Multiplicative thinking ability among primary school pupils

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Abstract - *Multiplicative thinking or reasoning is developed out of addition by learners. In spite of this, multiplicative reasoning or thinking is a complex operation, requiring higher order multiplicative thinking. Though developed out of addition, multiplication and division present formidable force to students as equal grouping (repeated addition) used as a model for teaching Multiplication and division, is blamed for its lack of other multiplicative ideas and its constriction of fuller meaning and understanding of these operations.*

This study carried an investigation into the development of children's progression from additive to multiplicative thinking or reasoning. The study involved 81 pupils (drawn from both experimental and control groups) in Cape Coast Metropolis. The study was a quasi experimental design, involving Control and Experimental groups. Participants were interviewed individually. The results indicate the following three levels that children were identified with as they progressed through multiplicative thinking from additive thinking: additive thinking, multiplicative thinking with no immediate success and multiplicative thinking with immediate success. It was concluded that using multiple contexts in teaching multiplication and division impacted well on pupils' multiplicative thinking.

Keywords: *multiplicative thinking or reasoning, multiplication and division, multiple contexts*

I. Introduction

Among the four basic operations in mathematics in the primary school curriculum are multiplication and division concepts. Though these are foundational concepts in mathematics learning which are taught after the treatment of addition and subtraction, research has shown that most used model of multiplication and division has been equal grouping. This leads to addition; and that any treatment of multiplication and division in this way creates 'an enduring effect and might limit later interpretation' [1].

Research studies point to the fact that pupils in the primary schools exhibit difficulties in these operations. For instance, [2] had revealed that primary school children in their majority do not use multiplication facts they learn, instead they like to use addition or recitation pattern eg. 3, 6, 9, 12, 18 … as they tally on their fingers (cited by [3]). Reference [4] has also affirmed this, testifying that children add instead of multiplying. For example, '12 eggs in a carton, 25 cartons how many eggs are there?' (The answer was $12 + 25 = 37$,[5]). Based on this they viewed children's difficulties in multiplication as result of their inability to understand multiplication.

In a collaborative action research study involving a non-public funded school with three second grade classrooms of 58 students, [6] found that most students only memorized the multiplication table. They depended on calculations (in their minds or using counters or other things that they had in their learning centres). Further, in their attempt to find out if students relate the situations with multiplication when given three problems related to daily life, they observed that 'not always students relate their modeling with context in correct way'. Consequently, they believed it was difficult for some of the graders to write the correct reasoning of problem solving. Earlier, [7] has also reported of many difficulties found with multiplication and division. Test introduced to years 3, 4, and 6 involving a national sample of 300 hundred students to find out the strength and weaknesses of children's mathematical skills identified multiplication and division as a leading weakness. Reference [6] also has raised concern about the use of repeated addition for multiplication. They stressed that these concepts are taught separately with multiplication preceding division. They revealed that students learn multiplication table after which they start with division. Reference [6], however, hinted that the evaluation carried out in the fifth grades classes by MASHT, revealed there are obstacles related to applications of multiplication and division operations.

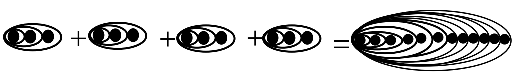
II. THEORETICAL FRAMEWORK

The following elements supposedly combine to constitute the theoretical framework of this study: the structure of operation and language involved; children's understanding and finally constructivism.

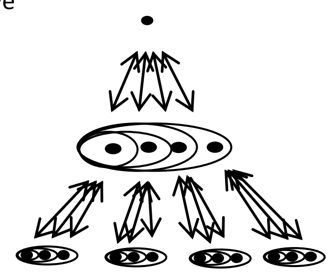
Structure of operation: In mathematics learning, the structure is of paramount importance. The structure provides for 'maximum meaning and understanding' [3]. It has been noted structure and the language involved in these operations are much of difficulty to the learners. As they point out, whilst operations of addition and subtraction only involve situation of either combined or disjoint sets of similar elements, the structure of multiplication and division is quite difficult. In multiplication, just as there must be two sets of different element types, each element in one set must of necessity relate to every element of the subset of the other. Tough among young children the development of their multiplication methods is enhanced with their familiarity with addition procedures [3], admittedly multiplicative thinking or reasoning is a complex operation and demands higher order thinking or reasoning [4].

Studying children's multiplicative thinking development, within the framework of Piaget's theory, [4] gave credent to the fact that children construct multiplicative thinking out of their ability to think in terms of addition. To them, children's additive thinking involves one level of association and one level of inclusion. Following Piaget, they agreed that additive thinking is inherent in number construction. This is achieved with repeated addition of one. Additive thinking is said to involve one level of abstraction. In explanation, for example, the child will only add each unit of three, which is made up of ones (figure 1). The child gets the group by combining successively '3 + (3 more ones), then 6 + (3 more ones), and finally 9 + (3 more ones)'.

However, multiplication entails two kinds of relationships. They identified them as many-to-one correspondence and composition of inclusion relations on more than one level (fig. 1a).



a. additive



b. Multiplicative

Figure 1:): **Additive thinking (3 + 3 + 3 + 3) compared with (2) Multiplicative thinking (4 x 3) as used by Kamii & Clark, 1996)**.

In the multiplicative thinking b), there is an abstraction at higher level in that one unit of three is made out of three units of one. The additive thinking would involve thinking of only units of one. The complexity of inclusion relations is represented horizontally at the level of units of one, where one is included in two, and two in three, and at the level of three, such that child includes 1 three in 2 threes, 2 threes in 3 threes and 3 threes in 4 threes. Vertically inclusion relation is seen where 3 ones are included in each units of 1 three, and 4 units of three are in the product. All the relationships must be simultaneously made and are illustrated by the arrows in both directions. Thus, thinking multiplicatively means one can think about these relationships at the same time. For instance, children think simultaneously about units of one and units of others when they have developed 'multiplicativeness'

Reference [5] talks about semantic structure of multiplication and division. Following (Nesher, as cited by [5]) they used semantic structure to mean the 'category to which a multiplicative problem is assigned'. In their view, semantic structure classification is 'arbitrary' since there is possibility of extension of the categories or they may be collapsed or refined. However, they admitted that this depends on the purpose of the investigation, and that is the prerogative of the researcher before the problems are given to students.

Constructivism: Teacher's knowledge about students and how they learn mathematics can have that impelling power to establish a conducive learning environment. Ideal learning environment therefore begins with the teacher's understanding of his students. In line with this, it is required that a teacher knows how his/her students learn just as he must of a necessity have content knowledge. References [8] and [9] advocate that a teacher must take appropriate choice of pedagogy to provide that learning opportunity which will allow students to construct their mathematical knowledge.

Constructivist theory of learning believes that students in the course of learning move from experience to knowledge by constructing their own meaning. In the process, they build up new learning from what they know already, and develop their learning through active involvement.

Constructivism challenges the idea that meaning resides in words, objects and actions (Von Glasserfied cited by [10]). The theory of constructivism advocates that knowledge is constructed by learners in different ways from each other (through social interaction). Constructivist learning theory focuses on the learner. In Constructivist learning process teachers and the learners play the role of active meaning makers. The meanings they derive from each other's word and actions as they interact are contextually based on learners' construction of their own meaning, building new learning from previous knowledge and developing their learning through active tasks.

Steffe (as cited in [11]) states:
"It is reasonable to expect that when the dialogic function is dominated in classroom discourse, pupils will treat their utterances and those of others as thinking devices. Instead of accepting them on as information to be received, encoded and shared, they will take an active stance towards them by questioning and extending them by incorporating them into their external and internal utterances". Source: [12] in [11].

Designing a classroom that reflects the constructivist learning approach encourages students to search for implications of results and generate multiple conclusions. It is in line with this that Jaworski (as cited by [10]) contended that teachers and pupils negotiate meaning of actions and words as they interact. Thus through 'negotiation' teachers and students may share knowledge and not through mere transmission of knowledge.

Contented that the mathematical experience and beliefs about what it means to know and do mathematics is greatly influenced by the classroom community as they negotiate the constraints of the classroom (Cobb as cited by [13]). Learning is believed to be both a process of active individual learning and social interactive process of classroom mathematical practice that operate collectively with equal significance.

Students learning have been described not as passive absorption of information or knowledge. Rather students are to take active parts in the acquisition of knowledge. Stated differently, learning is a shared responsibility in which the greater part of which devolves on the learner whilst the teacher provides an enabling environment by creating and sustaining learner's interest to succeed.

In conclusion, we draw the lesson that if constructivism will offer any hope for teaching then we need to establish firmly our teaching objectives and goals on the fact of children's thinking and not insistence on right and accurate responses of children. Again, we must help children come out with their own procedures and answers whilst helping to agree on the right ones.

III. A STATEMENT OF THE PROBLEM

Multiplication and division are components of early mathematics experienced in the primary curriculum. As foundational topics on which higher mathematics is built, their conceptual understanding is therefore crucial to the surviving tactics of mathematics learners. However, research points out to the difficulty learners are confronted with.

Multiplicative thinking, affirmed widely to be developed by young children through their familiarity with addition, is a complex operation and requires higher order thinking or reasoning which over reliance on additive thinking alone cannot provide. Also equal grouping model of teaching multiplication and division lacks other interpretive aspect of multiplication and division, and insufficient to equip learners with 'multiplicativeness'. This situation therefore warrants an investigation into children's progression from additive to multiplicative thinking.

IV. PURPOSE OF THE STUDY

The purpose of this study was to investigate children's progression in their developmental process of multiplicative thinking.

V. RESEARCH QUESTIONS

1. Are there any developmental levels related to children's development of multiplicative thinking?
2. Is there any significant difference between the experimental group and the control group in expressing verbally their multiplicative thinking processes?

VI. HYPOTHESIS

Null Hypotheses

1. There is no significant difference in the mean performance of pupils' verbal expression of multiplicative thinking processes among the experimental and the control groups.

VII. METHODOLOGY

The study was conducted using the quasi experimental research design, involving experimental group (n = 41) and control group (n = 40). This design was found suitable due to the fact that random assignment of students to experimental and control groups was highly impossible and also by the fact that it allowed the researcher to expose students to a treatment condition (using Multiple Contexts [MC] in teaching multiplication and division concepts).

The study comprised four intact class four pupils randomly selected from the upper primary schools in the Cape Coast Metropolis. The population is multicultural being made up predominantly of Fantes, Gas, Ewes and Hausas. The sample included three privately owned international mixed schools and one girl's only school (public school for girls) in the metropolis. The four schools involved in this study were assigned the letters A (n = 30), B (n = 37), C (n = 38) and D (n = 32) where n is the sample size. The total number of pupils involved in the study was 137, out which 41and 40 pupils respectively were further sampled from the experimental and control groups to be interviewed individually. Class Four was chosen for the study because it marks the beginning stage of the upper primary and this provided a good level platform for the children's participation. The intact class was used to ensure that each pupil in the class benefited from the study whilst saving the school authorities from that burden of having to look for different place to accommodate those who might not have been selected for the study. Again, this was to ensure a realistic study and that no child was left unattended to in the course of the study.

Two of the four classes were designated as the experimental groups and the other two as the control groups. The criterion used to select the classes into the various groups was based purely on their average performance in the pre-test. It was evident before the start of the study from the various class teachers that all the classes involved in the study had not yet been introduced to the concepts being investigated, so there were no scores for them. A pre-test was therefore given to each class under the same test conditions. The two classes with the high average score were selected to be the control groups whilst the other two with relatively low average scores were chosen to be the experimental groups. Of these participants, 81 pupils were sampled randomly, 41 from the experimental and 40 from the control groups to be interviewed individually.

The instrument used for the study was a structured Interview. It was an aspect of Piaget's clinical techniques. It was an adaptation from [4] (in [14]) which was devised by Sinclair [15]. For instance some materials were introduced to reflect Ghanaian context and experiences of the children. Round wooden tiles were used instead of chips. Paper sacs were introduced in this new task instead of fish. The instrument had also been used as a component of a research programme (Concepts in Secondary School Mathematics and Science – [16]).

The interview was in the form of tasks (1 through to 5) requiring the children to express verbally their multiplicative thinking processes. This ability would reveal their conceptual understanding of multiplication and division concepts. It was also to reveal their ability to retain facts. The materials used were 3 sacks of paper and 40 green round tiles (made of wood).

The instruments were administered to both experimental and control groups after an intensive teaching intervention process. This intervention process constituted the treatment, which was withheld from the control groups who were beneficiaries of traditional equal grouping strategy. Pupils' scores which formed the data were collated and subjected to analysis using Excel and the Statistical Package for Social Sciences- SPSS.

VIII. DATA ANALYSIS

Tasks 1 through to 5 related to how children are able to think multiplicatively. Multiplicative thinking refers to the ability to work flexibly with concepts, strategies and representations of multiplication and division as they occur in a wide range of contexts (Siemon's [17] draft as cited in [18]). Thinking multiplicatively is also inferred when there was a demonstrated ability to think simultaneously about unit of one, about unit of two, of three or of others.

The result from the responses of the children was analysed in two ways: the first one had to do with confining the analysis to responses coded correct only. The reason for this decision derived its importance from the fact that some responses were not justified. In fact, it is not sufficiently complete enough to depend on the right responses (in bold type e. g. 1, **2**, **3** for task 1 and **2**, 4, **6** for task 2) given by the children to delve into their processes of multiplicative thinking. Accordingly, though some responses were right, analysis concentrated on those that were justified by the owners through verbal expression to reveal their thinking. In the second part, on the other hand, the analysis carried on the responses of the children was to establish some developmental levels in children's progression from additive thinking to multiplicative thinking. Multiplicative thinking was inferred from their verbal expressions. Three of such levels were identified:

Level 1: Additive thinking with numerical sequence of + 1 or + 2, or + 2 for B and + 3 for C: The child at this level either relates what is in sack A to sack B and sack B to sack C or may also give 1 or 2 more to B, and to C, 1 or 2 more than B. The child also considers giving the number of times the interviewer predetermined. The child recognizes this and translates it by adding 2 and 3 to A and B respectively. The ensuing protocol clarifies this scenario.

Hawa (from control group): For 1 to sack A, she gave 2 to B and 4 to C [**1**, 2, 4]. When asked for explanation, she said; 'B contains 2 times what is in A. So B gets 2. C is 3 times bigger than A so if A is 1 then C is 4 because 1 + 3 = 4'. For 4 tiles to sack B, Hawa gave 2 to A and 6 to C [2, **4**, 6]. 'Why did you have to give 2 to A and 6 to C?' She answered, 'Because B is 2 times A, and A is 2, 2 + 2 is 4 and 4 + 2 = 6. C will get 6'. Thus Hawa's thinking was not in terms of 'multiplicativeness'. In reality, she happened to use multiplicative terminology superficially, but the processes going on in her mind (could be inferred) was additive thinking. A solid multiplicative thinker would not add as she did.

To another girl, Fiona (from Exp. 2), B gets 3 and C receives 5 [**1**, 3, 5] when sack A is given 1. She explained 'because sack B contains 2 times as many as A'. There was a counter question: 'if sack B contains 2 times as many as sack A, and A has1 tile why did you give 3 to B and 5 to C?' 'Sack C is 3 times as many as A because sack C is bigger than sack A. Fiona counted and gave 2 for sack A and 6 for sack C. 'Can you explain your answers?' The interviewer queried. 'Because if B has 4 tiles A does not have any tiles we have to give him 2 tiles'. She looked up and started: 'if two people are sharing four oranges they will get two two. Sack C has 6 because if you give 2 and 4 together you get 6'.

Level II: Multiplicative thinking but without immediate success: At this level, the child can think in terms of multiplication. The only thing lacking is that the child cannot give correct answers immediately, and sometimes depends on an aid. The following protocol illustrates this level:

Albertina (Exp. group): For 4 to B, she put her left hand on her cheek and said, 'please Sir', when she was asked to explain why she gave 2 to A and 8 to C [2, **4**, 8]. '…because sack B is 2 times as big as sack A and so 2 times 2 is 4'. She quickly came back with, 'please Sir, I made a mistake. It is 6'. She rearranged to get [2, **4**, 6]. She continued; 'sack C is 3 times as big as sack A'. For 14 to sack B, she gave 7 to A. She counted more tiles on to her laps but was asked to use the table. She grouped tiles into three sets of seven and put them all together. She said 16, assigning them to C. 'Did I hear 16?', the interviewer asked for clarity. '18', Albertina answered without counting. She waited for some time as if she was counting in her head, and said the answer is 21. Albertina seems to have developed multiplicative thinking by making 'three sets of seven' even though she was not successful immediately in giving the answer.

Level III: Multiplicative thinking with immediate success: - At this level the child gives correct multiplicative response to each question or task. Stated differently, this level identifies the solid multiplicative thinkers. The multiplicative thinkers give correct multiplicative response to each task. Excerpts from two participants' responses are given:

Hectoria (11 yrs, Exp. group). Hectoria gave multiplicative responses to all the five tasks. For 4 to sack B, she gave 2 to A and 6 to C. She explained; '2×2 will be 4. Sack B has 4, so A has to get 2'. She continued, '$2 \times 3 = 6$. So Sack C have 6'. When she was to answer task 5, where B received 14, she grouped and then counted. Thus '14 divided by 2 is 7'. She picked 7 new tiles into A. She then counted 3 groups of 7 new tiles into C. then she announced '21'.

Antoinette (9 yrs, Exp. group): For 9 to sack C, she gave 3 to sack A and said 3×3 is 9. She counted and arranged them on a table. 'Because B contains 2 times as A and C contains 3 times as A'. She then gave 6 to B. For 14 to B, she counted, grouped and gave the answers. 'Sack B has 2 times as many as sack A so if B is 14, then $14 \div 2$ is 7 so A will get 7'. For her answer for C, she then said 'Sack C is 3 times as many as A' and counted 21 for it. In both cases, the children actually demonstrated their multiplicative thinking. They indicated their concept of multiplication by grouping them into equal sets of elements.

Table I gives the distribution of levels of multiplicative development of the children within the experimental and the control groups. It also gives the overall percentages of the children in the levels identified.

TABLE I
MULTIPLICATIVE DEVELOPMENTAL LEVELS AMONG CHILDREN.

Level	Experimental	Control
I	2 (4.9)	3 (7.5)
II *	18 (43.9)	11(27.5)
III*	20 (48.8)	10 (25.0)

*Multiplicative thinkers: Exp: 92.7%; Control: 52.5%; All: (72.8%)
1. figures in parenthesis are in percentages
2. Control group (n = 40) , Experimental group:(n = 41) All (n = 81)
3. I - additive thinking
4.II- multiplicative thinking without immediate success
5. III – solid multiplicative thinkers
6. 2.4 percent (1 out of 41) and 40 (16 out 40) of the children in the experimental and contrails groups respectively did not make an attempt.

RESULTS AND DISCUSSION

In this study, the purpose was to investigate children's progression in their developmental processes of multiplicative thinking. To the question, 'are there any developmental levels related to children's development of multiplicative thinking?', the analysis of children's responses in terms of their multiplicative thinking development identified three levels that a child progresses through: *Additive thinking level*, *Multiplicative thinking with no immediate success level* and *Multiplicative thinking with immediate success level*. The important fact here is that there is a beginning level to the terminal level where solid multiplicative thinking status is acquired as depicted in figure IV.

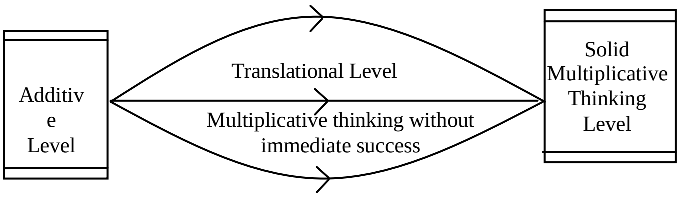


Fig. 28 Children's transitional levels in multiplicative development

In fact, children in the translational level seem to have need of 'learning clutches'. They are slow as identified by [4]. During the interview such children had to be reminded over and over again by asking them to count the members in their groupings. They could demonstrate 'multiplicativeness' by thinking in terms of unit of one, unit of two and other but failed in giving the product. The study shows 92.7 % (n = 41) of the experimental group being multiplicative thinkers with 48.8% being solid multiplicative thinkers (fig. III). These findings support Kamii and Clarke's [4] findings in a study in Alabama, a suburb of Birmingham which involved grades 1 - 5 children (n = 336). Their study identified 28.2% (n = 78) as solid multiplicative thinkers from grade four with 82.1% being multiplicative thinkers. Their grade four is comparable to primary four in this present study. Though the present study identified three developmental levels in multiplicative development, Kamii & Clarke's [4] findings included another level: *no serial correspondence or serial correspondence*. They found this level with grades 1 – 2 children which were non- included in this present study as it focused on the upper primary. From the findings, children within the experimental groups were more apt in showing 'multiplicativness' than their counterparts.

To answer the second question, the null hypothesis, 'there is no significant difference in the mean performance of pupils' verbal expression of multiplicative thinking processes among the experimental and the control groups' was explored. The interview responses formed the main data. The children's scores for correct categories were subjected to independent two-sample t-test. The two main groups: Experimental groups (n = 41) and the Control groups (n = 40) were compared. Table II presents the independent two-sample t-test results.

TABLE II

Independent Two-Sample T-Test Result For Verbal Expression Among Experimental Groups And Control Groups

Var	N	$\overline{X}$	SD	df	t-value	p-value	Decision
C	40	1.68	1.845				
E	41	3.32	1.795	79	4.06	0.001	Sig

$\alpha = 0.05$

The result indicates that, with an alpha = 0.05, the independent two-sample t-test showed that there was a statistically significant difference in the mean scores of the children in the control and experimental groups (t (79) = 4.06, p = 0.001) in expressing their multiplicative thinking processes verbally. The null hypothesis was therefore rejected and the alternative retained. The implication is that the use of MC in teaching multiplication and division appears to have impacted positively on primary school children's multiplicative thinking. Conclusively, the following recommendations are put forward:

1. Children are therefore not to be rushed through multiplication as a faster way of doing repeated addition; neither are their teachers expected to impose on them what they are not developmentally ready to learn.
2. Children found in the transition level should be handled carefully, and with appropriate materials to ensure smooth passage to be solid multiplicative thinkers. Teachers at the primary level of our educational system as a matter of urgency espouse the instructions incorporated with multiple contexts in teaching multiplication and division.

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