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Analysis of an N-policy M/E_k/1 Queueing system with Unreliable server, Multiple Vacations and Balking

P.Jayachitra[#1], A.James Albert[*2]

[#1] *Research Scholar, Department of Mathematics, Karpagam University, Coimbatore, Tamil Nadu, India,*

[*2] *Professor, Department of Mathematics, Hindusthan College of Engineering and Technology, Coimbatore, Tamil Nadu, India*

***Abstract*— Present investigation deals with optimal control of a non reliable and removable server in an N-policy M/E_k/1 queueing system with server multiple vacations and balking. The server can be turned off and takes a vacation whenever the system is empty. If the number of units waiting in the system is less than N, the server will take another vacation. When the queue length reaches N (N≥1), the server is immediately turned on and serves the waiting units until the number of units in the system becomes empty. Arriving customers may balk with certain probability and may depart without getting service due to impatience. Lack of service occur where the server is busy or due to sudden breakdown. By applying probability generating function technique expected number of units in the system is obtained and also derived various performance measures. Further the total expected cost function is developed to determine the optimal threshold N at a minimum cost. The sensitivity analysis is performed to examine the effect of various parameters in the system.**

***Keywords*— N-policy, M/E_k/1 queueing system, probability generating function, server vacations, balking.**

I. Introduction

Queueing models with server vacations accommodate the real-world situations more closely. Server vacations are useful for the system where the server wants to utilize his idle time for different purposes. Such model frequently occurs in the area of computer and communications and manufacturing systems. The analysis of queueing systems with impatient customers has become more prevalent in the literature due to the potential application of these systems in call canters, communication systems, production-inventory systems and many other related areas. Models with customers' impatience in queues have been studied by various authors in the past, where the source of impatience was either a long wait already experienced in the queue or a long wait anticipated by a customer upon arrival. There is an extensive amount of literature based on this kind of model. Haight[6] first considered an M/M/1 queue with balking. An M/M/1 queue with customers reneging was also proposed by Haight[7]. The combined effects of balking and reneging in an M/M/1/N queue have been investigated by Ancker and Gafarian[2,3]. Barer [4] carried out one of the early work on reneging where he considered deterministic reneging with single server Markovian arrival and service rates. Abou-El-Atc and Hariri[1] considered multiple server queueing system M/M/C/N with balking and reneging. Multi-server Markovian queuing system with balking and reneging was discussed by Choudhury and Medhi [5]. M/M/1 queueing system with customers' impatience in the form of random balking was analysed by Manoharan and Jose [10]. This paper provides an adequate service for its customers with tolerable waiting, whenever customers' impatience becomes sufficiently strong and customers leave before being served.

The concept of the *N* policy was first introduced by Yadin and Naor [12]. Wang and Huang [11] proposed a management policy for M/Ek/1 queueing systems under *N* policy. Jayachitra and James Albert [9] discussed Optimal control of an N-policy M/Ek/1 queueing system with server startup and breakdown. Also the authors [8] provided the Optimal management policy for Heterogeneous arrival M/Ek/1 queueing system with server breakdowns and multiple vacations under N-policy.

The purpose of this paper is three fold.

1. We establish the steady state results to obtain the probability distribution of the number of units in the system.
2. We determine the probability of empty system and expected number of units in the system.
3. We formulate the total expected cost for the system and determine the optimal management policy for such type of queueing models.

II. Model Description

For the purpose of analytical investigation, we consider the model with the following assumptions:

1. The arrival is Poisson process and the service times according to an Erlang distribution with mean 1/μ and stage parameter k. The Erlang type k distribution is made up of k independent and identical exponential stages, each with mean 1/kμ. A customer goes into the first stage of the service then progresses through the remaining stages and must complete

the last stage before the next customer enter the first stage. We assume that customers arriving at the service station form a single waiting line and are served in the order of their arrivals.

2. When the system is turned off, the server leaves the system for a random period of time called vacation which is exponentially distributed with parameter θ.
3. When the server is working, the server may breakdown at any time with a Poisson breakdown rate α.
4. When the server fails, it is immediately repaired at a repair rate β, where the repair times are exponentially distributed.
5. A customer may balk from the queue with probability b_0 when the server is in idle or may balk with a probability b_1 when the server is in service made due to impatience.

III. STEADY STATE RESULTS

In steady state the following notations are used.

$P_{0,0,0}$ = Probability that there are no customers in the system when the server is on vacation

$P_{n,i,0}$ = Probability that there are n customers in the system and the customers in service is in stage i while the server is on vacation.

$P_{n,i,1}$ = Probability that there are n customers in the system and the customer is in the stage i while the server is busy.

$P_{n,i,2}$ = Probability that there are n customers in the system and the customer in service is in stage i when the server is in operation but found to be broken down.

The steady state equations are given as follows

$$\lambda b_0 P_{1,k,0} = \lambda b_0 P_{0,0,0} \tag{1}$$

$$\lambda b_0 P_{n,k,0} = \lambda b_0 P_{n-1,k,0} \qquad (2 \le n \le N-1) \tag{2}$$

$$\lambda b_0 P_{0,0,0} = k\mu P_{1,1,1} \tag{3}$$

$$(\lambda b_1 + \theta) P_{n,k,o} = \lambda b_0 P_{n-1,k,0} \qquad (n \ge N) \tag{4}$$

$$(\lambda b_1 + k\mu + \alpha) P_{1,i,1} = k\mu P_{1,i+1,1} + \beta P_{1,i,2} \qquad (1 \le i \le k-1) \tag{5}$$

$$(\lambda b_1 + k\mu + \alpha) P_{1,k,1} = k\mu P_{2,1,1} + \beta P_{1,k,2} \tag{6}$$

$$(\lambda b_1 + k\mu + \alpha) P_{n,i,1} = \lambda b_1 P_{n-1,i,1} + k\mu P_{n,i+1,1} + \beta P_{n,i,2} \qquad (2 \le n \le N-1\ 1 \le i \le k-1) \tag{7}$$

$$(\lambda b_1 + k\mu + \alpha) P_{n,k,1} = \lambda b_1 P_{n-1,k,1} + k\mu P_{n+1,1,1} + \beta P_{n,k,2} \qquad (2 \le n \le N-1) \tag{8}$$

$$(\lambda b_1 + k\mu + \alpha) P_{n,k,1} = \lambda b_1 P_{n-1,k,1} + k\mu P_{n+1,1,1} + \theta P_{n,k,0} + \beta P_{n,k,2} \qquad (n \ge N) \tag{9}$$

$$(\lambda b_1 + k\mu + \alpha) P_{n,i,1} = \lambda b_1 P_{n-1,i,1} + k\mu P_{n,i+1,1} + \beta P_{n,i,2} \qquad (n \ge N, 1 \le i \le k-1) \tag{10}$$

$$(\lambda b_1 + \beta) P_{1,k,2} = \alpha P_{1,k,1} \tag{11}$$

$$(\lambda b_1 + \beta) P_{n,i,2} = \alpha P_{n,i,1} + \lambda b_1 P_{n-1,i,2} \qquad (1 \le i \le k-1)\ (n \ge 2) \tag{12}$$

$$(\lambda b_1 + \beta) P_{1,i,2} = \alpha P_{1,i,1} \qquad (1 \le i \le k-1) \tag{13}$$

$$(\lambda b_1 + \beta) P_{n,k,2} = \lambda b_1 P_{n-1,k,2} + \alpha P_{n,k,1} \qquad (n \ge 2) \tag{14}$$

Solving equations (1),(2),(3) and (4) recursively, we finally get

$$P_{n,k,0} = \begin{cases} P_{0,0,0}, & 1 \le n \le N-1 \\ \left(\dfrac{\lambda b_0}{\lambda b_1 + \theta} \right)^{n-(N-1)} P_{0,0,0}, & n \ge N \end{cases} \tag{15}$$

IV. PROBABILITY GENERATING FUNCTION

The technique of using the probability generating function may be applied in a recursive manner from equation (1) to (14) to obtain the analytic solution of $P_{0,0,0}$ in neat closed form expression. Define the probability generating function of $G_0(z)$, $G_1(z)$ and $G_2(z)$ respectively as follows:

$$G_0(z) = P_{0,0,0} + \sum_{n=1}^{\infty} z^n P_{n,k,0} \tag{16}$$

$$H_i(z)=\sum_{n=1}^{\infty} z^n P_{n,i,1} \qquad 1\le i\le k-1 \tag{17}$$

$$G_1(z)=\sum_{i=1}^{K} H_i(z) \tag{18}$$

$$G_i(z)=\sum_{n=1}^{\infty} z^n P_{n,i,2} \qquad 1\le i\le k-1 \tag{19}$$

$$G_2(z)=\sum_{i=1}^{K} G_i(z) \tag{20}$$

where $|z|\le 1$

Applying algebraic manipulation technique to equations (2),(3),(15) and using in (16) we get the following

$$G_0(z)=\left[\frac{1-z^N}{1-z}+\frac{\lambda b_0 z^N}{\lambda b_1+\theta-\lambda b_0 z}\right]P_{0,0,0} \tag{21}$$

Multiplying equation (5) by z and (7)and (10) by z^n and summing over n we get

$$H_{i+1}(z)=\frac{1}{k\mu}\left(\lambda b_1(1-z)+k\mu+\alpha\right)H_i(z)-\beta G_i(z) \tag{22}$$

Multiplying equation (6) by z (8) and (9) by z^n and summing over n we get

$$\left[\lambda b_1 z^2-(\lambda b_1+k\mu+\alpha)z\right]H_K(z)+k\mu H_1(z)+\beta z G_k(z)=k\mu\left[z-\frac{\theta z^{N+1}}{\lambda b_1+\theta-\lambda b_0 z)}\right]P_{0,0,0} \tag{23}$$

Again multiplying (11) and (13), (12) and (14) respectively by appropriate powers of z and summary over n we find

$$\left[\lambda b_1(1-z)+\beta\right]G_i(z)=\alpha H_i(z) \tag{24}$$

$$\left[\lambda b_1(1-z)+\beta\right]G_K(z)=\alpha H_K(z) \tag{25}$$

Solving equations (22) to (25) we get

$$G_1(z)=\frac{k\mu z\left[\dfrac{\theta z^N-(\lambda b_1+\theta-\lambda b_0 z)}{(\lambda b_1+\theta-\lambda b_0 z)}\right]}{k\mu(1-z)+\alpha-\beta\left(\dfrac{\alpha}{\lambda b_1(1-z)+\beta}\right)}\left[\frac{1-\left\{\dfrac{1}{k\mu}\left[\left(\lambda b_1(1-z)+k\mu+\alpha\right)-\beta\left(\dfrac{\alpha}{\lambda b_1(1-z)+\beta}\right)\right]\right\}^k}{1-z\left\{\dfrac{1}{k\mu}\left[\left(\lambda b_1(1-z)+k\mu+\alpha\right)-\beta\left(\dfrac{\alpha}{\lambda b_1(1-z)+\beta}\right)\right]\right\}^K}\right]P_{0,0,0} \tag{26}$$

$$G_2(z)=\frac{\alpha}{\lambda b_1(1-z)+\beta}G_1(z) \tag{27}$$

Now using the normalizing condition $G\,(1)=G_0(1)+G_1(1)+G_2(1)=1$, we obtain the value of the probability that the empty system

$$P_{0,0,0}=\frac{\theta\left[\mu\beta-\lambda b_1\left(\alpha+\beta\right)\right]}{\left[\mu\beta-\lambda\left(\alpha+\beta\right)(b_1-b_0)\right]\left(N\theta+\lambda b_0\right)} \tag{28}$$

V. EXPECTED NUMBER OF UNITS IN THE SYSTEM

Let L_V , L_b and L_d denote the expected number of customers in the system when the server is on vacations, busy and breakdowns states respectively.

The expression for L_V, L_b and L_d are given by

$$L_V=G_0'(1) \qquad L_b=G_1'(1) \qquad L_d=G_2'(1)$$

Differentiating equations (21),(26), (27) respectively and using L'Hospital rule twice we get,

$$L_V=\left[\frac{N(N-1)}{2}+\frac{\lambda b_0}{\theta^2}\left(N\theta+\lambda b_0\right)\right]P_{0,0,0} \tag{29}$$

$$L_b = \left\{ \frac{\beta(N\theta+\lambda b_0)}{\theta(\alpha+\beta)} \left[\frac{2\alpha\mu k(\lambda b_1)^2 + \beta\lambda^2(\alpha+\beta)(k+1)}{2\mu\beta k\left[\rho b_1(\alpha+\beta)-\mu\beta\right]^2} \right] + \frac{\lambda(\alpha+\beta)[N\theta+\lambda b_0]}{\theta\left[\lambda(\alpha+\beta)-\mu\beta\right]} \left\{ \left[\frac{\theta N(N-1)}{(N\theta+\lambda b_0)} + 2\left(\frac{\lambda b_0}{\theta}+1\right) \right] \left[-\left(1+\frac{\alpha}{\beta}\right) \right] + 2\alpha\frac{\lambda b_1}{\beta^2} \right\} \right\} P_{0,0,0} \tag{30}$$

$$L_d = \frac{\alpha}{\beta} L_b - \left[\frac{\alpha\lambda^2 b_1 b_0}{\beta\left[\mu\beta - \lambda(b_1-b_0)(\alpha+\beta)\right]} \right] \tag{31}$$

Thus we obtain the expected number of customers in the queueing system is given by

$$L_N = \sum_{i=0}^{2} G_i'(1) \tag{32}$$

VI. OTHER SYSTEM CHARACTERISTIC

Let I, B, and D denote the length of the server off, busy and breakdown periods respectively. Applying the memory less property of the Poisson process, we find that the mean length of the idle period is

$$E(I) = \frac{N}{\lambda b_0}$$

The expected length of the idle period, busy period and breakdown period are denoted by E(I),E(B) and E(D) respectively. The expected length of the busy cycle is given by

E(C) = E(I) + E(B)+E(D)

The long run fraction of time the server is in idle, busy and breakdown states are respectively given by

$$\frac{E(I)}{E(C)} = G_0(1) = \frac{\mu\beta - \lambda b_1(\alpha+\beta)}{\mu\beta - \lambda(\alpha+\beta)(b_1-b_0)}$$

$$\frac{E(B)}{E(C)} = G_1(1) = \frac{\beta\lambda b_0}{\mu\beta - \lambda\left[\alpha+\beta\right](b_1-b_0)}$$

$$\frac{E(D)}{E(C)} = G_2(1) = \frac{\alpha\lambda b_0}{\mu\beta - \lambda\left[\alpha+\beta\right](b_1-b_0)}$$

Thus we have the number of cycles per unit time

$$E(C) = \frac{1}{\lambda b_0 P_{0,0,0}}$$

VII. OPTIMAL CONTROL POLICY

Now we develop a cost model by considering *N* as decision variable. Our objective is to determine the optimal value of N, say N* so that the cost function is minimized. Let us denote the cost factors associated with different activities as follows.

C_h = holding cost per unit time for each customer present in the system.

C_f = setup Cost per busy cycle.

C_b = Cost per unit time for keeping the server on and in operation.

C_o = Cost per unit time for keeping the server is on vacation.

C_d =breakdown cost per unit time

C_k = Cost per unit time when a customer balks.

Using the definition of each cost element and its corresponding characteristics, the total expected cost function per unit time is given by

$$T(N) = C_h L_N + \frac{C_f}{E(C)} - C_o\frac{E(I)}{E(C)} + C_b\frac{E(B)}{E(C)} + C_d\frac{E(D)}{E(C)} + C_k\left\{\lambda(1-b_0)\frac{E(I)}{E(C)} + \lambda(1-b_1)\left[\frac{E(B)}{E(C)} + \frac{E(D)}{E(C)}\right]\right\} \tag{33}$$

We obtain the optimal value N*, which minimizes cost function by differentiating equation (33) with respect to N and setting the result to be zero. i.e., $\frac{\partial}{\partial N}(T(N)) = 0$. The solution N to (33) may not be an integer and the optimal positive integer value of N is one of the integers surrounding N* which gives a smaller cost T.

VIII. SENSITIVITY ANALYSIS

We now perform a sensitive analysis on the optimum value N* based on changes in specific values of the system parameters using MATLAB. In order to arrive at the conclusions, the following arbitrary values of the system parameters are considered.

Case 1: C_h=5, C_0=10, C_b=100, C_d=200, C_s=125, C_f=500, C_k=10
Case 2: C_h=5, C_0=50, C_b=300, C_d=350, C_s=400, C_f=1000, C_k=30
Case 3: C_h=50, C_0=100, C_b=500, C_d=550, C_s=800, C_f=1500, C_k=100

The optimal value of N and its minimum expected cost T(N*) for the above three cases are shown in the following table for various values of parameters.

It is observed that

i) N* and T(N*) increases when λ increases
ii) T(N*) decreases as Θ increases and N* slightly decreases as Θ increases.
iii) T(N*) increases when α changes from 0.1 to 0.3 for any cases. Intuitively N* is insensitive to changes in α.
iv) N* remains unchanged to the changes of β from 2.0 to 6.0 and T(N*) slightly decreases to the changes in β from 2.0 to 6.0
v) N* increases when μ changes from 1.0 to1.4. and T(N*) decreases to the changes in μ from 1.0 to 1.4 for all cases.
vi) T(N*) increases as b_0 increases and T(N*) decreases as b_1 increases.
vii) N* is slightly increases as b_0 increases and N*decreases as b_1 increases.
viii) N* is insensitive in every cases and T(N*) decreases as k increases for every cases.

Overall we conclude that

1. α, β and k do not affect N*
2. Θ, b_0 and b_1 rarely affect N*.
3. λ and μ affect N* significantly
4. C_h and C_f have much stronger effect on N*

TABLE

The optimal value of N and its minimum expected cost for various parameters

λ	Case(1) (N*,T(N*))	Case(2) (N*,T(N*))	Case(3) (N*,T(N*))	Θ	Case(1) (N*,T(N*))	Case(2) (N*,T(N*))	Case(3) (N*,T(N*))
0.4 0.6 0.8	5, 64.690 6, 144.591 7, 185.651	8, 83.7252 9, 205.133 10,268.352	2,320.483 3,812.853 4,869.505	0.1 0.3 0.5	5,70.910 5,65.187 4,64.417	8,92.478 8,84.565 7,83.240	2,420.516 3,329.111 3,315.615
α	Case(1) (N*,T(N*))	Case(2) (N*,T(N*))	Case(3) (N*,T(N*))	β	Case(1) (N*,T(N*))	Case(2) (N*,T(N*))	Case(3) (N*,T(N*))
0.1 0.2 0.3	5, 42.125 5, 44.175 5, 46.336	7, 86.525 7, 92.256 7, 98.624	3, 182.858 3, 194.260 3, 206.389	2 4 6	4, 37.641 4, 37.032 4, 36.831	5, 99.562 5, 96.564 5, 93.654	2, 169.008 2, 165.501 2, 164.349
μ	Case(1) (N*,T(N*))	Case(2) (N*,T(N*))	Case(3) (N*,T(N*))	k	Case(1) (N*,T(N*))	Case(2) (N*,T(N*))	Case(3) (N*,T(N*))
1.0 1.2 1.4	4,50.516 5,42.345 6,37.234	5,104.365 6,85.654 7,68.564	2,246.484 3,197.067 4,166.662	1 2 3	5,41.467 5,41.355 5,41.318	7,45.051 7,44.904 7,44.867	3,181.774 3,180.659 3,180.288
b_0	Case(1) (N*,T(N*))	Case(2) (N*,T(N*))	Case(3) (N*,T(N*))	b_1	Case(1) (N*,T(N*))	Case(2) (N*,T(N*))	Case(3) (N*,T(N*))

0.4	4, 40.748	6, 45.516	2, 178.618	0.2	5, 41.467	7, 46.058	3,181.774
0.45	5, 44.205	7, 52.825	3, 196.915	0.3	5, 40.748	6, 45.516	2, 178.618
0.5	5, 47.500	7, 59.834	3, 214.338	0.4	4, 40.053	6, 45.015	2, 175.601

IX. CONCLUSION

Optimal strategy analysis of an N-policy M/Ek/1 queueing system with unreliable server, server multiple vacations and balking have been studied. Some of the system performance measures have been derived. A cost function has been formulated to determine the optimal value of N. Sensitivity analysis has been carried out through numerical illustrations to examine the effect on the optimal value N and cost function of different parameters. This analysis will be helpful to decision makers and system designers for the choice of optimal control policy.

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