 1 means full membership.

Conclusion:

The main aim of this paper is to introduce the importance of Linguistics in Classical and Fuzzy Mathematics and not to provide the minute details**.** In this paper we have focused on: The basic building blocks of propositional logic and how they are used in building up Classical and Fuzzy Mathematical Theory. Proper knowledge of English as an instructional language is necessary in Mathematics for translating ideas in to symbols. Also it is necessary to understand proper use of punctuation marks and tenses to describe Mathematical concepts. It is also required to adopt consistent meaning of terms until the students understand the conceptual meaning.

Future Scope: In future we will introduce some of the major and important applications of fuzzy set theory and fuzzy logic for real life situations.

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