Applications on Differential Transform method for solving Singularly Perturbed Volterra integral equation, Volterra integral equation and integro-differential equation.

Raafat R. Rizkalla[a], Seham SH. Tantawy[b] , Mahmoud H.Taha[c]
[a,b,c]*Math.Depart., Faculty* of Education, Ain Shams University
[a]raafat3r@yahoo.com
[b]Seham15@hotmail.com
[c]mahmoud.h.taha@gmail.com

Abstract

In this paper, singularly perturbed Volterra integral equation, Volterra integral and integro-differential equations are solved using differential transform method. The approximate solution of some applications is calculated in the form of series with easily computable terms. Those terms was compared with the exact solution and graphed using Mathematica program. The efficiency of the method is shown from that it obtains the exact solutions in most cases studied with high accuracy.

Key words: Differential Transform method, Singularly perturbed volterra integro-differential equations, Volterra integral equations, Volterra Population model, Heat Radiation in a Semi-Infinite Solid.

1 Introduction:

The concept of the differential transform was first proposed by Zhou [1]. Integral equations and singularly perturbed integral equations arise in many scientific and engineering problems. A large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations. The potential theory contributed more than any field to give rise to integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of integral equations [2]. Many other applications in science and engineering are described by integral equations or integro-differential equations. The Volterra's population growth model [2], biological species living together, propagation of stocked fish in a new lake, the heat transfer and the heat radiation are among many areas that are described by integral equations [2]. In this paper we presented a variety of integral and integro-differential equations that were handled using differential transform method. The problems we handled led in most cases to the determination of exact solutions. Because it is not always possible to find exact solutions to problems of physical sciences, much work is devoted to obtaining qualitative approximations that highlight the structure of the solution. It is the aim of this paper to handle some integral applications taken from a variety of fields [1-7]. The obtained series will be handled to give insight into the behavior of the solution and some of the properties of the examined phenomenon. Polynomials are frequently used to approximate power series.

2 Differential transform method (DTM):

The transformation of the kth derivative of a function in one variable is as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f}{dx^k}(x)\right]_{x=x_0} \tag{1}$$

and the inverse transformation is defined by

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k. \tag{2}$$

The following theorems that can be deduced from Eqs. (1) and (2) are given as:

Theorem:

(1) $If \quad f(x) = g(x) \pm w(x), then\ F(k) = G(k) \pm W(k).$

(2) $If\ f(x) = cg(x), then\ F(k) = cG(k), where\ c\ is\ constant.$

(3) If $f(x) == \frac{d^n g(x)}{dx^n}, then\ F(k) = \frac{(k+n)!}{k!}G(k+n).$

(4) If $f(x) = \exp(\lambda x), then\ F(k) = \frac{\lambda^k}{k!}$

(5) If $f(x) = g(x)h(x), then\ F(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$

(6) If $f(x) = x^n, then\ F(k) = \delta(k-n)$ where, $\delta(k-n) = \begin{cases} 1 & k=n \\ 0 & k \neq n \end{cases}.$

(7) $If\ f(x) = g_1(x)g_2(x)...g_{n-1}(x)g_n(x),$

$$then\ F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)... \\ ...G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1})$$

$$(8)\ If\ f(x)=\int_{x_0}^{x} g(t)dt,\ then\ F(k)=\frac{G(k-1)}{k}, where\ k\geq 1.$$

$$(9)\ If\ f(x)=\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\cdots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\int_{x_0}^{x_1} g(t)dtdx_1dx_2dx_3...dx_{n-1}$$

$$then,\ F(k)=\frac{(k-n)!}{k!}G(k-n), where\ k\geq n.$$

$$(10)\ If\ f(x)=g(x)\int_{x_0}^{x} h(t)dt,$$

$$then\ F(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}G(k-k_1)H(k_1-1), where\ k\geq 1.$$

$$(11)\ If f(x)=\int_{x_0}^{x} g_1(t)g_2(t)dt,$$

$$then\ F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1}G_1(k_1)G_2(k-k_1-1), where\ k\geq 1.$$

$$(12)\ If\ f(x)=\int_{x_0}^{x} g_1(t)g_2(t)...g_{n-1}(x)g_n(x)dt,\quad then:$$

$$F(k)=\frac{1}{k}\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)G_2(k_2-k_1)...$$
$$...G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}-1).$$

$$(13)\ If\ f(x)=(x-t)^m, where\ \text{m is any rational number},$$

$$then\ \mathrm{F}(k)=(-1)^k\frac{m!}{(m-k)!k!}x^{m-k}$$

3 Applications and results:

In this section, we will demonstrate the effectiveness of the Differential Transform method on variety of mathematical and famous physical applications which was solved before using other methods, however; DTM shown remarkable results when compared to those methods.

3.1 Singularly perturbed Volterra integral equations:

The aim of our study is to introduce the differential transform method as an alternative to existing methods in solving the singularly perturbed Volterra integral equations

$$\varepsilon y(x) = g(x) + \int_0^x k(x, t, y(t))dt, \ 0 \leq t \leq x \tag{3}$$

where is a small parameter satisfying $0 < \varepsilon << 1$ and where g and K are given smooth functions on [0, X]. Under appropriate conditions g and K, for every $\varepsilon > 0$, Eq. (3) has unique continuous solutions on [0, T]. The singularly perturbed nature of Eq.(3) arises when the properties of the solution with $\varepsilon > 0$ are incompatible with those when $\varepsilon = 0$. For $\varepsilon > 0$, Eq.(3) is an integral equation of the second kind which typically is well posed whenever K is sufficiently well behaved. When $\varepsilon = 0$, Eq.(3) reduced to an integral equation of the first kind whose solution may well be incompatible with the case for $\varepsilon > 0$. The interest here is in those problems which do imply such an incompatibility in the behavior of y near $x = 0$. This suggests the existence of boundary layer near the origin where the solution undergoes a rapid transition [8-11].

Example (1):

Let us first consider the following singularly perturbed Volterra Integral Equation [3-16]:

$$\varepsilon y(x) = \int_0^x [1 + t - y(t)]dt \tag{4}$$

with exact solution

$$y(x) = x + 1 - \exp(\frac{-x}{\varepsilon}) - \varepsilon(1 - \exp(\frac{-x}{\varepsilon})) \tag{5}$$

Using the differential transform method in Eq.(1),(2), and the Theorem, we obtain the following recurrence relation:

$$\varepsilon Y(k) = \frac{\delta_{k-1,0}}{k} + \frac{\delta_{k-1,1}}{k} - \frac{Y(k-1)}{k} \tag{6}$$

$at\ k=0:\ Y(0)=0.$

$at\ k=1: Y(1)=\frac{1}{\varepsilon}[1-\frac{Y(0)}{1}]=\frac{1}{\varepsilon}.$

$at\ k=2: Y(2)=\frac{1}{\varepsilon}[\frac{1}{2}-\frac{Y(1)}{2}]=\frac{1}{\varepsilon}[\frac{1}{2}-\frac{1}{2\varepsilon}].$

$at\ k=3: Y(3)=\frac{1}{\varepsilon}[-\frac{Y(2)}{3}]=\frac{-1}{3\varepsilon}\frac{1}{\varepsilon}[\frac{1}{2}-\frac{1}{2\varepsilon}].$

$at\ k=4: Y(4)=\frac{1}{\varepsilon}[-\frac{Y(3)}{4}]=\frac{1}{4\varepsilon}\frac{1}{3\varepsilon}\frac{1}{\varepsilon}[\frac{1}{2}-\frac{1}{2\varepsilon}].$

$at\ k=5: Y(5)=\frac{1}{\varepsilon}[-\frac{Y(4)}{5}]=\frac{-1}{5\varepsilon}\frac{1}{4\varepsilon}\frac{1}{3\varepsilon}\frac{1}{\varepsilon}[\frac{1}{2}-\frac{1}{2\varepsilon}].$

Using the formula in Eq.(2), we get:

$$y(x)=\frac{1}{\varepsilon}x+\frac{1}{\varepsilon}\left[\frac{1}{2}-\frac{1}{2\varepsilon}\right]x^2-\frac{1}{3\varepsilon}\frac{1}{\varepsilon}\left[\frac{1}{2}-\frac{1}{2\varepsilon}\right]x^3+\frac{-1}{3\varepsilon}\frac{1}{\varepsilon}\left[\frac{1}{2}-\frac{1}{2\varepsilon}\right]x^4-\frac{1}{5\varepsilon}\frac{1}{4\varepsilon}\frac{1}{3\varepsilon}\frac{1}{\varepsilon}\left[\frac{1}{2}-\frac{1}{2\varepsilon}\right]x^5+\cdots \tag{7}$$

Then,

$$y(x)=1-\left[1-\frac{x}{\varepsilon}+\frac{x^2}{2\varepsilon^2}-\frac{x^3}{3!\varepsilon^3}+\frac{x^4}{4!\varepsilon^4}-\frac{x^5}{5!\varepsilon^5}+\cdots\right]-\varepsilon+x$$
$$+\varepsilon\left[1-\frac{x}{\varepsilon}+\frac{x^2}{2\varepsilon^2}-\frac{x^3}{3!\varepsilon^3}+\frac{x^4}{4!\varepsilon^4}-\frac{x^5}{5!\varepsilon^5}+\cdots\right] \tag{8}$$

Then,

$$y(x)=1+x-\exp(\frac{-x}{\varepsilon})-\varepsilon(1-\exp(\frac{-x}{\varepsilon})). \tag{9}$$

The graph of the solution is shown in figure (1). Which is the exact solution, this shows the efficiency and accuracy of the differential transform method also more accurate than the variational iteration method by He introduced in [16] in (2013), which obtained an approximate solution to the same problem, and it's third iteration is given by

$$y(x)=\frac{1}{\varepsilon}\left(x+\frac{x^4}{24\varepsilon^2}+\frac{x^3(1-\varepsilon)}{6\varepsilon^2}+\frac{x^2(-1+\varepsilon)}{2\varepsilon}\right)$$

A figure of the exact solution is shown below at $\varepsilon=0.75$.

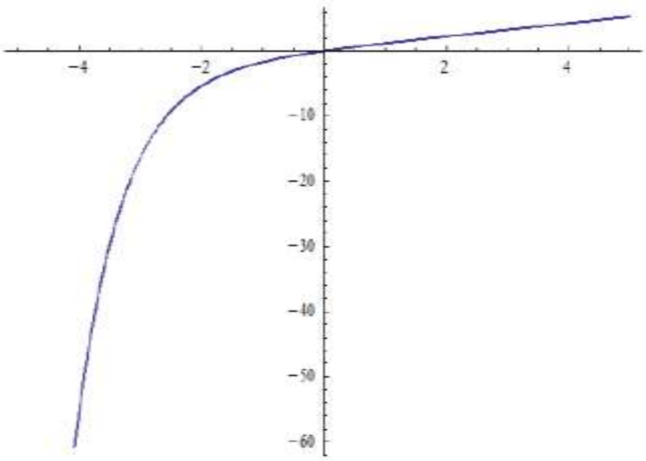


Fig.1: Exact solution of Eq.(3)

Example (2):

Now we consider the following singularly perturbed Volterra integral equation [9]:

$$\varepsilon y(x) = \int_0^x e^{x-t}\left[y^2(t) - 1\right] dt \tag{10}$$

$$\varepsilon y(x) = \int_0^x e^{x} y^2(t) dt - \int_0^x e^{-t} dt$$

Which has the exact solution

$$y(x) = \frac{2\left(1 - \exp(\alpha x)\right)}{\varepsilon(\alpha - 1)\exp(\alpha x) + \alpha + 1} \tag{11}$$

Where the parameter α is defined by

$$\alpha = \frac{1}{\varepsilon}\sqrt{4 + \varepsilon^2}$$

Using the Eq.(1),(2) and the Theorem, we will get the following recurrence relation :

$$Y(k) = \frac{1}{\varepsilon}\left[\sum_{s=1}^{k}\sum_{m=1}^{s}\sum_{n=1}^{m}\frac{1}{s}\frac{(-1)^{n-1}}{(n-1)!(k-s)!}Y(m-n)Y(s-m) - \sum_{r=1}^{k}\frac{1}{r}\frac{(-1)^{r-1}}{(r-1)!(k-r)!}\right]. \tag{12}$$

$at\ k=0: \ Y(0)=0.$

$at\ k=1: Y(1)=-\frac{1}{\varepsilon}.$

$at\ k=2: Y(2)=\frac{-1}{2\varepsilon}$

$at\ k=3: Y(3)=\frac{-7}{54\varepsilon}$

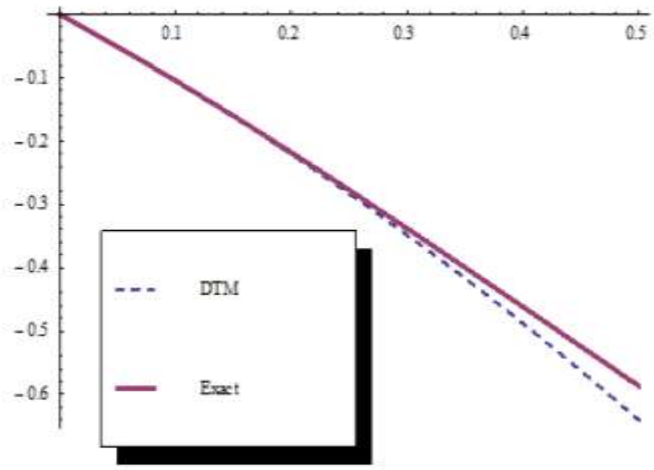
$at\ k=4: Y(4)=\frac{-1}{216\varepsilon}$

$at\ k=5: Y(5)=\frac{277}{27000\varepsilon}.$

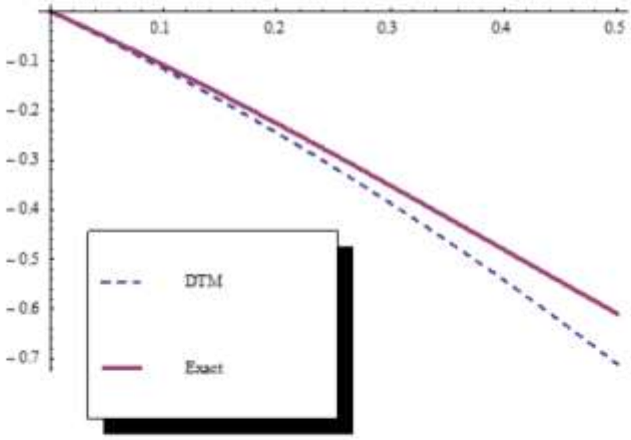
Using the formula in Eq.(2), we get:

$$y(x)=-\frac{1}{\varepsilon}x-\frac{1}{2\varepsilon}x^2-\frac{7}{54\varepsilon}x^3-\frac{1}{216\varepsilon}x^4+\frac{277}{27000\varepsilon}x^5+\cdots \tag{13}$$

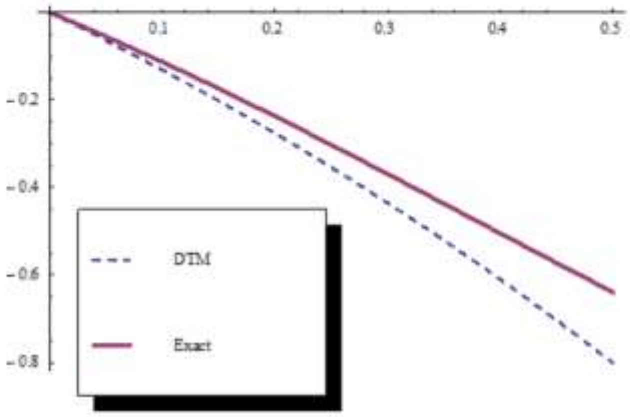
The evolution results for the exact solution in Eq.(11) and the solution obtained by the differential transform method in Eq.(13), for different values of ε, are shown in Figs. 1-3.



$Fig.2$: Plots of Eq.(13), when $\varepsilon=1$.



$Fig.3$: Plots of Eq.(13), when $\varepsilon = 0.9$.



$Fig.4$: Plots of Eq.(13), when $\varepsilon = 0.8$.

3.2 Volterra population model:

In this section we will study the Volterra model for population growth of a species within a closed system. The Volterra's population model is characterized by the nonlinear Volterra integro-differential equation

$$\frac{dP}{dt} = aP - bP^2 - cP\int_0^T P(x)dx, P(0) = P_0 \qquad (14)$$

where P = P(T) denotes the population at time T, a, b, and c are constants and positive parameters, a > 0 is the birth rate coefficient, b > 0 is the crowding coefficient, c > 0 is the toxicity coefficient, and P_o is the initial population. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long run.

Many time scales and population scales may be applied. However, we apply the scale time and population by introducing the non-dimensional variables

$t = cTb, u = bPa$

to obtain the non-dimensional Volterra's population growth model

$$r\,\frac{du}{dt} = u - u^2 - u\int_0^t u(x)dx, \qquad u(0) = u_0 \tag{15}$$

where u = u(t) is the scaled population of identical individuals at a time t, and the non-dimensional parameter $r = \frac{c}{ab}$ is a prescribed parameter. Volterra introduced this model for a population u(t) of identical individuals which exhibits crowding and sensitivity to the amount of toxins produced.

Equation (4) is transformed by using Theorem (2), (3), (5), and (10) to obtain the following recurrence relation:

$$r(k+1)U(k+1) = U(k) - \sum_{r=0}^{k} U(n)U(k-n) - \sum_{r=1}^{k} \frac{1}{r} U(k-r)U(r-1)$$

Then,

$$U(k+1) = \frac{1}{r(k+1)} U(k) - \sum_{r=0}^{k} U(n)U(k-n) - \sum_{r=1}^{k} \frac{1}{r} U(k-r)U(r-1), \qquad U(0) = u_0 \tag{16}$$

For simplicity reason choose $r = 0.1$, and $u(0) = 0.1$

Then Eq.(5) will be:

$$U(k+1) = \frac{1}{0.1(k+1)} U(k) - \sum_{r=0}^{k} U(n)U(k-n) - \sum_{r=1}^{k} \frac{1}{r} U(k-r)U(r-1), \qquad U(0) = 0.1 \tag{17}$$

Using Eq.(1) and (4)

$$U(1) = \frac{1}{0.1}[0.1 - 0.1^2] = 0.9$$

From Eq.(6):

$$At\ k = 1 : U(2) = \frac{1}{0.2}[U(1) - \sum_{n=0}^{1} U(n)U(1-n) - \sum_{r=1}^{1} U(1-r)U(r-1)] = 3.55$$

$$At\ k = 2 : U(3) = \frac{1}{0.3}[U(2) - \sum_{n=0}^{2} U(n)U(2-n) - \sum_{r=1}^{2} U(2-r)U(r-1)] = 6.316666667$$

$$At\ k=3: U(4)=\frac{1}{0.4}[U(3)-\sum_{n=0}^{3}U(n)U(3-n)-\sum_{r=1}^{3}U(3-r)U(r-1)]=-5.5375$$

$$At\ k=4: U(5)=\frac{1}{0.5}[U(4)-\sum_{n=0}^{4}U(n)U(4-n)-\sum_{r=1}^{4}U(4-r)U(r-1)]=-63.70916667$$

And so on,
and using the inverse transformation rule in Eq. (2), the following series solution is obtained.

$$u(t)=0.1+0.9t+3.55t^2+6.316666667t^3-5.5375t^4-63.7091667t^5+O(t^6) \qquad (18)$$

Same solution was obtained using two different methods in [2], however; DTM as a method of solving that type of equations proved to be time saver and facilitating the computational difficulties with great accuracy and more efficiency compared to the other methods in [2]. A graph of the solution is shown in figure (5).

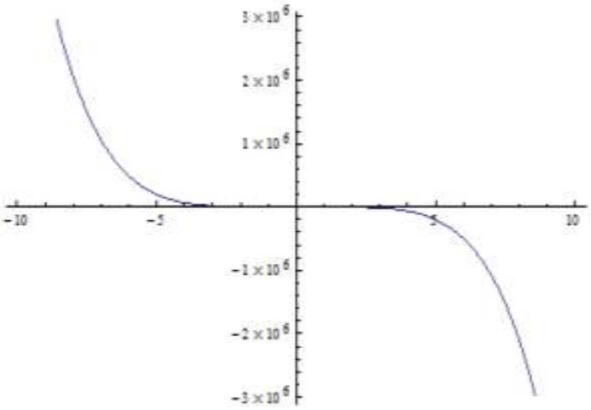


Fig.5: Approximate solution given in Eq.(18)

3.3 Heat Radiation in a Semi-Infinite Solid:

In this part we will study an Abel-type nonlinear Volterra integral equation as discussed in [2] and it's given by :

$$u(x)=\frac{1}{\sqrt{\pi}}\int_0^x \frac{f(t)-u^n(t)}{\sqrt{x-t}}dt \qquad (19)$$

where $u(x)$ gives the temperature at the surface for all time. The physical problem which motivated consideration of (19) is that of determining [17] the temperature in a semi-infinite solid, whose surface can dissipate heat by nonlinear radiation. At the surface, energy is supplied according to the given function f(t), while radiated energy [17] escapes in proportion to $u^n(t)$.

Equation (19) may be rewritten as

$$u(x)=\frac{1}{\sqrt{\pi}}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt-\frac{1}{\sqrt{\pi}}\int_0^x \frac{1}{\sqrt{x-t}}u^n(t)dt \tag{20}$$

The Differential Transform method handles such problems effectively. In what follows, we will select only two cases for f(x) and n.

Example (1):
As an application, we select $f(x)=2\sqrt{\frac{x}{\pi}}$ *and* $n=3$. based on this selection, Eq.(20) becomes

$$u(x)=x-\frac{1}{\sqrt{\pi}}\int_0^x \frac{1}{\sqrt{x-t}}u^3(t)dt \tag{21}$$

Let

$$g(x)=\frac{1}{\sqrt{x-t}}$$

applying property(12) *we* get: $G(K)=(-1)^k\frac{m!}{(m-k)!k!}x^{m-k}$

Using property (11) and (13) in the Theorem, the following recurrence relation may be obtained:

$$U(k)=\delta_{k,1}-\frac{1}{\sqrt{\pi}k}\sum_{n=1}^{k}\sum_{m=1}^{n}\sum_{l=1}^{m}(-1)^{l-1}\frac{(\frac{-1}{2}-l)!}{(\frac{-1}{2}-l)!*k!}x^{-\frac{1}{2}-l}U(m-l)U(n-m)U(k-n),\ k\geq 1 \tag{22}$$

Then:

$$\begin{aligned}
&at\ k=0:\ U(0)=0.\\
&at\ k=1: U(1)=1\\
&at\ k=2: U(2)=0\\
&at\ k=3: U(3)=0\\
&at\ k=4: U(4)=-\frac{1}{4\sqrt{\pi}}x^{\frac{-1}{2}}\\
&at\ k=5: U(5)=-\frac{1}{10\sqrt{\pi}}x^{\frac{-3}{2}}\\
&at\ k=6: U(6)=-\frac{1}{16\sqrt{\pi}}x^{\frac{-5}{2}}
\end{aligned}$$

Using the formula in Eq.(2), we get:

$$u(x)=x-0.370555x^{\frac{7}{2}}+0.197419x^6-0.0975588x^{\frac{17}{2}}+0.0382333x^{11}-0.00938229x^{\frac{27}{2}}-\cdots \tag{23}$$

An approximate solution of the same problem as obtained by Adomian decomposition method [2], a comparison between the two solutions can be shown in the figure (6) and table 1:

Table 1: Adomian Decomposition method (ADM) results compared to Differential Transform method results (DTM):

x	ADM	DTM
0.1	0.0998375	0.099883
0.2	0.198193	0.198687
0.3	0.292793	0.294661
0.4	0.381344	0.385771
0.5	0.462348	0.47008
0.6	0.537142	0.546072
0.7	0.622664	0.612863

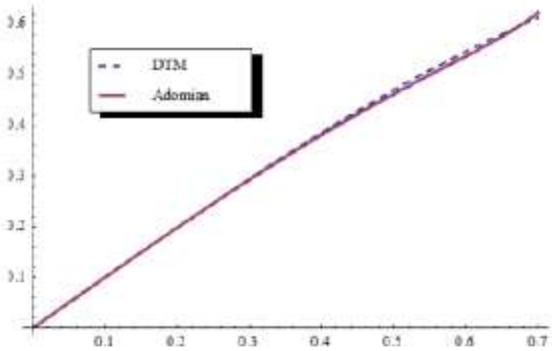


Fig.6: Comparison between solution obtained in Eq.(23), and in [2].

4 Conclusions:

In this work, we successfully apply the differential transform method to find exact and approximate solutions for linear and non-linear Volterra integral equations and integro-differential equation with separable kernels and some applications on it. The present method reduces the computational difficulties of the other traditional methods and all the calculations can be made by simple manipulations. Several examples were tested by applying the DTM and the results have shown remarkable performance. All results were tested using Mathematica program as well. Therefore, this method can be applied to many nonlinear integral, differential and integro-differential equation equations without linearization, discretization or perturbation.

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