Congruence on Clifford Semigroup of left Quotient

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Abstract
For a Clifford Semigroup Q of left quotients of its subsemigroup S, the paper investigates congruences on Q and concludes that its congruence is determined by the subsemigroup S. In particular, there is always a left reversible congruence on Q in addition to other congruences.

AMS Classification No: *20H10*

Key Words: *Congruence, Left Order, Green's Relation.*

1.0 Introduction

The construction of Clifford semigroup Q, of left quotients was explicitly handled in [2]. We recall that a semigroup S is called a left order in an over semigroup Q if every element of Q can be written in the form $a^{-1}b$, where $a , b \in S$, a is square cancelable and every square cancelable element of S is in a subgroup of Q. The properties of this constructed semigroup are also exposed in [2]. To further enhance the nice properties of this semigroup, the study of its congruences is inevitable. W. D. Munn and N. R. Reilly in [5] characterize congruences on bi-simple ω–semigroup in terms of its normal subgroup and conclude that congruence on it is either idempotent separating or group congruence. Our method differs a bit from this, as we explore the structural property of Q to construct congruence and use this to study its properties. To bring us closer to some properties of congruences on Q, we adopt the example in [4].

Preliminary concepts that would provide necessary insight to the main work are given in section two. Our major construction of congruence on Q is given in section three. Section four gives further properties of the constructed congruence. We follow [3] and [1] in our notations.

2.0 Preliminary Concepts.

In this section, we shall explain the results that will be used in the sections below. We start with some definitions.

2.1 Definition

A Clifford semigroup is a regular semigroup in which its idempotents lie in its centre. In other words, a semilattice of groups is called a Clifford semigroup.

2.2 Remark

We remark here that a Clifford semigroup is also defined as an inverse semigroup with central idempotent. This makes a Clifford semigroup to assume characteristics of an inverse semigroup.

2.3 Example

Consider the set $S = \{0 , e , f , 1 , a\}$ with multiplication defined as follows:

$0 . k = 0$, where $k = e , f , 1 , a.$

$a . a = a^2 = 1,$

$e . f = f = f . e = e$

$a . m = a$, where $m = e , f.$

Here $\{1 , a\}$ is the group of units and $\{1 , e , f , 0\}$ is the set of idempotents in S. With the defined multiplication, S is a Clifford semigroup.

2.4 Definitions

(i) Let S be a semigroup. Then S is said to be cancellative if for all a in S and any x , y in S,

$$ax = ay \text{ implies } x = y$$

and

$$xa = ya \text{ implies } x = y$$

(ii) A semigroup S is said to be separative if for any x, y in S,

$$x^2 = xy \text{ and } y^2 = yx \text{ imply } x = y$$

and

$$x^2 = yx \text{ and } y^2 = xy \text{ imply } x = y$$

(iii) A semigroup S is said to be right (resp. left) reversible if for any given elements a , b in S, there are elements u , v in S such that $ua = vb$ (resp. $au = bv$).

(iv) An element *a* of a semigroup *S*, is said to be square cancelable if and only if for all $x , y \in S^1$, $xa^2 = ya^2$ implies that $xa = ya$, and $a^2x = a^2y$ implies that $ax = ay$, where, $S^1 = \begin{cases} S, \text{if } S \text{ has an identity} \\ S \cup \{1\}, \text{ otherwise.} \end{cases}$

A survey of the concept of a semigroup of left quotients of a subsemigroup has already been treated by [7] and so we give a working definition of it here. A semigroup Q is called a semigroup of left quotients of its subsemigroup S if we can express every element of *Q* in the form $a^{-1}b$, with a , b in **S** and a^{-1} is the inverse of *a* in a subgroup of *Q* and also, every square-cancelable element of *S* is in a subgroup of *Q*. The answer to the question of when will a subsemigroup *S* of a Clifford semigroup *Q* be a left order was provided by [2] as is contained in the following theorem,

2.5 Theorem

A semigroup *S* is a left order in a semilattice *Y* of groups $G_a, a \in Y$, if and only if *S* is a semilattice *Y* of right reversible, cancellative semigroups $S_a, a \in Y$.

It is obvious from the above that the concepts of reversibility and cancellative are very useful in determining the structure of the semigroup *Q*. To understand these concepts well, we need the following:

2.6 Lemma[2]

Let *S* be semigroup, then the followings are equivalent:

(i) $a \mathscr{L}^* b$ $(a \mathscr{R}^* b)$, where a , b ∈ *S*.

(ii) Let $a , b \in S$, then for all $x , y \in S^1$, we have, $xa = ya$ if and only if $xb = yb$ ($ax = ya$;if and only if $bx = by$).

Where $\mathscr{L}^*$ (resp. $\mathscr{R}^*$) is defined in terms of *a* $\mathscr{L}$ (resp. $\mathscr{R}$), that is, for a subsemigroup S of an oversemigroup Q and a , b ∈ S, $a \mathscr{L}^* b$ (resp. $a \mathscr{R}^* b$) in S if and only if $a \mathscr{L} b$ (resp. $a \mathscr{R} b$) in Q.

In what follows, *Q* will be given the following structure:

Suppose $Q = \bigcup_{\alpha \in \Lambda} G_\alpha$ and $S = \bigcup_{\Lambda \in \alpha} S_\alpha$, where $S_\alpha = G_\alpha \cap S$, and for $\alpha \neq \beta$

$G_\alpha \cap G_\beta = S_\alpha \cap S_\beta = \varnothing$. We note that G_α is the $\mathscr{H}$-class of Q.

Using the definition above, *Q* is a Clifford semigroup, while S is a semilattice of right reversible and cancellative semigroup. It was shown in [6] that semilattices of reversible, cancellative semigroup are again reversible, cancellative.

3.0 Congruence Construction

Let $Q = \bigcup_{\alpha \in \Lambda} G_\alpha$ be the Clifford semigroup of left order of the semigroup

$S = \bigcup_{\alpha \in \Lambda} S_\alpha$. We define the relation ρ as follows: Let $a^{-1}b \; \rho \; c^{-1}d$ if and only if there exist $x , y \in S_{\alpha\beta}$ such that $xb = yc$, where $a , b \in S_\alpha$ and $c , d \in S_\beta$. The structure of *Q* guarantees the existence of *x, y* in $S_{\alpha\beta}$ as given in section two above. It is obvious that relation ρ is reflexive and symmetric. We only need to show that it is transitive. To do this, we assume that for every $a^{-1}b$, $c^{-1}d$, $p^{-1}q \in Q$, $a^{-1}b\rho c^{-1}d$ and $c^{-1}d\rho p^{-1}q$. Then there exist x , y in $S_{\alpha\beta}$ such that $xb = yc$ and m , n in $S_{\beta\gamma}$ such that $md = np$. Note that p , q are in S_γ. So,

$$xb = yc \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$md = np \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Since d , c are in S_β and S_β is right reversible, we have,

$$td = sc$$

where t , s are in S_β. Using $u \in S_\alpha$, to multiply both sides, we have,

$$utd = usc$$

But $ut , us \in S_{\alpha\beta}$. Putting $k = ut$, $v = us$, then,

$$kd = vc.$$

Therefore, from (1), there exist ℓ , r in $S_{\alpha\beta}$ such that

$$rb = \ell d$$

Using *g* in S_γ, to multiply both sides, we have,

$$grb = g \ell d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and from (2)

$$gmd = gnp \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Multiplying (3) by m and (4) by, λ we have,

$$mgrb = mg\ell d \quad \ldots\ldots\ldots\ldots\ldots(5)$$

$$\lambda gmd = \lambda gnp \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

so,

$$sb = hd \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(7)$$

and

$$hd = wp \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(8)$$

imply

$$sb = wp.$$

This implies that $a^{-1}b\rho p^{-1}q$. Thus, ρ is an equivalence relation. Let $a , b \in S_\beta$ and $m , n \in S_\gamma$. To conclude, we show that ρ is congruence. Let $a^{-1}b, m^{-1}n \in Q$, suppose

$$a^{-1}b\rho c^{-1}d \ldots\ldots\ldots\ldots\ldots\ldots.(9)$$

Then by definition of ρ, there exist x , y in $S_{\alpha\beta}$ such that

$$xb = yc.$$

Also, since ρ is reflexive, we have

$$m^{-1}n\rho m^{-1}n \ldots\ldots\ldots\ldots\ldots\ldots.(10)$$

That is, there exist s , t in S_γ such that

$$sn = tm.$$

Using (9) and (10), we have,

$$m^{-1}n\bullet a^{-1}b\rho m^{-1}n\bullet a^{-1}b$$

if and only if there exist h , k in $S_{\alpha\beta\gamma}$ such that

$$hvb = kqm$$

where v is in $S_{\gamma\alpha}$ and q is in $S_{\beta\gamma}$.

In [5], we note, that semilattice of a right reversible semigroup is again a right reversible semigroup. Now, let $a , b \in S_\alpha ; c , d \in S_\beta$ and m , n $\in S_\gamma$, then $a , b , c , d , m , n \in S$. Let $b , m \in S$, then by right reversibility of S, we have, $h' , k' \in S$, such that

$$h'b = k'm$$

so $k', h \in S_{\alpha\beta\gamma}$ and the equation

$$hvb = kqm$$

Note that,

$$hv \in S_{\alpha\beta\gamma\gamma\alpha} = S_{\alpha\beta\gamma\alpha} = S_{\alpha\beta\gamma}$$

and

$$kq \in S_{\alpha\beta\gamma\gamma\beta} = S_{\alpha\beta\gamma\beta} = S_{\alpha\beta\gamma}.$$

Therefore, letting $hv = h'$ and $kq = k'$, then we have the required relation. Thus, ρ is congruence. We have just proved the following:

3.1 Theorem

There is always a right reversible congruence on every Clifford semigroup of left quotient.

3.2 Example

1. Suppose $S_1 = \{(a , b) \mid a , b$ are even integers$\}$ and
$S_2 = \{(x , y) \mid x = 3^m \text{ and } y = 3^n, m , n \text{ are integers}, m \geq 1 , n \geq 1\}$.
Let $T = S_1 \cup S_2$, and define multiplication in T as follows:
For $(a , b) , (c , d) \in T$,

$$(a,b)(c,d) = \left(a\frac{b\vee c}{c}, d\frac{b\vee c}{c}\right),$$

where $b\vee c$ denotes the L. C. M. (Least common Multiple) of b and c. T is a semilattice Ω of the bisimple inverse semigroups S_1, S_2 with $\Omega = \{1 , 2\}$, where $1 .2 = 2. 1 = 1 . 1 = 1$ and $2 .2 = 2$. We note that for any element (a , b) of T, (b , a) is its inverse. Now, let us consider the set of all elements of T of the form $x^{-1}y$, where $x = (x_1 , x_2)$, $y = (y_1 , y_2)$. Define ρ on the set T as follows: Let $x = (x_1 , x_2) , y = (y_1 , y_2) , a = (a_1 , a_2) , b = (b_1 , b_2) , c = (c_1 , c_2) , d = (d_1 , d_2)$, where $a , b \in S_1 ; c , d \in S_2$. Now, $a^{-1}b\rho c^{-1}d$ if and only if there exists x , y in $S_{12} = S_1$ such that $xb = yc$. ρ is a congruence.

2. Let Q be the semigroup of rational numbers and S_α = {all even integers} , S_β = {all odd integers} with $\alpha\beta = \alpha$. So, $S = S_\alpha \cup S_\beta$. The set Q has elements of the form $a^{-1}b$, where, $a , b \in S$ and a^{-1} is the inverse of a in the subgroup of Q. We now define ρ on Q as follows: For $a^{-1}b, c^{-1}d \in Q$, $a^{-1}b\rho c^{-1}d$, if and only if there exists $x , y \in S_{\alpha\beta}$ such that $xb = yc$. Thus ρ is again a congruence. In this case, the ρ-classes are just the multiples of integers.

3.3 Remark

Looking at the definition of our congruence, we can say that the elements involved are those of S. This naturally leads us to the next result.

3.4 Theorem

A congruence on S determines congruence on Q if S is an inverse semigroup.

Proof

Let ρ be a congruence on S. We note that Q is a Clifford semigroup of left quotient of the semigroup S. ρ is reflexive in S and so for $x \in S$, $x\rho x$. Since S is an inverse semigroup we have $x^{-1}\rho x^{-1}$. Combining the two, we have $x^{-1}x\rho x^{-1}x$. But $x^{-1}x \in Q$. Therefore, ρ is reflexive in Q. Let $a\rho c$ in S, then $a^{-1}\rho c^{-1}$ in S. Also let $b\rho d$ in S, then combining, we have $a^{-1}b\rho c^{-1}d$, and this is in Q. Now, ρ is symmetric in S, so $c^{-1}\rho a^{-1}$ and $d\rho b$ in S. Thus, combining, we have $c^{-1}d\rho a^{-1}b$ in Q. Thus, $a^{-1}b\rho c^{-1}d$ in Q implies $c^{-1}d\rho a^{-1}b$ in Q. So, ρ is symmetry in Q. Suppose that $a\rho c$, $c\rho x$ and $d\rho y$ in S, then since S is an inverse semigroup, we have, $a^{-1}\rho c^{-1}$. Making the necessary combination, we have, $a^{-1}b\rho c^{-1}d$ and $c^{-1}d\rho x^{-1}y$ in Q. We now apply the concept of transitivity in S. Thus, $a^{-1}\rho c^{-1}$ and $c^{-1}\rho x^{-1}$ in S imply $a^{-1}\rho x^{-1}$ in S. Similarly, $b\rho d$ and $d\rho y$ in S implies $b\rho y$ in S. Therefore, combining again, we have $a^{-1}b\rho x^{-1}y$ belongs to Q. Thus, if $a^{-1}b\rho c^{-1}d$ and $c^{-1}d\rho x^{-1}y$ in Q, then $a^{-1}b\rho x^{-1}y$ in Q. So ρ is transitive in Q. We have thus established that ρ is an equivalent relation in Q. Using the arguments as above, if $a^{-1}b\rho c^{-1}d$ and $x^{-1}y \in Q$, then $x^{-1}y\ a^{-1}b\rho\ x^{-1}y\ c^{-1}d$. Similarly, $a^{-1}b\ x^{-1}y\ \rho\ c^{-1}d\ x^{-1}y$. Therefore, ρ is a congruence on Q.

3.5 Remark

The next construction of a normal subgroup will illuminate the theorem above. It should be noted that the normal subgroup is itself an inverse semigroup and being traditional for the authors to use it in determining congruence in a semigroup, a generalization of this fact as embodied in the above theorem can then be appreciated.

4.0 Construction of Normal Subgroup

Let us now use our congruence to construct a normal sub-semigroup of Q. Let us define the set A_ρ as follows:

$$A_\rho = \{ a^{-1}b : a^{-1}b\rho e_\alpha\}.$$

Note that $a^{-1}b \in Q$ and $Q = \bigcup_{\alpha\in\Lambda} G_\alpha$, where each G_α is the $\mathscr{H}$-class and is a group, $e_\alpha \in G_\alpha$. Thus,

$A_\rho = \{a^{-1}b \in Q: xb = ye_{\alpha;}$ where $x, y \in S_{\alpha\beta},\}$. We now show that A_ρ is a normal subgroup of Q. We shall maintain the above notations except otherwise stated.

Suppose $a^{-1}b, c^{-1}d, \in A_\rho$, where $a, b \in S_\alpha$; $c, d \in S_\beta$ then by definition, we have,

$$a^{-1}b\rho e_\alpha$$

and

$$c^{-1}d\rho e_\beta.$$

Here $e_\alpha \in G_\alpha$ and $e_\beta \in G_\beta$. So, combining the two equivalent relations, we have

$$(a^{-1}b)(c^{-1}d)\rho e_\alpha e_\beta$$

That is,

$$(xa)^{-1}yd\rho e_{\alpha\beta}$$

where $x, y \in S_{\alpha\beta}$. Since the product $(xa)^{-1}yd$ is in $G_{\alpha\beta}$ and $e_{\alpha\beta} \in G_{\alpha\beta}$, then $(a^{-1}b)(c^{-1}d) \in A_\rho$. Therefore, A_ρ is a closed subset of the semigroup Q. We remark here that, if $a^{-1}b \in Q$, then $b^{-1}a \in Q$. Now, let $a^{-1}b\rho e_\alpha$, then, $(a^{-1}b)^{-1}\rho e_\alpha^{-1}$, that is, $b^{-1}a\rho e_\alpha A_\rho$. Therefore, $(a^{-1}b)^{-1} = b^{-1}a \in A_\rho$. So, A_ρ is a subgroup of Q. Now, let $a^{-1}b \in A_\rho$ and $m^{-1}n \in Q$. So by definition of elements of A_ρ, we have

$$a^{-1}b\rho e_\alpha ;\ m^{-1}n\rho e_{\alpha\beta}$$

imply

$$(m^{-1}n)^{-1}\ \rho e_\beta^{-1},$$

that is,

$$(m^{-1}n)^{-1}\ \rho e_\beta \text{ or } (n^{-1}m)\rho e_\beta.$$

Then, combining the equivalent relations, we have

$$(m^{-1}n)a^{-1}bm^{-1}n\rho e_{\alpha\beta} \text{ and } a^{-1}b \in Q$$

That is,

$$(m^{-1}n)^{-1}a^{-1}\ bm^{-1}n\ A_\rho.$$

Thus, A_ρ is a normal subgroup of Q.

4.1 Remark

The kernel of ρ is actually the normal subgroup.

4.2 Theorem

Let Q be a Clifford semigroup of left quotient, and A_ρ is a normal subgroup of Q. Then, Q/A_ρ is also a Clifford semigroup of left quotient.

Proof

Let Q be a Clifford semigroup of left quotient of the semigroup S. Our duty is to show that S/A_ρ is a left order in Q/A_ρ. That is, we show that ${}^{S_\alpha}\!/_{A\rho_\alpha}$ is a semilattice of right reversible concellative semigroups S/A_ρ. Let

$$Q/A_\rho = \bigcup_{\alpha\in\Lambda}\left(G_\alpha / A_{\rho_\alpha}\right)$$

and

$$S/A_\rho = \bigcup_{\alpha\in\Lambda}\left(S_\alpha / A_{\rho_\alpha}\right)$$

where

$$S_\alpha / A_{\rho_\alpha} = S/A_\rho \cap G_\alpha / A_{\rho_\alpha}.$$

Any element of S_α/A_{ρ_α} is expressed in the form $a_\alpha A_{\rho\alpha}$. Now, for $a_\alpha \in S_\alpha$,

$$a_\alpha A_{\rho_\alpha} \in S_\alpha / A_{\rho_\alpha}$$

so

$$S_\alpha / A_{\rho\alpha} \neq \emptyset.$$

Let $a_\alpha^{-1}b_\alpha \in G_\alpha$, where $a_\alpha, b_\alpha \in S_\alpha$, then

$$a_\alpha^{-1}b_\alpha A_{\rho\alpha} \in G_\alpha / A_{\rho\alpha}.$$

Note that $G_\alpha/A_{\rho\alpha}$ is a group and the identity of G_α/A_{ρ_α} is the form $e_\alpha A_{\rho\alpha}$ with $e_\alpha \in G_\alpha$, the identity of G_α . So

$$a_\alpha^{-1}b_\alpha A_{\rho\alpha} = a_\alpha^{-1}b_\alpha e_\alpha A_{\rho\alpha}.$$

Since every element of Q can be written in the form $a^{-1}b$, where $a, b \in S$, then every element of Q/A_ρ can be written in the form $a^{-1}bA_\rho$. S_α is right reversible and cancellative. Thus S_α/A_{ρ_α} is also right reversible and cancellative. We recall that S is a semilattice Y of the semigroups S_α, $\alpha \in Y$. Therefore, S/A_ρ is also a semilattice Y of the semigroups $S_\alpha/A_{\rho\alpha}$. Thus, Q/A_ρ is a Clifford semigroup of left quotients of the semigroup S/A_ρ

4.2 Theorem

Let S be a left order in the Clifford semigroup Q. If ρ is a congruence on Q, then S/ρ is a left order in Q/ρ.

Proof

To do this, we have to first show that $(a\rho)^{-1}$ is the inverse of $a\rho$ in a subgroup of Q/ρ. Note that every element of S/ρ is of the form $a\rho$, where $a \in S$. In this case, we note that

$$(a\rho)^{-1} = a^{-1}\rho.$$

so,

$$\begin{aligned}(a\rho)^{-1}a\rho(a\rho)^{-1} &= a^{-1}\rho a\rho a^{-1}\rho\\ &= (a^{-1}aa^{-1})\rho\\ &= a^{-1}\rho\\ &= (a\rho)^{-1}\end{aligned}$$

and

$$\begin{aligned}(a\rho)(a\rho)^{-1}a\rho &= (aa^{-1}a)\rho\\ &= a\rho.\end{aligned}$$

Now, S being a left order in Q implies that for $x \in Q$,

$$x = a^{-1}b, a, b \in S$$

and

$$a^{-1} \in T \leq Q. \text{ where } \leq \text{ denote subgroup.}$$

So,

$$x\rho = (a^{-1}b)\rho$$

$$= a^{-1}\rho b\rho$$
$$= (a\rho)^{-1} b\rho \in Q/\rho$$

where $a\rho$, $b\rho \in S/\rho$, $(a\rho)^{-1} \in T/\rho \leq Q/\rho$.

Thus, every element of Q/ρ can be written in the form

$$a^{-1}\rho b\rho = (a^{-1}b)\rho.$$

Since S is a left order in Q, very square-cancellable element of S is in a subgroup of Q. That is, if a is square cancelable in S, that is

$$a^2x = a^2y \text{ implies that } ax = ay \text{ , for } x , y \in S$$

and

$$xa^2 = y^2a \text{ implies that } xa = ya \text{ , for } x , y \in S.$$

Then,

$$a \in T \leq Q.$$

Now, let

$$(a\rho)^2(x\rho) = (a)^2(y\rho), \text{ for } a\rho , y\rho \in S/\rho .$$

That is,

$$(a^2\rho)(x^2\rho) = (a^2\rho)(y\rho).$$

That is,

$$(a^2x)\rho = (a^2y)\rho.$$

That is,

$$(ax)\rho = (ay)\rho, \text{ since } a \text{ is square cancelable.}$$

That is,

$$(a\rho)(x\rho) = (a\rho)(y\rho).$$

Also,

$$(x\rho)(a\rho)^2 = (y\rho)(a\rho)^2.$$

Then,

$$(x\rho)(a\rho) = (y\rho)(a\rho).$$

Thus, $a\rho$ is square cancelable in S/ρ . By definition of square cancellativity $a\rho\mathscr{H}^*a^2\rho$ in S/ρ . We also note that if $a\rho\mathscr{H}^*a^2\rho$ in S/ρ , then $a\rho\mathscr{H}a^2\rho$ in Q/ρ . Since aρ is in a subgroup $T/\rho \leq Q/\rho$ if and only if $a\rho\mathscr{H}a^2\rho$ in Q/ρ , then every square-cancellable element of S/ρ is in a subgroup of Q/ρ . Thus S/ρ is a left order in Q/ρ .

4.3 Note

Let $a^{-1}b \in Q$, then, $a^{-1}b \bullet a^{-1}b = (xa)^{-1}yb$, where $a \in S_\alpha$, $b \in S_\beta$ and $x , y \in S_{\alpha\beta}$. Now, if x = y, then,

$$a^{-1}b \bullet a^{-1}b = a^{-1}x^{-1}xb$$
$$= a^{-1}b.$$

That, is, $a^{-1}b$ an idempotent in Q if x = y. Taking this into consideration, we note that elements of the form $a^{-1}a$, $b^{-1}b$ are all idempotents in Q if the condition above holds. Since generally, $a^{-1}a$ is an idempotent in Q, then,

$$((a\rho)^{-1}a\rho)^2 = ((a^{-1}\rho)(a\rho))^2$$
$$= ((a^{-1}a)\rho)^2$$
$$= (a^{-1}a)^2\rho$$
$$= (a^{-1}a)\rho.$$

Thus, elements of the form $(a^{-1}a)\rho$ are the idempotents in Q/ρ.

4.4 Definition

The set of idempotents of Q/ρ is called the kernel of Q/ρ.

4.5 Lemma

The kernel of Q/ρ is a sub-semigroup of Q/ρ.

Proof

Let $a^{-1}a\rho$, $b^{-1}b\rho \in Q/\rho$ then,

$$(a^{-1}a\rho)(b^{-1}b\rho) = (a^{-1}a)(b^{-1}\rho)$$
$$= ((xa)^{-1}yb\rho, \quad \text{where } x, y \in S_{\alpha\beta}.$$

So, $[(xa)^{-1}yb]\rho \in Q/\rho$. Therefore, Q/ρ is closed.

Let $a^{-1}a\rho$, $b^{-1}b\rho$, $m^{-1}m\rho \in Q/\rho$ then,

$$\begin{aligned}(a^{-1}a\rho b^{-1}b\rho)\, m^{-1}m\rho &= [(a^{-1}ab^{-1}b)\, m^{-1}m]\rho \\ &= [(a^{-1}ab^{-1}b)\, m^{-1}m]\rho \\ &= [(a^{-1}a(b^{-1}b\, m^{-1}m)]\rho \\ &= [(a^{-1}a)\rho(b^{-1}b\, m^{-1}m)]\rho \\ &= [(a^{-1}a)\rho[(b^{-1}b)\rho(m^{-1}m)\rho].\end{aligned}$$

Thus, the kernel is a sub-semi group of Q/ρ.

4.6 Remark

By definition, two idempotents are either equal or different. If $a \in S_\alpha$ and $a^{-1}a$ is an idempotent in Q, then $a^{-1}a\rho$ is an idempotent in Q/ρ. Considering the natural mapping of Q onto Q/ρ, we note that idempotent in Q is mapped into idempotent in Q/ρ.

4.7 **Notation** We shall denote the kernel of Q/ρ by $\{A_{\rho\alpha} : \alpha \in Y\}$. Viewed as elements of Q, each $A_{\rho\alpha}$ contains an idempotent. Clifford and Preston [1] have proved that $A_{\rho\alpha}$ is a normal subgroup. So idempotents in Q are contained in some element of A. Elements of $A_{\rho\alpha}$ are of the form $a^{-1}a\rho$, where $a \in S_\alpha$.

4.8 Lemma

Let $x^{-1}y \in Q$, then $(x^{-1}y)^{-1} A_{\rho_\alpha} x^{-1}y \subseteq A_{\rho_\alpha}$ for some α in Y.

Proof.

We note that $(x^{-1}y)^{-1} = y^{-1}x$, and that $x^{-1} A_{\rho\alpha}\, x$ is an idempotent, since every element of $A_{\rho\alpha}$ is an idempotent. So,

$(x^{-1}y)^{-1} A_{\rho\alpha}\, x^{-1}y = y^{-1}x\, A_{\rho\alpha}\, x^{-1}y \subseteq A_{\rho_\alpha}$ for some α in Y,

since $y^{-1}x\, A_{\rho\alpha}\, x^{-1}y$ is an idempotent.

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