Fixed and Common Fixed Point Theorems in two M- Fuzzy Metric Spaces

T.Veerapandi [1], T.Thiripura Sundari [2], J.Paulraj Joseph [3]

Associate Professor of Mathematics(Retd) ,P.M.T. College
Melaneelithanallur-627953, India

Department of Mathematics,Sri K.G.S Arts College
Srivaikuntam, India

Professor and H.O.D of Mathematics,Manonmaniam Sundaranar University
Tirunelveli, India

Abstract: *The purpose of this paper is to obtain some common fixed point theorems in two M- fuzzy metric spaces.*

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1.INTRODUCTION

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh[9] which laid the foundation of fuzzy mathematics. Since then to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [2] andKramosil and Michalek [3] introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and E-infinity theory which were given and studied by El Naschie [4-7]. Dhage [1] introduced the notion of generalized metric or D-metric spaces and proved several fixed point theorems in it. Recently Sedghi and Shobe [8] introduced D*-metric space as a probable modification of D-metric space and studied some topological properties . In this paper we prove some common fixed point theorems in two M-Fuzzy Metric Spaces.

Definition:1.1[8]. Let X be a nonempty set. A generalized metric (or D' - metric) on X is a function: D': $X^3 \to [0,\infty)$, that satisfies the following conditions for each x, y, z, a $\in$ X

(i) D'(x, y, z) $\geq$ 0,
(ii) D'(x, y, z) = 0 iff x = y = z,
(iii) D'(x, y, z) = D'(p{x, y, z}), s(symmetry) where p is a
permutation function,
(iv) D'(x, y, z) $\leq$ D'(x, y, a) + D'(a, z, z).

The pair (X, D'), is called a generalized metric (or D' - metric) space.

Examples of D' - metric are
(a) D' (x, y, z) = max {d(x, y), d(y, z), d(z, x) },
(b) D' (x, y, z) = d(x, y) + d(y, z) + d(z, x).
Here, d is the ordinary metric on X.

Definition: 1.2 A fuzzy set M in an arbitrary set X is a function with domain X and values in [0, 1].

Definition: 1.3 A binary operation *: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions

(i) * is associative and commutative,
(ii) * is continuous,
(iii) a * 1 = a for all a $\in$ [0, 1],
(iv) a*b $\leq$ c*d whenever a $\leq$ c and b $\leq$ d, for each a, b, c, d $\in$ [0, 1].

Two typical examples for continuous t-norm are a*b = ab and a*b = min {a, b}.

Definition: 1.4 A 3-tuple (X, M, *) is called a M-fuzzy metric space. if X is an arbitrary non-empty set, * is a continuous t-norm, and M is a fuzzy set on $X^3 \times (0,\infty)$, satisfying the following conditions for each x, y, z, a $\in$ X and t, s > 0

(FM – 1) M (x, y, z, t) > 0
(FM – 2) M (x, y, z, t) = 1 iff x = y = z
(FM – 3) M (x, y, z, t) = M (p{x, y, z}, t), where p is a
permutation function
(FM – 4) M (x, y, a, t) * M (a, z, z, s) $\leq$ M (x, y, z, t+s)
(FM – 5) M (x, y, z, $\cdot$) : $(0,\infty) \to [0,1]$is continuous
(FM – 6) $\lim_{t \to \infty}$ M (x, y, z, t) = 1.

Example: 1.5 Let X be a nonempty set and D* is the D* - metric on X. Denote a*b = a.b for all a, b $\in$ [0,1]. For each t $\in$ $(0,\infty)$, define

$$M (x, y, z, t) = t / (t+D^*(x,y.z))$$

for all x, y, z $\in$ X, then (X, M, *) is a M- fuzzy metric space.

Lemma: 1.6 Let (X, M, *) be a M- fuzzy metric space. . Then for every t > 0 and for every x, y $\in$ X we have

$M(x, x, y, t) = M(x, y, y, t)$.

Proof:

For each $\in > 0$ by triangular inequality

We have

(i) $M(x, x, y, \in + t) \geq M(x, x, x, \in) * M(x, y, y, t)$
$= M(x, y, y, t)$

(ii) $M(y, y, x, \in + t) \geq M(y, y, y, \in) * M(y, x, x, t)$
$= M(y, x, x, t)$.

By taking limits of (i) and (ii) when $\in \to 0$,

we obtain $M(x, x, y, t) = M(x, y, y, t)$

Lemma: 1.7 Let $(X, M, *)$ is a fuzzy metric space. If we define $M : X^3 \times (0,\infty) \to [0, 1]$ by $M(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t)$ for every x, y,z in X, then $(X, M, *)$ is a M-fuzzy metric space.

Lemma: 1.8 Let $(X, M, *)$ be a M- fuzzy metric space. Then $M(x, y, z, t)$ is non-decreasing with respect to t, for all x, y, z in X.

Proof:

For each $x, y, z, a \in X$ and $t, s > 0$ we have

$M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$

If set a=z we get

$M(x, y, z, t) * M(z, z, z, s) \leq M(x, y, z, t + s)$

That is $M(x, y, z, t + s) \geq M(x, y, z, t)$.

Definition: 1.9 Let $(X, M, *)$ be a M- fuzzy metric space. For $t > 0$, the open ball $B_M(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_M(x, r, t) = \{y \in X: M(x, y, y, t) > 1 - r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_M(x, r, t) \subseteq A$.

Definition: 1.10 Let $(X, M, *)$ be a M- fuzzy metric space. and $\{x_n\}$ be a sequence in X

(a) $\{x_n\}$ is said to converge to a point $x \in X$ if $\lim_{n \to \infty} M(x, x, x_n, t) = 1$ for all $t > 0$

(b) $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n \to \infty} M(x_{n+p}, x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

Remark: 1.11 A M- fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark: 1.12 since $*$ is continuous, it follows from (FM-4) that the limit of the sequence is uniquely determined.

Lemma: 1.13 [4] Let $\{x_n\}$ be a sequence in a M- fuzzy metric space. $(X, M, *)$ with the condition(FM-6). If there exists a number $q \in (0,1)$ such that

$M(x_n, x_n, x_{n+1}, t) \geq M(x_{n-1}, x_{n-1}, x_n, t/q)$

for all $t > 0$ and $n = 1, 2, 3, \ldots$, then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.14 [4] Let $(X, M, *)$ be a M- fuzzy metric space. with condition (FM-6). If for all $x, y, z \in X$, $t > 0$ with positive number $q \in (0,1)$ and $M(x, y, z, qt) \geq M(x, y, z, t)$, then $x = y = z$.

Definition: 1.15 A point x in X is a common fixed point of two maps $T_1, T_2: X \to X$ if $T_1(x) = T_2(x) = x$.

2.MAIN RESULTS

Theorem 2.1: Let $(X, M_1,*)$ and $(Y, M_2,*)$ be two complete M- fuzzy metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X satisfying

$$2M_1(Sy,Sy,STx,qt) \geq M_1(x,x,STx,t).M_1(x,x,Sy,t)+M_2(y,y,Tx,t) \quad \text{---- (1)}$$

$$2M_2(Tx,Tx,TSy,qt) \geq M_2(y,y,TSy,t).M_2(y,y,Tx,t)+M_1(x,x,Sy,t) \quad \text{----- (2)}$$

for all x in X and y in Y where $q < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further $Tz = w$ and $Sw = z$.

Proof: Let x_0 be an arbitrary point in X. Define two sequences $\{x_n\}$ and $\{y_n\}$ in X and Y, respectively, as follows:

$$x_n = (ST)^n x_0, \quad y_n = T(x_{n-1})$$

for $n = 1,2,\ldots$. By (1) we have

$$\begin{aligned}
2M_1(x_n, x_n, x_{n+1}, qt) &= 2M_1((ST)^n x_0, (ST)^n x_0, (ST)^{n+1} x_0), qt) \\
&= 2M_1(S(T(ST)^{n-1}x_0, S(T(ST)^{n-1}x_0, ST(ST)^n x_0, qt) \\
&= 2M_1(ST(x_{n-1}), ST(x_{n-1}), STx_n, qt) \\
&= 2M_1(Sy_n, Sy_n, STx_n, qt) \\
&\geq M_1(x_n, x_n, STx_n, t). M_1(x_n, x_n, Sy_n, t) + M_2(y_n, y_n, Tx_n, t) \\
&= M_1(x_n, x_n, x_{n+1}, t). M_1(x_n, x_n, x_n, t) + M_2(y_n, y_n, y_{n+1}, t) \\
&\geq M_1(x_n, x_n, x_{n+1}, qt) + M_2(y_n, y_n, y_{n+1}, t)
\end{aligned}$$

Which implies

$$M_1(x_n, x_n, x_{n+1}, qt) \geq M_2(y_n, y_n, y_{n+1}, t) \text{ ------------- (3)}$$

Similarly, by (2)

$$\begin{aligned}
2M_2(y_n, y_n, y_{n+1}, qt) &= 2M_2(Tx_{n-1}, Tx_{n-1}, Tx_n, qt) \\
&= 2M_2(Tx_{n-1}, Tx_{n-1}, TSy_n, qt) \\
&\geq M_2(y_n, y_n, TSy_n, t) .M_2(y_n, y_n, Tx_{n-1}, t) + M_1(x_{n-1}, x_{n-1}, Sy_n, t) \\
&= M_2(y_n, y_n, y_{n+1}, t). M_2(y_n, y_n, y_n, t) + M_1(x_{n-1}, x_{n-1}, x_n, t) \\
&\geq M_2(y_n, y_n, y_{n+1}, qt) + M_1(x_{n-1}, x_{n-1}, x_n, t)
\end{aligned}$$

Which implies

$M_2(y_n, y_n, y_{n+1},qt) \geq M_1(x_{n-1}, x_{n-1}, x_n,t)$ ----------- (4)

Therefore, by (3) and (4)

$M_1(x_n,x_n,x_{n+1},qt) \geq M_2(y_n,y_n,y_{n+1},t)$

$\geq M_1(x_{n-1}, x_{n-1},x_n,t/q)$

$\vdots$

$\geq M_1(x_0, x_0, x_1,t/q^n) \to 1$ as $n\to\infty$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since $(X, M_1,*)$ is complete, $\{x_n\}$ converges to a point z in X. Similarly we prove $\{y_n\}$ converges to a point w in Y.

Again by (2) we have

$2M_2(Tz,Tz,y_{n+1},qt) = M_2(Tz,Tz,TSy_n,qt)$

$\geq M_2(y_n,y_n,TSy_n,t).M_2(y_n,y_n,Tz,t) + M_1(z,z,Sy_n,t)$

$= M_2(y_n,y_n,y_{n+1},t).M_2(y_n,y_n,Tz,t) + M_1(z,z,x_n,t)$ --(5)

Letting $n \to \infty$ in (5) we have

$2M_2(Tz,Tz,w,qt) \geq M_2(w,w,Tz,qt) + 1$

That is $M_2(Tz,Tz,w,qt) \geq 1$

which implies that $M_2(Tz,Tz,w,qt) =1$ so that Tz = w.

On the other hand, by (1) we have

$2M_1(Sw, Sw, x_{n+1},qt) = 2M_1(Sw,Sw,STx_n,t)$

$\geq M_1(x_n,x_n,STx_n,t).M_1(x_n,x_n,Sw,t) + M_2(w,w,Tx_n,t)$

$=M_1(x_n,x_n,x_{n+1},t).M_1(x_n,x_n,Sw,t)+M_2(w,w,y_{n+1},t)$-- (6)

Letting $n\to\infty$ in (6), it follows that Sw = z.

Therefore we have STz = Sw = z and TSw = Tz = w, which means that the point z is a fixed point of ST and the point w is a fixed point of TS.

To prove the uniqueness of the fixed point z, let z′ be the second fixed point of ST.

By (1) we have

$2M_1(z ,z, z',qt) = 2M_1(Sw, Sw, STz',qt)$

$\geq M_1(z',z',STz',t). M_1(z',z',Sw,t) + M_2(w,w,Tz',t)$

$= M_1(z',z',z',t). M_1(z',z',z,t) + M_2(w,w,Tz',t)$

$\geq M_1(z',z',z,qt) + M_2(w,w,Tz',t)$

Which implies that

$M_1(z ,z, z',qt) \geq M_2(w,w ,Tz',t)$ ----- (7)

Similarly by (2), we have

$2M_2(w,w,Tz',qt) = 2M_2(Tz,Tz,TSTz',qt)$

$\geq M_2(Tz',Tz',TSTz',t).M_2(Tz',Tz'Tz,t)+ M_1(z,z,STz',t)$

$\geq M_2(Tz',Tz',w,qt)+ M_1(z,z,z',t)$

Which implies that

$M_2(w,w,Tz',qt) \geq M_1(z,z,z',t)$ ------ (8)

Therefore by (7) and (8)

$M_1(z, z, z',qt) \geq M_2(w, w,Tz',t) \geq M_1(z,z,z',t/q)$ (since q< 1),

which is a contradiction.

Thus z = z′.

So the point z is the unique fixed point of ST in X. Similarly, we prove the point w is also a unique fixed point of TS in Y.

Theorem 2.2: Let $(X, M_1,*)$ and $(Y, M_2,*)$ be two complete M-fuzzy metric spaces with continuous t-norm * is defined by

a*b = min {a, b}. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$M_1(SAx, SAx, TBx',qt) \geq \min\{ M_1(x,x,x',t), M_1(x,x,SAx,t),$

$M_1(x', x',TBx',t), M_1(x,x,TBx',2t).M_1(x', x',SAx,2t),$

$M_2(Ax,Ax,Bx',t)\}$ ------ (1)

$M_2(BSy,BSy, ATy',qt) \geq \min\{ M_1(y,y,y',t), M_2(y,y,BSy,t),$

$M_2(y', y',ATy',t), M_2(y,y,ATy',2t).M_2(y', y',BSy,2t),$

$M_1(Sy,Sy,Ty',t) \}$ -------(2)

for all x, x′ in X and y, y′ in Y. If one of the mappings A, B, S and T is continuous, then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$Ax_{2n-2} = y_{2n-1}, Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}; Ty_{2n} = x_{2n}$

for n = 1, 2, 3 . . . . .

Now we have

$M_1(x_{2n+1}, x_{2n+1}, x_{2n}, qt) = M_1(SAx_{2n}, SAx_{2n}, TBx_{2n-1},qt)$

$\geq \min\{ M_1(x_{2n}, x_{2n}, x_{2n-1},t), M_1(x_{2n}, x_{2n}, SAx_{2n},t),$

$M_1(x_{2n-1}, x_{2n-1}, TBx_{2n-1},t), M_1(x_{2n}, x_{2n}, TBx_{2n-1},2t).$

$M_1(x_{2n-1}, x_{2n-1}, SAx_{2n},2t), M_2(Ax_{2n}, Ax_{2n},Bx_{2n-1},t)\}$

$= \min\{ M_1(x_{2n}, x_{2n},x_{2n-1},t), M_1(x_{2n}, x_{2n}, x_{2n+1},t),$

$M_1(x_{2n-1}, x_{2n-1}, x_{2n},t), M_1(x_{2n}, x_{2n},x_{2n},2t).$

$M_1(x_{2n-1}, x_{2n-1},x_{2n+1},2t), M_2(y_{2n+1}, y_{2n+1},y_{2n},t)\}$

$\geq \min\{ M_1(x_{2n}, x_{2n},x_{2n-1},t), M_1(x_{2n}, x_{2n},x_{2n+1},t),$

$M_1(x_{2n-1}, x_{2n-1},x_{2n},t), M_1(x_{2n-1}, x_{2n-1}, x_{2n},t)*$

$M_1(x_{2n}, x_{2n},x_{2n+1},t), M_2(y_{2n+1}, y_{2n+1},y_{2n},t)\}$

$\geq \min\{M_1(x_{2n-1}, x_{2n-1},x_{2n},t) , M_2(y_{2n+1}, y_{2n+1},y_{2n},t)\}$

Now

$M_2(y_{2n+1}, y_{2n+1},y_{2n},qt) = M_2(y_{2n}, y_{2n},y_{2n+1},qt)$

$= M_2(BSy_{2n-1}, BSy_{2n-1}, ATy_{2n}, qt)$

$\geq \min\{ M_2(y_{2n}, y_{2n}, y_{2n-1},t), M_2(y_{2n-1}, y_{2n-1},BSy_{2n-1},t),$

$M_2(y_{2n}, y_{2n}, ATy_{2n},t), M_2(y_{2n-1}, y_{2n-1},ATy_{2n},2t).$

$M_2(y_{2n}, y_{2n}, BSy_{2n-1}, 2t), M_1(Sy_{2n-1}, Sy_{2n-1}, Ty_{2n}, t)\}$

$\geq \min\{ M_2(y_{2n}, y_{2n}, y_{2n-1}, t), M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t),$

$M_2(y_{2n}, y_{2n}, y_{2n+1}, t), M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t)*$

$M_2(y_{2n}, y_{2n}, y_{2n+1}, t), M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t)\}$

$\geq \min\{ M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t), M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t)\}$ ---------- (3)

Hence

$M_1(x_{2n+1}, x_{2n+1}, x_{2n}, qt) \geq \min\{ M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t), M_2(y_{2n+1}, y_{2n+1}, y_{2n}, t) \}$

$\geq \min\{ M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t), M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t/q),$

$M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t/q)\}$

$\geq \min\{ M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t), M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t/q)\}$ ---------- (4)

Similarly we have

$M_1(x_{2n}, x_{2n}, x_{2n-1}, qt) \geq$

$\min\{ M_1(x_{2n-2}, x_{2n-2}, x_{2n-1}, t), M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t)\}$

$M_2(y_{2n}, y_{2n}, y_{2n-1}, qt) \geq$

$\min \{M_2(y_{2n-2}, y_{2n-2}, y_{2n-1}, t), M_1(x_{2n-1}, x_{2n-1}, x_{2n-2}, t)\}$ ------------ (5)

Hence

$M_1(x_{2n}, x_{2n}, x_{2n-1}, qt) \geq$

$\min\{ M_1(x_{2n-2}, x_{2n-2}, x_{2n-1}, t), M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t)\}$

$\geq \min\{M_1(x_{2n-2}, x_{2n-2}, x_{2n-1}, t), M_2(y_{2n-2}, y_{2n-2}, y_{2n-1}, t/q),$ ---------- (6)

from inequalities (3), (4), (5) and (6), we have

$M_1(x_{n+1}, x_{n+1}, x_n, qt)$

$\geq \min \{ M_1(x_n, x_n, x_{n-1}, t), M_2(y_n, y_n, y_{n-1}, t/q)\}$

$\vdots$

$\geq \min\{M_1(x_1, x_1, x_0, t/q^{n-1}), M_2(y_1, y_1, y_0, t/q^n) \} \to 1$ as $n \to \infty$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since $(X, M_1, *)$ is complete, it converges to a point z in X. Similarly we can prove that the sequence $\{y_n\}$ is a Cauchy sequence in Y and it converges to a point w in Y.

Suppose A is continuous, then

$$\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove SAz = z. .

Suppose $SAz \neq z$.

We have

$M_1(SAz, SAz, z, qt) = \lim_{n\to\infty} M_1(SAz, SAz, TBx_{2n-1}, qt)$

$\geq \lim_{n\to\infty} \min\{ M_1(z,z,x_{2n-1},t), M_1(z,z,SAz,t),$

$M_1(x_{2n-1}, x_{2n-1}, TBx_{2n-1}, t), M_1(z,z,TBx_{2n-1}, 2t).$

$M_1(x_{2n-1}, x_{2n-1}, SAz, 2t), M_2(Az, Az, Bx_{2n-1}, t)\}$

$= \lim_{n\to\infty} \min\{ M_1(z,z,z,t), M_1(z,z,SAz,t),$

$M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t), M_1(z,z,x_{2n}, 2t).$

$M_1(x_{2n-1}, x_{2n-1}, SAz, 2t), M_2(Az, Az, y_{2n}, t)\}$

$= \min\{ 1, M_1(z,z,SAz,t), 1, 1, M_1(z,z,SAz,2t), 1\}$

$\geq M_1(z,z,SAz,t)$ (since $q<1$)

which is a contradiction.

Thus SAz = z.

Hence Sw = z. (Since Az = w)

Now we prove BSw = w.

Suppose $BSw \neq w$.

We have

$M_2(BSw, BSw, w, qt) = \lim_{n\to\infty} M_2(BSw, BSw, y_{2n+1}, qt)$

$= \lim_{n\to\infty} M_2(BSw, BSw, ATy_{2n}, qt)$

$\geq \lim_{n\to\infty} \min\{M_2(w,w,y_{2n},t), M_2(w,w,BSw,t), M_2(y_{2n}, y_{2n}, ATy_{2n}, t),$

$M_2(w,w,ATy_{2n}, 2t).$

$M_2(y_{2n}, y_{2n}, BSw, 2t), M_1(Sw, Sw, Ty_{2n}, t)\}$

$= \min\{1, M_2(w,w,BSw,t), 1, M_2(w,w,BSw,2t), 1\}$

$\geq M_2(w,w,BSw,t)$ (Since $q < 1$) which is a contradiction.

Thus BSw = w

Hence Bz = w. (Since Sw = z)

Now we prove TBz = z.

Suppose $TBz \neq z$.

$M_1(z,z,TBz,qt) = \lim_{n\to\infty} M_1(x_{2n+1}, x_{2n+1}, TBz, qt)$

$= \lim_{n\to\infty} M_1(SAx_{2n}, SAx_{2n}, TBz, qt)$

$\geq \lim_{n\to\infty} \min\{ M_1(x_{2n}, x_{2n}, z, t),$

$M_1(x_{2n}, x_{2n}, SAx_{2n}, t), M_1(z,z,TBz,t),$

$M_1(x_{2n}, x_{2n}, TBz, 2t). M_1(z,z,SAx_{2n}, 2t),$

$M_2(Ax_{2n}, Ax_{2n}, Bz, t)\}$

$= \min\{1, 1, M_1(z,z,Bz,t), M_1(z,z,TBz,2t), 1\}$

$\geq M_1(z,z,TBz,t)$ (Since $q< 1$)

which is a contradiction.

Thus TBz = z.

Hence Tw = z. (Since Bz = w)

Now we prove ATw = w.

Suppose $ATw \neq w$.

$M_2(w,w,ATw,qt) = \lim_{n\to\infty} M_2(y_{2n}, y_{2n}, ATw, qt)$

$= \lim_{n\to\infty} M_2(BSy_{2n-1}, BSy_{2n-1}, ATw, qt)$

$\geq \lim_{n\to\infty} \min\{ M_1(y_{2n-1}, y_{2n-1}, w, t),$

$M_2(y_{2n-1}, y_{2n-1}, BSy_{2n-1}, t), M_2(w,w,ATw,t),$

$M_1(y_{2n-1}, y_{2n-1}, ATw, 2t).M_2(w, w, BSy_{2n-1}, 2t)$, $M_1(Sy_{2n-1}, Sy_{2n-1}Tw, t)\}$

$\geq M_2(w,w,ATw,t)$ (Since $q < 1$)

which is a contradiction.

Thus ATw = w.

The same results hold if one of the mappings B, S and T is continuous.

Remark: 2.3 In the above theorem if A = B and S = T, we have the following corollary.

Corollary: 2.4 Let (X,M, *) and (Y, M_2,*) be two complete M-fuzzy metric spaces . Let A be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities.

$M_1(TAx, TAx, TAx',qt) \geq \min\{ M_1(x,x,x',t), M_1(x,x,TAx,t), M_1(x', x',TAx',t), M_1(x,x,TAx',2t). M_1(x', x',TAx,2t), M_2(Ax,Ax,Ax',t)\}$

$M_2(ATy,ATy, ATy',qt) \geq \min\{ M_1(y,y,y',t), M_2(y,y,ATy,t), M_2(y', y',ATy',t), M_2(y,y,ATy',2t). M_2(y', y',ATy,2t), M_1(Ty,Ty,Ty',t)\}$

for all x, x′ in X and y, y′ in Y. If one of the mappings A and T is continuous, then TA have a fixed point z in X and AT have a fixed point w in Y. Further, Az = w and Tw = z.

Theorem 2.5: Let (X, M_1,*) and (Y, M_2,*) be two complete M-fuzzy metric spaces . Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$4 M_1(SAx, SAx,TBx',qt) \geq M_1(x,x,x',t) + M_1(x,x,SAx,t) + M_1(x', x',TBx',t) + [M_1(x,x,SAx,t). M_1(x', x',TBx',t)] / M_1(x,x,x',t)$ ---------- (1)

$4 M_2(BSy, BSy, ATy',qt) \geq M_2 (y, y, y',t) + M_2 (y, y, BSy,t) + M_2 (y', y',ATy',t)+ [M_2(y, y,BSy,t). M_2(y', y',ATy',t)] / M_2(y, y, y',t)$ ---------- (2)

for all x, x′ in X and y, y′ in Y where $0< q < 1$. If one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$Ax_{2n-2} = y_{2n-1}, Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}; Ty_{2n} = x_{2n}$ for n = 1, 2, 3 . . . . .

Now from (1) we have

$4 M_1(x_{2n+1}, x_{2n+1}, x_{2n},qt) = M_1(SAx_{2n}, SAx_{2n}, TBx_{2n-1},qt)$
$\geq M_1(x_{2n},x_{2n},x_{2n-1},t) + M_1(x_{2n}, x_{2n},SAx_{2n},t) + M_1(x_{2n-1}, x_{2n-1},TBx_{2n-1},t) + [M_1(x_{2n}, x_{2n}, SAx_{2n},t). M_1(x_{2n-1}, x_{2n-1},TBx_{2n-1},t)] / M_1(x_{2n}, x_{2n},x_{2n-1},t)$
$= M_1(x_{2n}, x_{2n},x_{2n-1},t) + M_1(x_{2n}, x_{2n},x_{2n+1},t) + M_1(x_{2n-1}, x_{2n-1}, x_{2n},t) + [M_1(x_{2n}, x_{2n},x_{2n+1},t) . M_1(x_{2n-1}, x_{2n-1}, x_{2n},t)] / M_1(x_{2n}, x_{2n},x_{2n-1},t)$
$= 2 M_1(x_{2n}, x_{2n},x_{2n-1},t) + 2 M_1(x_{2n}, x_{2n},x_{2n+1},t)$
$\geq 2 M_1(x_{2n}, x_{2n},x_{2n-1},t) + 2 M_1(x_{2n}, x_{2n},x_{2n+1},qt)$

Which implies $M_1(x_{2n+1}, x_{2n+1}, x_{2n},qt) \geq M_1(x_{2n}, x_{2n},x_{2n-1},t)$ ------ (3)

Similarly we prove that

$M_1(x_{2n}, x_{2n}, x_{2n-1},qt) \geq M_1(x_{2n-1}, x_{2n-1},x_{2n-2},t)$ ------ (4)

From inequalities (3) and (4) we have

$$M_1(x_{n+1}, x_{n+1}, x_n,qt) \geq M_1(x_n, x_n,x_{n-1},t) \geq M_1(x_{n-1}, x_{n-1},x_{n-2},t/q) \geq \cdots \geq M_1(x_1, x_1,x_0,t/q^{n-1}) \to 1$$

as $n\to\infty$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, it converges to a point z in X.

Similarly $\{y_n\}$ is a Cauchy sequence in Y and it converges to a point w in Y .

Suppose A is continuous, then

$\lim_{n\to\infty} Ax_{2n} = Az = w.$

Now we prove SAz = z.

We have

$4 M_1(SAz, SAz, z,qt) = \lim_{n\to\infty} 4 M_1(SAz, SAz,TBx_{2n-1},qt)$
$= \lim_{n\to\infty} 4 M_1(SAz, SAz,x_{2n},qt)$
$\geq \lim_{n\to\infty} M_1(z, z, x_{2n-1},t) + M_1(z,z, SAz,t) + M_1(x_{2n-1}, x_{2n-1},TBx_{2n-1},t) + [M_1(z, z, SAz,t). M_1(x_{2n-1}, x_{2n-1}, TBx_{2n-1},t)] / M_1(z, z, x_{2n-1},t)$
$\geq 2 + 2 M_1(z, z, SAz, qt)$

Therefore $M_1(SAz, SAz, z,qt) \geq 1$

Which implies $M_1(SAz, SAz, z,qt) = 1$ for each $t > 0$.

Thus SAz = z.

Hence Sw = z. (Since Az = w)

Now we prove BSw = w.

We have

$4\ M_2(BSw, BSw, w,qt) = \lim_{n\to\infty} 4\ M_2(BSw, BSw, ATy_{2n},qt)$

$\geq \lim_{n\to\infty} M_2(w,w,y_{2n},t) + M_2(w, w,BSw,t)+$
$M_2(y_{2n}, y_{2n}, ATy_{2n},t) + [M_2(w,w, BSw,t).$
$M_2(y_{2n}, y_{2n}, ATy_{2n} ,t)] / M_2(w,w,y_{2n},t)$

Therefore $M_2(BSw, BSw, w,qt) \geq 1$

Which implies $M_2(BSw, BSw, w,qt) = 1$ for each $t > 0$.

Thus BSw = w.

Hence Bz = w. (Since Sw = z)

Now we prove TBz = z.

We have

$4\ M_1(z,z,TBz,qt) = \lim_{n\to\infty} 4\ M_1(SAx_{2n}, SAx_{2n}, TBz,qt)$

$\geq \lim_{n\to\infty} M_1(x_{2n}, x_{2n},z,t) + M_1(x_{2n}, x_{2n},SAx_{2n},t) +$
$M_1(z,z,TBz,t) +[M_1(x_{2n}, x_{2n},TBz,t).$
$M_1(z,z,SAx_{2n},t)] / M_1(x_{2n}, x_{2n},z,t)$

Which implies $M_1(z,z,TBz,qt) = 1$ for each $t > 0$.

Thus TBz = z .

Hence Tw = z. (Since Bz = w)

Now we prove ATw = w.

We have

$4\ M_2(w,w,ATw,qt) = \lim_{n\to\infty} 4\ M_2(BSy_{2n-1}, BSy_{2n-1},ATw,qt)$

$\geq \lim_{n\to\infty} M_2(y_{2n-1},y_{2n-1},w,t) + M_2(y_{2n-1}, y_{2n-1}, BSy_{2n-1},t)$
$+ M_2(w, w,ATw,t) +[M_2(y_{2n-1}, y_{2n-1},ATw,t).$
$M_2(w,w,BSy_{2n-1},t)] / M_2(y_{2n-1},y_{2n-1},w,t)$

Which implies $M (w,w,ATw,qt) = 1$ for each $t > 0$.

Thus ATw = w.

The same results hold if one of the mappings B, S and T is continuous.

Uniqueness: Let z′ be another common fixed point of SA and TB in X, w′ be another common fixed point of BS and AT in Y.

We have

$4M_1(z,z,z',qt) = M_1(SAz,SAz,TBz',qt)$
$\geq M_1(z,z,z',t)+ M_1(z,z,SAz,t)+ M_1(z', z',TBz',t)+$
$[M_1(z,z,SAz,t). M_1(z', z',TBz',t)] / M_1(z,z,z',t)$
$= M_1(z,z,z',t) + 1+ 1+ 1 / M_1(z,z,z',t)$
$= M_1(z,z,z',t) + 2 + 1 / M_1(z,z,z',t)$
$\geq M_1(z,z,z',t) + 2 + 1$
$= M_1(z,z,z',t) + 3$

Therefore $M_1(z,z,z',qt) \geq 1$

That is $M_1(z,z,z',qt) = 1$ for each $t > 0$.

Thus z = z′.

So the point z is the unique fixed point of ST. Similarly we prove the point w is also a unique fixed point of TS.

Remark: 2.6 In the above theorem if A = B and S = T, we have the following corollary.

Corollary: 2.7 Let (X,M, ∗) and (Y, M_2,*) be two complete M-fuzzy metric spaces . Let A be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities.

$M_1(TAx, TAx, TAx',qt) \geq M_1(x,x,x',t) + M_1(x,x,TAx,t) +$
$M_1(x', x',TAx',t) + [M_1(x,x,TAx,t).$
$M_1(x', x',TBx',t)] / M_1(x,x,x',t)$

$M_2(ATy,ATy, ATy',qt) \geq M_2 (y, y, y',t) + M_2 (y, y, ATy,t) +$
$M_2 (y', y',ATy') + [M_2(y, y, ATy,t).$
$M_2(y', y',ATy',t)] / M_2(y, y, y',t)$

for all x, x′ in X and y, y′ in Y. If one of the mappings A and T is continuous, then TA have a unique fixed point z in X and AT have a unique common fixed point w in Y. Further, Az = w and Tw = z.

Theorem 2.8: Let (X, M_1,∗) and (Y, M_2, ∗) be two complete M- fuzzy metric spaces. Let A and B be mappings from X to Y and S and T be mappings from Y to X satisfying the following inequalities

$M_1(x, x,x',t).M_1(SAx,SAx,TBx',qt) \geq \min\{M_1(x, x, x', t).$
$M_1(x', x',TBx',t), M_1(x, x,SAx,t).M_1(x, x',TBx',t),$
$M_2(Ax, Ax,Bx',t). M_1(x, x',TBx', t),$
$M_1(x, x', x',t). M_1(x, x',TBx', t)\}.$ . . (1)

$M_2(y, y, y',t). M_2(BSy',ATy,ATy,qt)\geq \min\{M_2(y, y, y',t).$
$M_2(y', y', BSy', t), M_2(y, y,ATy, t).M_2(y, y', BSy', t),$
$M_2(y', y',BSy', t).M_1(Sy', Sy', Ty, t),$
$M_2(y, y, y', t).M_2(y, y', BSy', t)\}$ . . . (2)

for all x, x′ in X and y, y′ in Y and 0 < q < 1. If one of the mappings A, B, S and T is continuous then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y. Further Az = Bz = w and Sw = Tw = z.

Proof: Let x_0 be an arbitrary point in X. We define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$Ax_{2n-2} = y_{2n-1}$; $Sy_{2n-1} = x_{2n-1}$; $Bx_{2n-1} = y_{2n}$; $Ty_{2n} = x_{2n}$ for n = 1, 2, 3, . . .

Now we have

$M_1\{x_{2n}, x_{2n}, x_{2n-1}, t) \cdot M_1(SAx_{2n}, SAx_{2n}, TBx_{2n-1}, qt) \geq \min$

$\{M_1(x_{2n}, x_{2n}, x_{2n-1}, t) \cdot M_1(x_{2n-1}, x_{2n-1}, TBx_{2n-1}, t),$

$M_1(x_{2n}, x_{2n}, SAx_{2n}, t).M_1(x_{2n}, x_{2n-1}, TBx_{2n-1}, t),$

$M_2(Ax_{2n}, Ax_{2n}, Bx_{2n-1}, t).M_1(x_{2n}, x_{2n-1}, TBx_{2n-1}, t),$

$M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \cdot M(x_{2n}, x_{2n-1}, TBx_{2n-1}, t)\}$

$\geq \min\{M_1(x_{2n}, x_{2n}, x_{2n-1}, t) \cdot M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t),$

$M_1(x_{2n}, x_{2n}, x_{2n+1}, t) \cdot M_1(x_{2n}, x_{2n-1}, x_{2n}, t),$

$M_2(y_{2n+1}, y_{2n+1}, y_{2n}, t). \ M_1(x_{2n}, x_{2n-1}, x_{2n}, t),$

$M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \cdot M_1(x_{2n}, x_{2n-1}, x_{2n}, t)\}.$

which implies

$M_1(x_{2n+1}, x_{2n+1}, x_{2n}, qt) \geq \min\{M_2(y_{2n}, y_{2n}, y_{2n+1}, t), M_1(x_{2n}, x_{2n}, x_{2n-1}, t)\}$ . . . (3)

Also we have

$M_2(y_{2n}, y_{2n}, y_{2n-1}, t) \cdot M_2(BSy_{2n-1}, ATy_{2n}, ATy_{2n}, qt) \geq$

$\min\{M_2(y_{2n}, y_{2n}, y_{2n-1}, t) \cdot M_2(y_{2n-1}, y_{2n-1}, BSy_{2n-1}, t),$

$M_2(y_{2n}, y_{2n}, ATy_{2n}, t).M_2(y_{2n}, y_{2n-1}, BSy_{2n-1}, t),$

$M_2(y_{2n-1}, y_{2n-1}, BSy_{2n-1}, t) \cdot M_1(Sy_{2n-1} Sy_{2n-1} Ty_{2n}, t),$

$M_2(y_{2n}, y_{2n}, y_{2n-1}, t) \cdot M_2(y_{2n}, y_{2n-1}, BSy_{2n-1}, t)\}$

$\geq \min\{M_2(y_{2n}, y_{2n}, y_{2n-1}, t)\ M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t),$

$M_2(y_{2n}, y_{2n}, y_{2n+1}, t).\ M_2(y_{2n}, y_{2n-1}, y_{2n}, t),$

$M_2(y_{2n-1}, y_{2n-1}, y_{2n}, t) \cdot M_1(x_{2n-1}, x_{2n-1}, x_{2n}, t),$

$M_2(y_{2n}, y_{2n}, y_{2n-1}, t) \cdot M_2(y_{2n}, y_{2n-1}, y_{2n}, t)\}$

which implies

$M_2(y_{2n}, y_{2n+1}, y_{2n+1}, qt) \geq \min\{M_2(y_{2n-1}, y_{2n}, y_{2n}, t), M_1(x_{2n-1}, x_{2n}, x_{2n}, t)$ . . . (4)

Using (3) and (4) we have

$M_1(x_n, x_n, x_{n+1}, qt) \geq \min\{M_1(x_{n-1}, x_{n-1}, x_n, t), M_2(y_n, y_n, y_{n+1}, t/q)\}$. . . (5)

$M_2(y_n, y_n, y_{n+1}, qt) \geq \min\{M_2(y_{n-1}, y_{n-1}, y_n, t), M_1(x_{n-1}, x_{n-1}, x_n, t)\}$. . . (6)

Using inequalities (5) and (6) we have

$M_1(x_n, x_n, x_{n+1}, qt) \geq \min\{M_1(x_{n-1}, x_{n-1}, x_n, t), M_2(y_n, y_n, y_{n+1}, t)\}$

$\vdots$

$$\geq \min\{M_1(x_0, x_0, x_1, \frac{t}{q^{n-1}}), M_2(y_1, y_1, y_2, \frac{t}{q^{n-1}})\}$$

$\to 1$ as $n \to \infty$

Thus $\{x_n\}$ is a Cauchy Sequences in X. Similarly we prove $\{y_n\}$ is a Cauchy sequence in Y respectively. Since ,(X, M_1, $*$) and (Y, M_2, $*$) are complete, $\{x_n\}$ converges to a point z in X and $\{y_n\}$ converges to a point w in Y.

Suppose A is continuous, then $\lim_{n\to\infty} Ax_{2n} = Az = y_{2n+1} = w$

Applying inequality (1), we have

$M_1(z, z, x_{2n-1}, t) \cdot M_1(SAz, SAz, TBx_{2n-1}, qt) \geq$

$\min\{M_1(z, z, x_{2n-1}, t) \cdot M_1(x_{2n-1}, x_{2n-1}, TBx_{2n-1}, t),$

$M_1(z, z, SAz, t).M_1(z, x_{2n-1}, TBx_{2n-1}, t),$

$M_2(Az, Az, Bx_{2n-1}, t) \cdot M_1(z, x_{2n-1}, TBx_{2n-1}, t),$

$M_1(z, x_{2n-1}, x_{2n-1}, t) \cdot M_1(z, x_{2n-1}, TBx_{2n-1}, t)\}$

Taking limit as $n \to \infty$, we have

$M_1(SAz, SAz, z, qt) \geq M_1(z, z, SAz, t)$

which is a contradiction since $q < 1$.

Thus SAz = z.

Hence Sw = z (Since Az = w)

Applying inequality (2) we have

$M_2(y_{2n}, y_{2n}, w, t).\ M_2(BSw, ATy_{2n}, ATy_{2n}, qt)$

$\geq \min\{M_2(y_{2n}, y_{2n}, w, t).M_2(w, w, BSw, t),$

$M_2(y_{2n}, w, BSw, t).M_2(y_{2n}, y_{2n}, ATy_{2n}, t),$

$M_2(w, w, BSw, t)\ .\ M_1(Sw, Sw, Ty_{2n}, t),$

$M_2(y_{2n}, y_{2n}, w, t) \cdot M_2(y_{2n}, w, BSw, t)\}$

Taking limit as $n\to\infty$, we have

$M_2(BSw, w, w, qt) \geq M_2(w, w, BSw, t)$

Thus BSw = w.

Hence Bz = w (Since Sw = z).

Applying inequality (1) again, we have

$M_1(x_{2n}, x_{2n}, z, t) \cdot M_1(SAx_{2n}, SAx_{2n}, TBz, qt)$

$\geq \min\{M_1(x_{2n}, x_{2n}, z, t) \cdot M_1(z, z, TBz, t),$

$M_1(x_{2n}, x_{2n}, SAx_{2n}, t)\ M_1(x_{2n}, z, TBz, t),$

$M_2(Ax_{2n}, Ax_{2n}, Bz, t) \cdot M_1(x_{2n}, z, TBz, t),$

$M_1(x_{2n}, z, z, t) \cdot M_1(x_{2n}, z, TBz, t)\}$

Taking limit as $n\to\infty$, we have

$M_1(z, z, TBz, qt) \geq M_1(z, z, TBz, t)$ (since $q < 1$)

which is a contradiction

Thus TBz = z

Hence Tw = z (Since Bz = w)

Applying inequality (2), we have

$M_2(w, w, y_{2n-1}, t) \cdot M_2(BSy_{2n-1}, ATw, ATw, qt)$

$\geq \min\{M_2(w, w, y_{2n-1}, t).M_2(y_{2n-1}, y_{2n-1}, BSy_{2n-1}, t),$

$M_2(w, w, ATw, t).M_2(w, y_{2n-1}, BSy_{2n-1}, t),$

$M_2(y_{2n-1}, y_{2n-1}, BSy_{2n-1}, t) . M_1(Sy_{2n-1}, Sy_{2n-1}, Tw, t),$

$M_2(w, w, y_{2n-1}, t) \cdot M_2(w, y_{2n-1}, BSy_{2n-1}, t)\}$

Taking limit as $n \to \infty$, we have

$$M_2(w, ATw, ATw, qt) \geq M_2 (w, w, ATw, t)$$

which is a contradiction, since $q < 1$.

Thus $ATw = w$.

The same results hold if one of the mappings B, S and T is continuous.

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