Dual-Matrix Approach for Solving the Transportation Problem

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Abstract

An Algorithm has been developed for solving special types of transportation problem having total demand more than or equal to the total supply. This algorithm is useful for solving both balanced and unbalanced transportation problems. The dual-matrix approach is used in the present algorithm.

Keywords: Transportation problem, dual-matrix approach, stepping-stone method, simplex method, linear programming models.

Introduction

The minimum cost planning plays an important role for solving the transportation problems from origins to different destinations, such as from factories to warehouses, or from warehouses to supermarkets, etc. The shipping cost from one location to another is usually a linear function of the number of units shipped. Simplex method which utilizes linear programming models can be effectively used for solving transportation problems. Ping and CHU(2002) showed that dual-matrix approach is more efficient in comparison to simplex method in the context of transportation.

Dual- matrix approach was applied successfully earlier for solving the transportation problems[2] (ping and CHU, 2002) having demand is less than or equal to the supply. However the present study indicates that even the Ping and CHU’s approach also suffers from certain constraints having demand is more than or equal to the supply. The new approach similar to dual-matrix approach represents the algorithm for solving transportation problem having demand is greater than or equal to the supply.

The new approach considers the dual of the transportation problem instead of primal and obtains the optimal solution of the dual by the use of matrix operations. The new algorithm is detailed in the paper, and finally numerical example is given to illustrate the approach.

The Model and its Dual

The transportation problem is usually presented as matrix as shown in Figure 1. The unit transportation cost generally indicated on the northeast corner in each cell.
This problem can be expressed as a linear programming model as follows:

$$\text{minimize } \phi = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\left.\begin{aligned} &\text{Subject to } \sum_{j=1}^{n} x_{ij} \ge a_i \quad (i = 1,\ 2,\ \ldots,\ \text{m}) \qquad (1)\\ &\sum_{i=1}^{n} x_{ij} \le b_j \quad (j = 1,\ 2,\ \ldots,\ \text{n}) \qquad (2)\\ &x_{ij} \ge 0 \ (i = 1,\ 2,\ \ldots,\ \text{m},\ \ j = 1,\ 2,\ \ldots,\ \text{n}) \end{aligned}\right\} \ldots\ldots\text{LP}(1)$$

Here all a_i ’s and b_j ’s are assumed to be positive, and called supplies and demands respectively, as shown in Figure1, The cost c_{ij} are all nonnegative. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, is a condition for balanced transportation problem. If this condition is not met, a dummy origin or destination is generally introduced to make the problem balanced. The new dual-matrix approach presented now does not require a transportation problem to be balanced. The approach can be applied to both balanced and unbalanced problems, no dummy origin or destination is introduced, and so time and space are saved.

The dual matrix approach considers the dual model of the transportation problem, and the main operations are calculated on a matrix. Another advantage is that degeneracy does not exist in the dual-matrix approach.

If model LP1 is considered as the primal, then its dual can be formulated as follows:

$$\text{maximize } \psi = \sum_{i=1}^{m} a_i u_i - \sum_{j=1}^{n} b_j v_j$$

$$\left.\begin{array}{ll} \text{Subject to } u_i - v_j \le c_{ij} \quad (i = 1,\ 2,\ \dots,\ \text{m},\quad j = 1,\ 2,\ \dots,\ \text{n}) & (3) \\ \qquad u_i, v_j \ge 0 \ (i = 1,\ 2,\ \dots,\ \text{m},\quad j = 1,\ 2,\ \dots,\ \text{n}) & \end{array}\right\} \text{.....LP(2)}$$

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this condition is not met, a dummy origin or destination is generally introduced to make the problem balanced. The new dual-matrix approach presented now does not require a transportation problem to be balanced. The approach can be applied to both balanced and unbalanced problems, no dummy origin or destination is introduced, and so time and space are saved.

The primal problem has m+n constraints and m.n variables. The dual has m+n variables and m.n constraints. Because of fewer variables in dual problem, the dual problem will be solved by the dual-matrix approach quickly. However, all m.n constraints are not necessary to be presented explicitly as in LP2. All these constraints are kept in original transportation matrix, as indicated in Figure 1.

The Dual-Matrix Approach

The main idea of the dual-matrix approach is to obtain a first feasible solution to the dual problem and its corresponding matrix. Then the duality theory is used to check the optimality condition and to get the leaving cell. All non-basic cells are evaluated in order to get the entering cell. Finally, the entering cell replaces the leaving cell and the matrix is updated. The dual-matrix approach is presented as follows:

Step 0 Initialization

Step 0.1 Set $A = (a_1, a_2, \dots, a_m, -b_1, -b_2, \dots, -b_n)$

Step 0.2 Set $v_j = 0;\ (j = 1, 2, \dots, n)$ and let $u_t = \overline{c_{ij}} = \min\{c_{ij},\ j = 1, 2, \dots, n\};\ i = 1, 2, \dots, m$.

Ties can be broken arbitrarily. The corresponding cell to $\overline{c_{ij}}$ are $(i, j_i)(i = 1, 2, \dots, m)$, respectively.

Step 0.3 Let the basic cell set $\Gamma = \{(1, j_1),\ (2, j_2),\ \dots,\ (m, j_m),\ (0,1),\ (0,2),\ \dots, (0, n)\}$.

The cells $(0,1),\ (0,2),\ \dots,\ (0, n)$ are called virtual cells because they do not exist in the original transportation problem matrix.

Step 0.4 Let the matrix $D = [d_{ij}],\ i, j = 1, 2, \dots\dots, m+n;$

$$\text{where } d_{ij} = \begin{cases} 1 & i, j = 1, 2, \dots, m; \\ -1 & i = 1, 2, \dots, m;\ j = m + j_1,\ m + j_2, \dots, m + j_m \\ -1 & i, j = m+1,\ m+2, \dots, m+n \\ 0 & otherwise \end{cases}$$

And compute the objective $\psi = \sum_{i=1}^{m} a_i u_i - \sum_{j=1}^{n} b_j v_j$

Step1 **Determination of leaving cell:**

Step 1.1 Compute $Y = AD$

Step 1.2 Find the smallest value y_k in the elements of Y, that is the value of the k^{th} element in Y is the smallest. Ties can be broken arbitrarily.

Step 1.3 If $y_k \geq 0$, the solution is optimal(both the dual and primal), stop. Otherwise, leaving cell is k^{th} cell in Γ that is (i_k, j_k).

Step2 **: Determination of entering cell:**

Step 2.1 Let

$$P = \begin{bmatrix} p_1 \\ p_2 \\ . \\ . \\ . \\ p_i \\ . \\ . \\ . \\ p_m \end{bmatrix} = \begin{bmatrix} d_{1,k} \\ d_{2,k} \\ . \\ . \\ . \\ d_{i,k} \\ . \\ . \\ . \\ d_{m,k} \end{bmatrix} \quad Q = \begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ q_j \\ . \\ . \\ . \\ q_n \end{bmatrix} = \begin{bmatrix} d_{m+1,k} \\ d_{m+2,k} \\ . \\ . \\ . \\ d_{m+j,k} \\ . \\ . \\ . \\ d_{m+n,k} \end{bmatrix}$$

Step 2.2 For all non basic cells, if $q_j - p_i \leq 0$, then the dual problem is not bounded, and the original primal problem has no feasible solution, and stop. Otherwise, compute $\theta_{ij} = c_{ij} - u_i + v_j$ if $q_j - p_i > 0$

Step 2.3 Find the smallest value θ_{st} in all θ_{ij}, and the cell (s,t) is the entering cell. Ties can be broken arbitrarily.

Step3 **Updating**

Step3.1 Update the matrix D

Step3.1.1 For the elements of column k in D

$$\hat{d}_{lk} = -d_{lk} \qquad l = 1, 2, ..., m+n$$

Step3.1.2 For the elements of the other columns in D

$$\hat{d}_{lr} = d_{lr} + \left(d_{t+m,r} - d_{sr}\right)\hat{d}_{lk} \begin{cases} r = 1, 2, ..., k-1, k+1, ..., m+n; \\ l = 1, 2, ..., m+n \end{cases}$$

Step3.2 Update the basic cell set Γ: replace the k^{th} cell (i_k, j_k) Γ with the entering cell (s,t)

Step3.3 Update the objective value: Compute

$$\hat{u}_i = u_i - \theta_{st} p_i \qquad i = 1, 2, ..., m$$
$$\hat{v}_j = v_j - \theta_{st} q_j \qquad \text{j} = 1, 2, ..., n$$

and the objective $\psi = \sum_{i=1}^{m} a_i u_i - \sum_{j=1}^{n} b_j v_j$

go to step 1.

Destination								
Origin		D_1	D_2	$\dots$	D_j	$\dots$	D_n	Supply
	O_1	c_{11} x_{11}	c_{12} x_{12}	$\dots$	c_{1j} x_{1j}	$\dots$	c_{1n} x_{1n}	a_1
	O_2	c_{11} x_{11}	c_{22} x_{22}	$\dots$	c_{2j} x_{2j}	$\dots$	c_{2n} x_{2n}	a_2
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
	O_i	c_{i1} x_{i1}	c_{i2} x_{i2}	$\dots$	c_{ij} x_{ij}	$\dots$	c_{in} x_{in}	a_i
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
	O_m	c_{m1} x_{m1}	c_{m2} x_{m2}	$\dots$	c_{mj} x_{mj}	$\dots$	c_{mn} x_{mn}	a_m
	Demand	b_1	b_2	$\dots$	b_j	$\dots$	b_n	

Figure 1: transportation problem Matrix

The initialization procedure is to obtain an initial feasible solution. By setting $v_j = 0$ and u_i being the smallest cost in the row i, obviously they meet the constraint set(3) in the dual problem. The matrix D is an $(m+n)\times(m+n)$ matrix, which can be divided into four sub-matrices as follows:

1. The upper left sub matrix is an $m\times m$ identity matrix.
2. The upper right sub-matrix is an $m\times n$ matrix: If the cell (i, j) is a basic cell (corresponding to $\overline{c_{ij}}$) then the element (i, j) in this sub-matrix is -1. All other elements in this sub-matrix are 0.
3. The lower left sub-matrix is an $n\times m$ zero matrix.
4. The lower right sub-matrix is an $n\times n$ negative identity matrix.

During the main procedure of the dual-matrix approach, step 1 is to get the leaving cell, similar to getting a leaving variable in the simplex method. As matter of fact, the initial feasible solution in the dual-matrix approach is a very good starting point. From the objective function in the dual LP2, it is obvious that v_j should be smaller, the better. The smallest is 0 for all v_j. On the other hand, u_i should be larger, the better. However due to the constraints set (3) a u_i can only be the minimum value of c_{ij} in the row i. Step 2 is to obtain the entering cell by evaluating all non-basic cells, which is similar to stepping-Stone method (Charnes & Cooper, 1954). The equation in step 2.2 is the same as the one in stepping-stone method (MODI, precisely) except the sign of u_i. Finally the matrix D and other relevant data are updated accordingly. To explain this approach mathematically, a cell (i, j) can be represented as an equation, that is $u_i - v_j = c_{ij}$. The mathematical background of the dual-matrix approach is to find $m+n$ equations, i.e., $m+n$ basic cells from constraints set(3) in the dual. If these equations cannot maximize the objective of the dual, that is solution is not optimal, find another cell (equation), i.e., the entering cell, to replace one equation set until an optimal solution is found. So, the algorithm of this approach is very simple as shown previously.

Numerical Example

The following unbalanced transportation problem is considered.

	Destination					
		D_1	D_2	D_3	D_4	Supply
Origin	O_1	25 x_{11}	17 x_{12}	25 x_{13}	14 x_{14}	300
	O_2	15 x_{21}	10 x_{22}	18 x_{23}	24 x_{24}	500
	O_3	16 x_{31}	20 x_{32}	8 x_{33}	13 x_{34}	600
	Demand	300	300	500	500	

Step 0 Initialization

$A=(300,\ 500,\ 600,\ -300,\ -300,\ -500,\ -500)$ $v_1=v_2=v_4=0$ and $u_1=14,\ u_2=10,\ u_3=8$, $\Gamma=\{(1,4),\ (2,2),\ (3,3),\ (0,1),\ (0,2),\ (0,3),\ (0,4)\}$

$$D=\begin{bmatrix}1&0&0&0&0&0&-1\\0&1&0&0&-1&0&0\\0&0&1&0&0&-1&0\\0&0&0&-1&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&-1\end{bmatrix}$$

and $$\psi=\sum_{i=1}^{m}a_iu_i-\sum_{j=1}^{n}b_jv_j=14900$$

Step1 Determination of leaving cell:

$Y=AD=(300,\ 500,\ 600,\ 300,\ -200,\ -100,\ 200)$ so k=5 and the leaving cell is (0,2) in Γ

Step2 Determination of entering cell

$$P=\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}=\begin{bmatrix}d_{1,5}\\d_{2,5}\\d_{3,5}\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}\quad Q=\begin{bmatrix}q_1\\q_2\\q_3\\q_4\end{bmatrix}=\begin{bmatrix}d_{4,5}\\d_{3,5}\\d_{6,5}\\d_{7,5}\end{bmatrix}=\begin{bmatrix}0\\-1\\0\\0\end{bmatrix}$$

Among all the non-basic cells $(1,1)(1,2)(1,3)(3,1)(3,2)(3,4)$ have non positive (q_j-p_i)'s, while the cells $(2,1),(2,3),(2,4)$ have positive (q_j-p_i)'s so, $\theta_{st}=\min\{\theta_{21},\theta_{23},\theta_{24}\}=\min\{12-10+0,18-10+0,24-10+0\}=\min\{5,8,14\}=5$.

Now the entering cell $(s,t)=(2,1)$.

Step3 Updating

$$\begin{bmatrix}\hat{d}_{1.5}\\ \hat{d}_{2.5}\\ \hat{d}_{3.5}\\ \hat{d}_{4.5}\\ \hat{d}_{5.5}\\ \hat{d}_{6.5}\\ \hat{d}_{7.5}\end{bmatrix}=-\begin{bmatrix}0\\ -1\\ 0\\ 0\\ -1\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 0\\ 0\\ 1\\ 0\\ 0\end{bmatrix}\begin{bmatrix}\hat{d}_{1.2}\\ \hat{d}_{2.2}\\ \hat{d}_{3.2}\\ \hat{d}_{4.2}\\ \hat{d}_{5.2}\\ \hat{d}_{6.2}\\ \hat{d}_{7.2}\end{bmatrix}=\begin{bmatrix}d_{1.2}\\ d_{2.2}\\ d_{3.2}\\ d_{4.2}\\ d_{5.2}\\ d_{6.2}\\ d_{7.2}\end{bmatrix}+\left(d_{4.2}-d_{7.2}\right)\begin{bmatrix}\hat{d}_{1.5}\\ \hat{d}_{2.5}\\ \hat{d}_{3.5}\\ \hat{d}_{4.5}\\ \hat{d}_{5.5}\\ \hat{d}_{6.5}\\ \hat{d}_{7.5}\end{bmatrix}\begin{bmatrix}0\\ 1\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}+\left(0-1\right)\begin{bmatrix}0\\ 1\\ 0\\ 0\\ 1\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\\ -1\\ 0\\ 0\end{bmatrix}$$

Similarly other elements in D can be updated and

$$D=\begin{bmatrix}1&0&0&0&0&0&-1\\ 0&0&0&-1&1&0&0\\ 0&0&1&0&0&-1&0\\ 0&0&0&-1&0&0&0\\ 0&-1&0&-1&1&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&-1\end{bmatrix}$$

$$\Gamma=\left\{(1,4),(2,2),(3,3),(0,1),(2,1),(0,3),(0,4)\right\}$$

$$\begin{bmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\end{bmatrix}=\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}-\theta_{st}\begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix}=\begin{bmatrix}14\\ 10\\ 8\end{bmatrix}-5\begin{bmatrix}0\\ -1\\ 0\end{bmatrix}=\begin{bmatrix}14\\ 15\\ 8\end{bmatrix},$$

$$\begin{bmatrix}\hat{v}_1\\ \hat{v}_2\\ \hat{v}_3\\ \hat{v}_4\end{bmatrix}=\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}-\theta_{st}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}-5\begin{bmatrix}0\\ -1\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 5\\ 0\\ 0\end{bmatrix}$$

$$\psi=\sum_{i=1}^{m}a_iu_i-\sum_{j=1}^{n}b_jv_j=15,000$$

$$Y=AD=\left(300,300,600,100,200,-100,200\right)$$

Since one element of Y is negative so the problem is not optimal, So we go to Step 1 and repeat the procedure. After one iteration, final solution can be written as:

$$D=\begin{bmatrix}1&0&0&0&0&0&-1\\ 0&0&0&-1&1&0&0\\ 0&0&0&0&0&1&-1\\ 0&0&0&-1&0&0&0\\ 0&-1&0&-1&1&0&0\\ 0&0&-1&0&0&1&-1\\ 0&0&0&0&0&0&-1\end{bmatrix}$$

$$\Gamma=\left\{(1,4),(2,2),(3,3),(0,1),(2,1),(0,3),(0,4)\right\}$$

$$\begin{bmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\end{bmatrix}=\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}-\theta_{st}\begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix}=\begin{bmatrix}3\\ 4\\ 6\end{bmatrix}-1\begin{bmatrix}0\\ -1\\ 0\end{bmatrix}=\begin{bmatrix}3\\ 5\\ 6\end{bmatrix},\quad \begin{bmatrix}\hat{v}_1\\ \hat{v}_2\\ \hat{v}_3\\ \hat{v}_4\\ \hat{v}_5\end{bmatrix}=\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\\ v_5\end{bmatrix}-\theta_{st}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\\ q_5\end{bmatrix}=\begin{bmatrix}0\\ 2\\ 0\\ 0\\ 0\end{bmatrix}-1\begin{bmatrix}-1\\ 0\\ 0\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 0\\ 0\\ 0\end{bmatrix}$$

$$\psi=\sum_{i=1}^{m}a_iu_i-\sum_{j=1}^{n}b_jv_j=15,500$$

$$Y=AD=(300,300,500,100,200,100,100)$$

So the optimal solution is obtain with the objective $\psi=\phi=15,500$, with $x_{14}=300, x_{22}=300, x_{33}=500, x_{01}=100, x_{21}=200, x_{34}=100, x_{04}=100$

Conclusions

This method is more efficient for the problems having total demand more than the total supply. The dual-matrix approach can be applied to both balanced and unbalanced transportation problems. An unbalanced transportation problem is not required to be converted into a balanced problem. Another advantage is that it does not have the degeneracy problem and no path tracing.

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