On LPS Riemannian Manifold – Ii

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Abstract

In this paper, Lorentzian para sasakian Riemannian manifold have been studied. The first section is introductory. Basic definition and known results are defined. Second section deals with LPS-Riemannian manifold and the third section is devoted for KLPS-Riemannian manifold. Some interesting results have been investigated.

1. Introduction

Definition (1.1) Let an n dimensional differentiable manifold Mn is called a Lorentzian para sasakian (LP-sasakian) manifold if it admits (1,1) tensor field F, a vector field T, a 1 form A and a Lorentzian metric 'g' satisfying –

(1.1) (a) $A(T) = -1$

(b) $F^2 X = X + A(X) T,$

(c) $g(FX, FY) = -g(X, Y) + A(X) A(Y).$

(d) $g(X, T) = -A(X)$

(e) $F(X) = D_X T$

(f) $(D_X F)Y = g(X,Y) + A(X) A(Y) T + (X+A(X)T) A(Y)$

where D denoted the covariant differentiation with respect to g.
In an LP-sasakian manifold Mn with structure (F,T,A,G) we can easily show that -

(1.2) (a) $F(T) = 0$

(b) $A(FX) = 0$

(c) $\text{rank}(F) = n-1$

more over if we put

(1.3) $'F(X,Y) = g((\overline{X}, Y))$

where 'F is a skew symmetric

Definition (1.2) An LP- Sasakian manifold on which the fundamental 2 – form 'F such that

(1.4) $2\,'F = dA$ [1]

is satisfied is called LP-sasakian manifold [2]

Some Results on APST – Riemannian manifold

(1.5) $'F(X, Y) = \frac{1}{2}[(D_X A)(Y) - (D_Y A)(X)].$

(1.6) $(d\,'F) = 0 \Leftrightarrow (D_X\,'F)(Y,Z) + (D_Y\,'F)(Z,X) + (D_Z\,'F)(X,Y) = 0.$

(1.7) $'F(X,Y) = (D_X A)(Y) = -(D_Y A)(X)$

(1.8) $(D_Z\ 'F)(X,Y) = A\ (K(X,Y,Z))$

(1.9) $D_X\ 'F(Y,T) = -g\ (\overline{X}, \overline{Y})$

(1.10) $(D_Z\ 'F)\ (\overline{X}, \overline{Y}) - (D_Z\ 'F)\ (X, Y) - g\ (Z, X)\ A\ (Y) + g\ (Z, Y)\ A\ (X) = 0.$

(1.11) $(D_Z\ 'F)\ (X, Y) = A\ (Y)\ (D_Z\ 'F)\ (X, T) - A\ (X)\ (D_Z\ 'F)\ (Y, T).$

(1.12) $\underset{0}{N}\ (X, Y) = (D_{\overline{X}}\ F)\ (Y) - (D_{\overline{Y}}\ F)\ (X) + (D_X F)\ (\overline{Y}) - (D_Y F)\ (\overline{X}) -$

$A\ (Y)\ (\overline{X}) + A\ (X)\ (\overline{Y})$

(1.13) $(D_Z\ 'F)\ (\overline{X}, Y) + (D_Z\ 'F)\ (X, \overline{Y}) = A(X)\ g(\overline{Y}, Z) - A\ (Y)\ g\ (\overline{X}, Z\quad)$

(1.14) $D_X T = F(X) = \overline{X}$

(1.15) $D_X\ 'F(Y,Z) = A\ (K\ (Y,Z,X))$

2. SOME THEOREMS ON LPS – Riemannian manifold

Let us put

(2.1) $T\ (X, Y) = (D_{\overline{X}}\ F)\ (Y) - \overline{(D_X F)(Y)} + (D_X A)\ (Y)\ T$

$= (D_{\overline{X}}\ F)\ (Y) + (D_X F)\ (\overline{Y}) - (D_X A)\ (Y)\ T - A\ (Y)\ (\overline{X}) + (D_X A)\ (Y)\ T$

$= (D_{\overline{X}}\ F)\ (Y) + (D_X F)\ (\overline{Y}) - A\ (Y)\ (\overline{X}).$

(2.2) a) $'N\ (X, Y, Z) \overset{\text{def}}{=} g\ (\underset{0}{N}\ (X, Y), Z)$

b) $'T\ (X, Y, Z) \overset{\text{def}}{=} g\ (T\ (X, Y), Z).$

using (2.1),

(2.3) $T\ (X, Y) - T\ (Y, X) = (D_{\overline{X}}\ F)\ (Y) + (D_X F)\ (\overline{Y}) - A\ (Y)\ (\overline{X})\ -$

$(D_{\overline{Y}}\ F)\ (X) - (D_Y F)\ (\overline{X}) + A\ (X)\ (\overline{Y})$

from (1.12), (2.3) we get,

(2.4) $T\ (X, Y) - T\ (Y,X) = \underset{0}{N}\ (X, Y).$

or

(2.5) $\underset{0}{N}\ (X, Y) = T\ (X, Y) - T(Y, X) = -\ \underset{0}{N}\ (Y, X)$

Thus we have

Theorems (2.1) ON KLPS – Riemannian manifold,

$$\underset{0}{N}(X, Y) = T(X, Y) - T(Y, X) = -\underset{0}{N}(Y, X).$$

from (2.2) (b)

$$'T(X, Y, Z) = g(T(X, Y), Z)$$

$$= g((D_{\bar{X}} F) Y, Z) + g((D_X F) \bar{Y}, Z) - A(Y)\, g(\bar{X}, Z)$$

(2.6) $'T(X, Y, Z) = (D_{\bar{X}}\ 'F)(Y,Z) + (D_X\ 'F)(\bar{Y}, Z) - A(Y)\ 'F(X, Z)$

(2.7) $'T(X, Z, Y) = (D_{\bar{X}}\ 'F)(Z,Y) + (D_X\ 'F)(\bar{Z}, Y) - A(Z)\ 'F(X,Y)$

Adding (2.6) and (2.7), we get.

(2.8) $'T(X, Y, Z) + 'T(X, Z, Y) = (D_{\bar{X}}\ 'F)(Y,Z) + (D_X\ 'F)(\bar{Z},Y) -$

$A(Y)\ 'F(X, Z) - A(Z)\ 'F(X,Y).$

using (1.11), (2.8) becomes

(2.9) $'T(X, Y, Z) + 'T(X, Z, Y) = -A(Z)\ 'F(X,Y) - 'F(X,Z)\, A(Y) - A(Y)\ 'F(X,Z) - A(Z)\ 'F(X,Y)$

$= -2\{ 'F(X,Y)\, A(Z) + 'F(X,Z)\, A(Y) \}$

Thus we have

Theorem (2.2). On KLPS Riemannian manifold,

If, **$'T(X, Y, Z) = g(T, X, Y), Z)$,**

then

$'T(X, Y, Z) + 'T(X, Z, Y) = -2\,['F(X, Y)\, A(Z) + 'F(X, Z)\, A(Y)]$.

From (2.2) (a)

$$'N(X, Y, Z) = g(\underset{0}{N}(X, Y), Z)$$

$$= g((D_{\bar{X}} F) Y, Z) - g((D_{\bar{Y}} F) X, Z) + g((D_X F) \bar{Y}, Z) -$$

$$g((D_Y F) \bar{X}, Z) - A(Y)\, g(\bar{X}, Z) + A(X)\, g(\bar{Y}, Z),$$

(2.10) $'N(X, Y, Z) = (D_{\bar{X}}\ 'F)(Y, Z) - (D_{\bar{Y}}\ 'F)(X, Z) + (D_X\ 'F)(\bar{Y}, Z) -$

$(D_Y\ 'F)(\overline{X}, Z) - A(Y)\ 'F(X, Z) + A(X)\ 'F(Y,Z).$

(2.11) $'N(Y,Z,X) = (D_{\overline{Y}}\ 'F)(Z,X) - (D_{\overline{Z}}\ 'F)(Y, X) + (D_Y\ 'F)(\overline{Z}, X) - (D_Z 'F)(\overline{Y}, X) - A(Z)\ 'F(Y,X) + A(Y)\ 'F(Z, X).$

(2.12) $'N(Z, X, Y) = (D_{\overline{Z}}\ 'F)(X, Y) - (D_{\overline{X}}\ 'F)(Z, Y) + (D_Z\ 'F)(\overline{X}, Y) - (D_X\ 'F)(\overline{Z}, Y) - A(X)\ 'F(Z, Y) + A(Z)\ 'F(X, Y).$

Adding (2.10) and (2.12) and subtract (2.11), we get

(2.13) $'N(X,Y,Z) + 'N(Z, X, Y) - 'N(Y, Z, X) = (D_{\overline{X}}\ 'F)(Y, Z) - (D_{\overline{Y}}\ 'F)(X, Z) + (D_X\ 'F)(\overline{Y}, Z) - (D_Y\ 'F)(\overline{X}, Z) - A(Y)\ 'F(X, Z) + A(X)\ 'F(Y, Z) + (D_{\overline{Z}}\ 'F)(X, Y) - (D_{\overline{X}}\ 'F)(Z, Y) + (D_Z\ 'F)(\overline{X}, Y) - (D_X\ 'F)(\overline{Z}, Y) - A(X).\ 'F(Z, Y) + A(Z)\ 'F(X, Y) - (D_{\overline{Y}}\ 'F)(Z, X) + (D_{\overline{Z}}\ 'F)(Y, X) - (D_Y\ 'F)(\overline{Z}, X) + (D_Z 'F)(\overline{Y}, X) + A(Z)\ 'F(Y, X) - A(Y)\ 'F(Z, X).$

$= (D_{\overline{X}}\ F)(Y, Z) + (D_X\ 'F)(\overline{Y}, Z) - (D_Y\ 'F)(\overline{X}, Z) - (D_{\overline{X}}\ F)(Z,Y) + (D_Z\ 'F)(\overline{X}, Y) - (D_X\ 'F)(\overline{Z}, Y) - (D_Y\ 'F)(\overline{Z}, X) + (D_Z\ 'F)(\overline{Y}, X) + A(X)\ 'F(Y,Z) - A(X)\ 'F(Z,Y)$

Using (1.11), (2.13) becomes.

$= 2(D_{\overline{X}}\ 'F)(Y, Z) + A(X)\ 'F(Y, Z) - A(X)\ 'F(Z,Y)$

$= 2(D_{\overline{X}}\ F)(Y,Z) + 2A(X)\ 'F(Y, Z).$

$= 2\ 'T(X,Y,Z) + 2A(Z)\ 'F(X,Y) + 2A(Y)\ 'F(X,Z) + 2A(X)\ 'F(Y,Z)$

$= 2[\ 'T(X,Y,Z) + A(X)\ 'F(Y,Z) + A(Y)\ 'F(X,Z) + A(Z)\ 'F(X,Y)]$

thus we have,

Theorem (2.3) On KLPS Riemannian manifold, the $\underset{0}{N}$ **Vanishes**

if and only if.

$'T(X,Y) = -\{ A(X)\, 'F(Y,Z) - A(Y)\, 'F(Z,X) + A(Z)\, 'F(X,Y) \}$

$'T(X,Y) = -\{ A(X)\, \overline{Y} - A(Y)\, \overline{Z} + A(Z)\, \overline{X} \}$

3. AFFINE CONNEXION

On LPS – Riemannian manifold, Riemannian connexion D and affine connexion B is connected as,

(3.1) $B_X Y = D_X Y + H(X, Y)$,

(3.2) $B_X T = D_X T + H(X,T)$

Using (1.14)

$$B_X T = \overline{X} + H(X, T)$$

$$F(B_X T) = F(\overline{X}) + F(H(X,T))$$

$$F(B_X T) = \overline{\overline{X}} + F(H(X, T))$$

$$F(B_X T) = X + A(X)\, T + F(H(X, T))$$

$$F(B_{\overline{X}} T) = \overline{X} + A(\overline{X})\, T + F.(H(\overline{X}, T))$$

$$F(B_{\overline{X}} T) = \overline{X} + F(H(\overline{X}, T))$$

thus we have

Theorem (3.1) On LPS – Riemannian manifold,

(3.3) $$F(B_{\overline{X}} T) = \overline{X} + F(H(\overline{X}, T)).$$

we have from (3.1)

$$B_X Y = D_X Y + H(X, Y)$$

(3.4) $$B_X \overline{Y} = D_X \overline{Y} + H(X, \overline{Y})$$

(3.5) $$(B_X F)(Y) + F(B_X Y) = (D_X F)(Y) + F(D_X Y) + H(X, \overline{Y})$$

$$(B_X F)(Y) + F(B_X Y - D_X Y) = (B_X F)(Y) + H(X, \overline{Y})$$

$$(B_X F)(Y) + \overline{H(X,Y)} = (D_X F)(Y) + H(X, \overline{Y})$$

$$(B_X F)(Y) = (D_X F)(Y) + H(X, \overline{Y}) - \overline{H(X,Y)}$$

(3.6) $$g((B_X F))\, Y,Z) = g((D_X F)\, Y, Z) + g(H(X, \overline{Y}), Z) - g(\overline{H(X,Y)}, Z)$$

$= \quad (D_X\ 'F)(Y,Z) + 'H(X, \overline{Y}, Z) + 'H(X, Y, \overline{Z})$

Using (1.15) becomes

(3.7) $(B_X\ 'F)(Y,Z) \quad = \quad A(K(Y,Z,X)) + 'H(X, \overline{Y}, Z) + 'H(X, Y, \overline{Z})$

Thus we have

Theorem (4.2) On LPS – Riemannian manifold, we have

$(B_X\ 'F)(Y,Z) \quad = \quad A(K(Y,Z,X))$

If $'H(X, \overline{Y}, Z) + 'H(X,Y, \overline{Z}) \quad = \quad 0$

Let us assume

$(D_XD_YA)(Z) = (D_X\ 'F)(Y,Z) = g(X,Y)\,A(Z) + g(X,Z)\,A(Y) + 2A(X)\,A(Y)\,A(Z)$

$L(X,Y,Z) = g(X,Y)\,A(Z) + g(X,Z)\,A(Y) + 2A(X)\,A(Y)\,A(Z)$

$L(Y,Z,X) = g(Y,Z)\,A(X) + g(Y,Z)\,A(Z) + 2\,A(Y)\,A(Z)\,A(X)$

$L(X,Y,Z) – L(Y,Z,X) = g(X,Z)\,A(Y) – g(Y,Z)\,A(X)$

$L(X,Y,T) – L(Y,T,X) = 0$

$L(T,Y,Z) – L(Y,Z,T) = A(Z)\,A(Y) – g(Y,Z)$

$L(T, \overline{Y}, Z) – L(\overline{Y}, Z, T) = -(F(Y,Z)$

Theorem : On LPS – Riemannian manifold, we have $L(X,Y,T) = L(Y,T,X)$

and $L(T, \overline{Y}, Z) – L(\overline{Y}, Z, T) = -\ 'F(Y,Z)$

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