Dislocated Quasi-Metric Space and Some Common Fixed Point Theorems for contraction mapping

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Abstract: *the purpose of this paper is to establish some common fixed point theorems for contraction mappings. The results presented in the paper generalize and extend the result of Shailesh Patel and Mitesh Patel [8] and D.Panthi, K.Jha and G.Purru[10].*

Key Word: *Dislocated quasi –metric, fixed point, common fixed point.*

1.   Introduction and Preliminaries

Fixed point theorems are irrevocable in the theory of non linear analysis. In this direction one of the initial and crucial results is the Banach contraction mapping principle [1]. Banach (1922) proved fixed point theorem for contraction mappings in complete metric space. It is well- known as a Banach fixed point theorem. After this pivotal result, theory of fixed point theorems has been studied by many authors in many directions. In some papers, authors define new contractions and discuss the existence and uniqueness of fixed point of for such spaces. The concept of dislocated metric space was introduced by P. Hitzler [5] in which the self distance of points need not to be zero necessarily. They also generalized famous Banach's contraction principle in dislocated metric space. Dislocated metric space play a vital rule in topology, logical programming and electronic engineering. Recently Zeyada et al.[8] develops the notation of complete dislocated quasi metric spaces and generalized the result of Hitzler [5]in dislocated quasi metric space. After many papers have been published containing fixed point results in dislocated quasi metric spaces (see [2],[3],[6],[7]).

K. Jha et al [9] and R. Shrivastav et al [11] have also proved some results in these spaces.

In this paper, we prove common fixed point theorems in dislocated quasi metric spaces for contraction mapping. Our results extend and generalize the existing results of theorem 3.1 and 3.2 of [8] and theorems 12of [10].

First, we recall some definitions and other results that will be needed in the sequel.

Definition 1.1 [8] : Let X be a non − empty set let $d: X \times X \rightarrow [0, \infty]$ be a function satisfies the following conditions:

(i) $d(x, y) = d(y, x) = 0$ implies $x = y$.

(ii) $d(x, y) \leq d(x, z) + d(z, y)$

for all $x, y, z \in X$.

Then d is called a dislocated quasi - metric on X. If d satisfies $d(x, x) = 0$, then it is called a quasi – metric on X. If d satisfies $d(x, y) = d(y, x)$, then it is called dislocated metric.

Definition 1.2[8]: Let (X, d) be a dislocated quasi - metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X. Then,

(i) $\{x_n\}_{n \geq 1}$ dislocated quasi - converges to x if $\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0$

(ii) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for given $\varepsilon > 0 \; \exists \; n_0 \in N$ such that $d(x_m, x_n) < \varepsilon$ or $(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$

(iii)   A dislocated quasi – metric space (X, d) is

called a complete if every Cauchy sequence in X is a dislocated convergent

Definition 1.3[8]: Let (X, d_1) and (X, d_2) be a dislocated metric space and $T: X \to Y$ be a function. Then T is continuous to $x_o \in X$, if for each sequence $\{x_n\}_{n \geq 1}$ which is $d_1 - q$ convergent to x_0, the sequence $\{f(x_n)\}$ is $d_2 - q$ convergent to $T(x_0)$ in Y.

Definition 1.4[8]: Let (X, d) be a dislocated quasi - metric space. A map $T: X \to X$ is a contraction, if there exist $0 \leq \lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all} x, y, \in X.$$

Lemma 1.5[8]: d_q −limit in a d_q − metric space are unique.

Lemma 1.6[8]: let (X, d) be a dislocated quasi - metric space and let $T: X \to X$ be a continuous function, then $\{(T^n(x_0))\}$ is a Cauchy sequence for each $x_0 \in X$.

2.   Main Results

The results which will give are generalization of the theorems 3.1 and 3.2 of [8] theorem 12 of [`10].

Theorem 3.1: Let (X, d) be a complete dq-metric space and suppose there exist non negative constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 1$. Let $f_1, f_2: X \to X$ be a continuous self mappings satisfying the following condition:

$$d(f_1 x, f_2 y) \leq \alpha_1 d(x, y) + \alpha_2 d(x, f_1 x) + \alpha_3 d(y, f_2 y)$$

$$+ \alpha_4 [d(x, f_1 x) + d(y, f_2 y)]$$

$$+ \alpha_5 [d(x, f_2 y) + d(y, f_1 x)]......................... (1)$$

for all $x, y \in X$ .Then f_1 and f_2 have a unique common fixed point in X.

Proof: Let x_0 be arbitrary point in$X$, we define the sequence $\{x_n\}$ as follows:

$$x_1 = f_1 x_0, \text{ and } x_{2n+1} = f_1 x_{2n} = f_1 x^{2n+1}$$

Similarly

$$x_{2n+2} = f_2 x_{2n+1} = f_1 x^{2n+2}$$

Now we consider

$$d(x_{2n}, x_{2n+1}) \leq d(f_1 x_{2n-1}, f_2 x_{2n})$$

$$\leq \alpha_1 \ d(x_{2n-1}, x_{2n})$$

$$+ \alpha_2 d(x_{2n-1} f_1 x_{2n-1})$$

$$+ \alpha_3 d(x_{2n}, f_2 x_{2n})$$

$$+ \alpha_4 [d(x_{2n-1} f_1 x_{2n-1}) + d(x_{2n}, f_2 x_{2n})]$$

$$+ \alpha_5 [d(x_{2n-1}, f_2 x_{2n}) + d(x_{2n}, f_1 x_{2n-1})]$$

$$\leq \alpha_1 \ d(x_{2n-1}, x_{2n}) + \alpha_2 d(x_{2n-1} x_{2n}) + \alpha_3 d(x_{2n}, x_{2n+1})$$

$$+ \alpha_4 [d(x_{2n-1} x_{2n}) + d(x_{2n}, x_{2n+1})]$$

$$+ \alpha_5 [d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n}]$$

$$\leq (\alpha_1 + \alpha_2 + \alpha_4) d(x_{2n-1}, x_{2n}) + (\alpha_3 + \alpha_4) d(x_{2n}, x_{2n+1})$$

$$+ \alpha_5 [d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n+1})]$$

$$= (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) d(x_{2n-1}, x_{2n})$$

$$+ (\alpha_3 + \alpha_4 + \alpha_5) d(x_{2n}, x_{2n+1})$$

Therefore

$$d(x_{2n}, x_{2n+1}) \leq \frac{(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)}{1 - (\alpha_3 + \alpha_4 + \alpha_5)} d(x_{2n-1}, x_{2n})$$

$$d(x_{2n}, x_{2n+1}) \leq \lambda d(x_{2n-1}, x_{2n})$$

where $\lambda = \frac{(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)}{1 - (\alpha_3 + \alpha_4 + \alpha_5)}$

Similarly, we can show that

$$d(x_{2n-1}, x_{2n}) \leq \lambda d(x_{2n-2}, x_{2n-1}).$$

In this way; we get

$$d(x_{2n}, x_{2n+1}) \leq \lambda^n d(x_0, x_1)$$

Since $0 \leq \lambda < 1$, so for $n \to \infty$, $\lambda^n \to 0$ we have $d(x_{2n}, x_{2n+1}) \to 0$.

Hence $\{x_{2n}\}$ is a Cauchy sequence in the complete dislocated quasi –metric space in X.

So, there exist $u \in X$ such that $\{x_{2n}\} \to u$ i.e dislocated converges to $u \in X$.

Since f_1 continuous, we have

$$f_1(u) = \lim_{n \to \infty} f_1 x_{2n} = \lim_{n \to \infty} x_{2n+1} = u$$

Thus $f_1(u) = u$.

Similarly, taking the continuity of f_2.We can show that

$$f_2(u) = u.$$

Hence, u is the common fixed point of f_1 and f_2.

Uniqueness: Suppose that f_1 and f_2 have to common fixed points of u and v for $u \neq v$.

consider

$d(u,v) = d(f_1 u, f_2 v) \leq \alpha 1. d(u,v) + \alpha 2. d(u, f_1 u) + \alpha 3. d(v, f_2 v)$

$+ \alpha 4. [d(u, f_1 u) + d(v, f_2 v)] + \alpha 5. [d(u, f_2 v) + d(v, f_1 u)]$.......................... (2)

Since u and v are common fixed points of f_1 and f_2.

Now from condition (1) implies that $d(u, f_1 u) = 0$ and $(v, f_2 v) = 0$.

Then equation (2) becomes

$d(u,v) \leq \alpha 1. d(u,v)$

$\qquad + \alpha 5. [d(u, f_2 v) + d(v, f_1 u)]$.........(3)

Similarly,

$d(v,p) \leq \alpha_1. d(v,u)$

$\qquad + \alpha 5. [d(v, f_1 v) + d(u, f_1 f u)]$ .... (4)

Subtracting (4) from (3) we get,

$|d(u,v) - d(v,u)| \leq |\alpha_1. |d(u\ v) - d(v,u)|$.......................(5)

Since $\alpha_1 < 1$. So the above inequality is possible if

$$d(u,v) - d(v,u) = 0.$$

By combining equation (3), (4) and (5) we get

$$d(u,v) = 0 \text{ and } d(v,u) = 0.$$

Using (d_2) we have $u = v$. Hence f_1 and f_2 have unique common fixed point of u and v in X.

Theorem 3.2: Let (X, d) be a complete dq-metric space and Let $f_1, f_2 : X \to X$ be a continuous self mappings satisfying the following condition:

$$d(f_1 x, f_2 y) \leq \alpha \max\{d(x, y), d(x, f_1 x), d(y, f_2 y)\}$$

$$+ \beta \max\{d(x, f_2 y), d(x, y)\} +$$

$\mu d(x, y)$ for all $x, y \in X$ .If $0 \leq \alpha, \beta < 1$ such that

$\alpha + \beta + \mu < 1$.Then f_1 and f_2 have a unique common fixed point in X.

Proof: Let x_0 be arbitrary point in X, we define the sequence $\{x_n\}$ as follows:

$x_1 = f_1 x_0$, and $x_{2n+1} = f_1 x_{2n} = f_1 x^{2n+1}$

Similarly

$$x_{2n+2} = f_2 x_{2n+1} = f_1 x^{2n+2}$$

Now we consider

$$d(x_{2n}, x_{2n+1}) \leq d(f_1 x_{2n-1}, f_2 x_{2n})$$

$\leq \alpha \max\{d(x_{2n-1}, x_{2n}), d(x_{2n-1} f_1 x_{2n-1}), d(x_{2n}, f_2 x_{2n})\}$

$+ \beta \max\{d(x_{2n-1} f_2 x_{2n}), d(x_{2n-1}, x_{2n})\} + \mu d(d(x_{2n-1}, x_{2n})$

$\leq \alpha \max\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\}$

$+ \beta \max\{d(x_{2n-1} x_{2n+1}), d(x_{2n-1}, x_{2n})\} + \mu d(d(x_{2n-1}, x_{2n})$

Therefore $d(x_{2n}, x_{2n+1}) \leq (\alpha + \beta + \mu) d(x_{2n-1}, x_{2n})$

$$= hd(x_{2n-1}, x_{2n})$$

where $h = (\alpha + \beta + \mu)$ with $(\alpha + \beta + \mu) < 1$.

Similarly, we can show that

$$d(x_{2n}, x_{2n+1}) \leq hd(x_{2n-2}, x_{2n-1})$$
$$\dots\dots\dots\dots\dots\dots$$
$$\leq h^2 d(x_{2n-2}, x_{2n-1})$$

On continuing this process n times

$$d(x_{2n}, x_{2n+1}) \leq h^{2n} d(x_0, x_1)$$

Since $0 < h < 1$ as $n \to \infty, h^{2n} \to 0$. thus $\{x_{2n}\}$ is a Cauchy sequence in a complete dislocated metric space in X.there exist a point $u \in X$ such that $\{x_{2n}\} \to u \, i.e$ dislocated converges to $u \in X$. Therefore the subsequence $\{f_1 x_{2n}\} \to u$ and $\{f_2 x_{2n+1}\} \to u$.

Since f_1 and f_2 are continuous function, so we have $f_1(u) = u$ and $f_2(u) = u$.

Thus u is a common fixed point of f_1 and f_2.

Uniqueness of common fixed point: Let u and v be a common fixed point of f_1 and f_2. Then

$$d(u, v)) \leq d(f_1 u, f_2 v)$$

$$\leq \alpha \max\{d(u,v), \alpha 2. d(u, f_1 u), \alpha 3. d(v, f_2 v)\}$$

$$+\beta \max\{d(u, f_2 v), d(u,v)\} +\mu. d(u,v)$$

$$=\alpha \max\{d(u,v), \alpha 2. d(u, u), \alpha 3. d(v, v)\}$$

$$+\beta \max\{d(u,v), d(u,v)\} +\mu. d(u,v)$$

Replacing v by u, we get $d(u,u) \leq hd(u,u)$.

Since $0 < h < 1$, we have $d(u,u) = 0$.

Similarly, we have $d(v, v) = 0$.

In this way, we have $d(u,v) \leq hd(u,v)$.

Since $0 < h < 1$, we have $d(u,v) = 0$.

Similarly, we have $d(v, u) = 0$ and so $u = v$ is unique

common fixed point of f_1 and f_2.

Hence the proof is completed

Theorem 3.3: Let (X, d) be a complete dq-metric space and Let $T_1, T_2 : X \to X$ be a continuous self mappings satisfying the following condition:

$$d(T_1 x, T_2 y) \leq a_1 d(x, y) + a_2 \frac{d(x, T_1 x), d(y, T_2 y)}{d(x,y)}$$

$$+ a_3 [d(x, T_1 x) + d(y, T_2 y)]$$

$$+a_4 [d(x, T_2 y) + d(y, T_1 x)]$$

$$+ a_5 [d(x, T_1 x) + d(x, y)]$$

$$+a_6 [d(x, T_1 x) + d(y, T_2 y)]$$

.... (7)

for all $x, y \in X$, $a_1, a_2, a_3, a_4, a_5, a_6 > 0$ with $0 < a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 < 1$. Then T_1 and T_2 have a unique common fixed point in X.

Proof: Let x_0 be arbitrary point in X, we define the sequence $\{x_n\}$ as follows:

$$x_1 = f_1 x_0, \text{ and } x_{2n+1} = f_1 x_{2n} = f_1 x^{2n+1}$$

Similarly

$$x_{2n+2} = f_2 x_{2n+1} = f_1 x^{2n+2}$$

Now we consider

$$d(x_{2n}, x_{2n+1}) \leq d(T_1 x_{2n-1}, T_2 x_{2n})$$

$$\leq a_1 d(x_{2n-1}, x_{2n})$$

$$+ a_2 \frac{d(x_{2n-1}, T_1 x_{2n-1}), d(x_{2n}, x_{2n+1})}{d(x_{2n-1}, x_{2n})}$$

$$+ a_3 [d(x_{2n-1}, T_1 x_{2n-1}) + d(x_{2n}, T_2 x_{2n})]$$

$$+a_4 [d(x_{2n-1}, T_2 x_{2n}) + d(x_{2n}, T_1 x_{2n-1})]$$

$$+ a_5 [d(x_{2n-1}, T_1 x_{2n-1}) + d(x_{2n-1}, x_{2n})]$$

$+a_6[d(x_{2n}, T_2 x_{2n}) + d(x_{2n-1}, x_{2n})]$

$= a_1 d(x_{2n-1}, x_{2n}) + a_2 \frac{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})}{d(x_{2n-1}, x_{2n})}$

$+a_3[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]$

$+a_4[d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n})]$

$+a_5[d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, x_{2n})]$

$+a_6[d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})]$

$\leq (a_1 + a_3 + 2a_4 + 2a_5 + a_6)\, d(x_{2n-1}, x_{2n})$

$+ \ (a_2 + a_3 + 2a_4 + a_6)\, d(x_{2n}, x_{2n+1})$

Hen $\quad d(x_{2n}, x_{2n+1}) \quad \leq \quad \frac{a_1 + a_3 + 2a_4 + 2a_5 + a_6)}{1 - (a_2 + a_3 + 2a_4 + a_6)}$ $d(x_{2n-1}, x_{2n})$

Thus we have $d(x_{2n}, x_{2n+1}) \leq h\, d(x_{2n-1}, x_{2n})$

where $h = \frac{a_1 + a_3 + 2a_4 + 2a_5 + a_6}{1 - (a_2 + a_3 + 2a_4 + a_6)}$ with $0 < h < 1$.

Similarly, we can show that

$d(x_{2n-1}, x_{2n}) \leq h\ d(x_{2n-2}, x_{2n-1}).$

On continuing this process n times

$d(x_{2n}, x_{2n+1}) \leq h^{2n} d(x_0, x_1)$

Since $0 \leq \lambda < 1$, so for $n \to \infty$, $\lambda^n \to 0$ we have $d(x_{2n}, x_{2n+1}) \to 0$.

Hence $\{x_{2n}\}$ is a Cauchy sequence in the complete dislocated quasi –metric space in X.

So, there exist $u \in X$ such that $\{x_{2n}\} \to u$ *i.e* dislocated converges to $u \in X$. Also the subsequence $\{T_1 x_{2n}\} \to u$ and $\{T_2 x_{2n+1}\} \to u$.

Since T_1 continuous, therefore we have

$T_1(u) = \lim_{n \to \infty} T_1 x_{2n} = \lim_{n \to \infty} x_{2n+1} = u$.

Thus $T_1(u) = u.$

Similarly, taking the continuity of f_2. We can show that

$T_2(u) = u.$

Hence, u is the common fixed point of f_1 and f_2.

Uniqueness: Suppose that T_1 and T_2 have to common fixed points of u and v for $u \neq v$.

Consider

$d(u, v) = d(T_1 u, T_2 v)$

$\leq a_1 d(u, v) + a_2 \frac{d(u, T_1 u), d(v, T_2 v)}{d(x, y)}$

$+ a_3[d(u, T_1 u) + d(v, T_2 v)]$

$+a_4[d(u, T_2 v) + d(v, T_1 u)]$

$+ a_5[d(u, T_1 u) + d(u, v)]$

$+a_6[d(u, T_1 u) + d(v, T_2 v)]$
.... (8)

$\leq a_1 d(u, v) + a_2 \frac{d(u, u), d(v, v)}{d(u, v)}$

$+ a_3[d(u, u) + d(v, v)]$

$+ a_4[d(u, v) + d(v, u)]$

$+ a_5[d(u, u) + d(u, v)]$

$+a_6[d(u, u) +$
$d(v, v)]$................................. .(9)

Since u and v are common fixed point of T_1 and T_2, condition (7) implies that $d(u, u) = 0$

And $d(v, v) = 0$. Thus equation (9) becomes

$d(u, v) \leq (a_1 + a_4 + a_5 + a_6) d(u, v)$

$+a_4 d(v, u)$ .... (10)

Similarly we get

$d(v, u) \leq (a_1 + a_4 + a_5 + a_6) d(v, u)$

$+a_4 d(u, v)$ ....(11)

Subtracting (11) from (10) we get

$|d(u, v) - d(v, u)| \leq |a_1 + a_5 + a_6|\, |d(v, u) - d(u, v)|$

Since $a_1 + a_5 + a_6 < 1$, so the above inequality is possible

If $d(u, v) - d(v, u) = 0$ .... (12)

By combining equation (10,(11) and (12) one can get

$(u, v) = 0$ and $d(v, u) = 0$.

Using (i) We have $u = v$. Hence T_1 and T_2 have a unique

common fixed point in X.

Hence the proof is completed.

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