

On the topology generated by tgr -closed sets

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Abstract

Although the family of all tgr -closed sets in a topological space (X, τ) (briefly $tgrC(X, \tau)$) is not a topology on X , it can be considered as a subbase for some topology. In this paper we define the topology generated by tgr -closed sets (which is denoted by τ_{tgr}), and study some properties of spaces whose topologies are generated by the family of tgr -closed sets. We prove that $\tau_{tgr} \subseteq \tau$ if and only if τ is tgr -locally indiscrete. Also we define the topology generated by t^*gr -closed sets. We also prove that $\tau_{tgr} = \tau_{t^*gr}$ if (X, τ) is locally indiscrete.

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1 Introduction

General topologists have introduced and investigated many different classes of open sets and closed sets. Later they were interested on the topology generated by these classes. The topology generated by preopen sets was first introduced in 1987, where the properties of its closure operator were studied. Also the topology generated by semi regular sets was introduced in 1994 and some characterizations of extremally disconnected spaces were obtained. New classes of closed sets were introduced in [9] called tgr -closed, t^*gr -closed sets respectively. In this paper we study the topology generated by tgr -closed sets and some of its properties.

2 Preliminaries

Throughout this paper X and Y are always topological spaces with no separation axioms assumed unless otherwise stated. If $A \subseteq X$ the interior and the closure of A will be denoted by $int(A)$ and $cl(A)$ respectively.

Definition 2.1 [9] Let (X, τ) be a topological space, a set $A \subseteq X$ is called:

- (i) *tgr-closed* iff $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is *t-set*.
- (ii) *t*gr-closed* iff $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is *t*-set*.

Definition 2.2 [3],[5],[7],[8] A topological space (X, τ) is called:

- (1) *Hyperconnected* iff every nonempty open set is dense.
- (2) *Locally indiscrete* iff every open set is closed.
- (3) *Extremally disconnected* iff $cl(U)$ is open set whenever U is open set.
- (4) *Weakly Hausdorff* iff whenever x and y are distinct points in X , there exist regular closed sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.
- (5) *Compact* iff every open cover for (X, τ) has a finite subcover.
- (6) *S-closed* iff every regular closed cover has a finite subcover.
- (7) *Strongly S-closed* iff every closed cover has a finite subcover.
- (8) *Lindeloff* iff every open cover has a countable subcover.
- (9) *S-Lindeloff* iff every regular closed cover has a countable subcover.

Definition 2.3 [3],[4],[5],[8] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called:

- (1) *Continuous* iff the preimage of each open set in Y is open set in X .
- (2) *Contra continuous* iff the preimage of each open set in Y is closed in X .
- (3) *RC-continuous* iff the preimage of each open set in Y is regular closed in X .
- (4) *Open* iff the image of each open set in X is open in Y .
- (5) *Closed* iff the image of each closed set in X is closed in Y .
- (6) *Contra open* iff the image of each open set in X is closed in Y .
- (7) *Contra closed* iff the image of each closed set in X is open in Y .

3 On the topology generated by *tgr-closed* sets

In this section we introduce a new topology from a given topological space (X, τ) , we generate this topology from the family of *tgr-closed* sets in (X, τ) , we will give some examples and discuss some basic properties and characterizations.

Definition 3.1 Let (X, τ) be a topological space, the topology generated by the *tgr-closed* sets is the topology which have the collection of *tgr-closed* sets as a subbase and it is denoted by τ_{tgr} .

Example 3.2 Consider the real line with the cofinite topology. The nonempty *tgr-closed* sets are the infinite sets (see [9]), the topology generated by the *tgr-closed* sets is the discrete topology. To prove this, we will show that each singleton set belongs to τ_{tgr} . Let $a \in \mathbb{R}$ be arbitrary, then $(-\infty, a]$ and $[a, \infty)$ are *tgr-closed* sets and belong to τ_{tgr} , hence $(-\infty, a] \cap [a, \infty) = \{a\} \in \tau_{tgr}$ since τ_{tgr} is a topology. Therefore, τ_{tgr} is the discrete topology.

Theorem 3.3 If (X, τ) is locally indiscrete topological space, then $\tau \subseteq \tau_{tgr}$.

Proof. Let $U \in \tau$, then U is closed in τ since (X, τ) is locally indiscrete. Then, U is a clopen set in (X, τ) and so $U = cl(int(U))$, ie U is regular closed and hence it is *tgr-closed* set in (X, τ) , therefore $U \in \tau_{tgr}$.

Theorem 3.4 Let (X, τ) be a topological space. If (X, τ) is hyperconnected, then $\tau \subseteq \tau_{tgr}$.

Proof. Suppose that (X, τ) is a hyperconnected space. Let $U \in \tau$, if U is nonempty open set in (X, τ) , then U is dense in (X, τ) , hence U is a tgr -closed set in (X, τ) (see [9]), this implies that U is a subbasic open set in (X, τ_{tgr}) , therefore $\tau \subseteq \tau_{tgr}$.

Theorem 3.5 Let (X, τ) be a topological space, then $RC(X) \subseteq \tau_{tgr}$.

Proof. The result follows from the fact that every regular closed set is tgr -closed set (see [9]).

Definition 3.6 A topological space (X, τ) is called tgr -locally indiscrete if every tgr -open set is closed, or equivalently every tgr -closed set is open.

Theorem 3.7 The following are equivalent for a topological space (X, τ) :

- (1) $\tau_{tgr} \subseteq \tau$.
- (2) (X, τ) is tgr -locally indiscrete.

Proof. Suppose that $\tau_{tgr} \subseteq \tau$. Let V be a tgr -closed set in (X, τ) , then it is a subbasic open set in τ_{tgr} and $V \in \tau_{tgr} \subseteq \tau$, hence V is open set in (X, τ) and (X, τ) is tgr -locally indiscrete. Conversely, suppose that (X, τ) is tgr -locally indiscrete space. Let $V \in \tau_{tgr}$, then $V = \bigcup_{\alpha} \bigcap_{i=1}^n V_{\alpha_i}$ where each V_{α_i} is tgr -closed set in (X, τ) , but (X, τ) is tgr -locally indiscrete, so each V_{α_i} is open in (X, τ) . Hence $V = \bigcup_{\alpha} \bigcap_{i=1}^n V_{\alpha_i}$ is open set in (X, τ) and $\tau_{tgr} \subseteq \tau$.

Proposition 3.8 Let (X, τ) be a topological space. If $\tau_{tgr} \subseteq \tau$, then (X, τ) is extremally disconnected.

Proof. Suppose that $\tau_{tgr} \subseteq \tau$, then by theorem (3.7) we get that (X, τ) is tgr -locally indiscrete, to prove that (X, τ) is extremally disconnected, let U be a regular closed set, then U is tgr -closed set in (X, τ) (see [9]), so $U \in \tau_{tgr} \subseteq \tau$, therefore U is open set in (X, τ) . Hence, (X, τ) is extremally disconnected.

Remark 3.9 The converse of the above theorem need not be true. Consider the real line with the cofinite topology, this space is extremally disconnected but the cofinite topology does not contain the generated topology which is the discrete topology.

Theorem 3.10 Let (X, τ) be a topological space. If (X, τ_{tgr}) is compact space, then (X, τ) is S -closed space.

Proof. Let $\{V_{\alpha} : \alpha \in \Delta\}$ be a cover for (X, τ) of regular closed sets, then each V_{α} is tgr -closed set in (X, τ) , and $V_{\alpha} \in \tau_{tgr}$ for each $\alpha \in \Delta$, so $\{V_{\alpha} : \alpha \in \Delta\}$ is an open cover for (X, τ_{tgr}) . Since, (X, τ_{tgr}) is compact, there exist a finite set $\Omega \subseteq \Delta$ such that $\{V_{\alpha} : \alpha \in \Omega\}$ is a cover for (X, τ_{tgr}) , hence (X, τ) has a finite subcover of regular closed sets and it is S -closed space.

Remark 3.11 The converse of the above theorem need not be true in general as shown in the following example. Consider R with the cofinite topology, this space is S -closed since the only regular closed sets are ϕ, X . However the generated topology is the discrete topology and the discrete topology is compact iff X is finite, so R with the discrete topology is not compact. Hence, R with the cofinite topology is S -closed but the generated topology is not compact.

Theorem 3.12 Let (X, τ) be a topological space. If (X, τ_{tgr}) is Lindeloff space, then (X, τ) is S-Lindeloff.

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover for (X, τ) of regular closed sets, then each V_α is tgr -closed set in (X, τ) , and $V_\alpha \in \tau_{tgr}$ for each $\alpha \in \Delta$, so $\{V_\alpha : \alpha \in \Delta\}$ is an open cover for (X, τ_{tgr}) . Since, (X, τ_{tgr}) is Lindeloff, there exist a countable set $\Omega \subseteq \Delta$ such that $\{V_\alpha : \alpha \in \Omega\}$ is a cover for (X, τ_{tgr}) , hence (X, τ) has a countable subcover of regular closed sets and it is S-Lindeloff space.

Theorem 3.13 Let (X, τ) be a topological space. If (X, τ) is weakly Hausdorff, then (X, τ_{tgr}) is T_1 -space.

Proof. Suppose that (X, τ) is weakly Hausdorff. Let x, y be two distinct points in X , so there exist U, V two regular closed sets in (X, τ) such that $x \in U, y \notin U$ and $y \in V, x \notin V$ (since (X, τ) is weakly Hausdorff). Therefore U, V are two tgr -closed sets in (X, τ) , so U, V are subbasic open sets in (X, τ_{tgr}) such that $x \in U, y \notin U$ and $y \in V, x \notin V$, hence (X, τ_{tgr}) is T_1 -space.

Remark 3.14 The converse of the above theorem need not be true. Consider the real line with the cofinite topology, then τ_{tgr} is the discrete space which is T_1 -space, but the cofinite space is not weakly Hausdorff since the only nonempty regular closed set is X .

Theorem 3.15 If (X, τ) is weakly Hausdorff topological space, then the following are equivalent:

- (1) (X, τ_{tgr}) is strongly S-closed.
- (2) X is finite.

Proof. It is clear that if X is finite, then (X, τ_{tgr}) is strongly S-closed. Conversely, suppose that (X, τ_{tgr}) is strongly S-closed space and (X, τ) is weakly Hausdorff, then by theorem (3.13) (X, τ_{tgr}) is T_1 space. Hence we get a closed cover β for (X, τ_{tgr}) where $\beta = \{\{x\} : x \in X\}$, and since (X, τ_{tgr}) is strongly S-closed, β must have a finite subcover, this finite subcover is β itself. Hence, X is finite.

Theorem 3.16 If (X, τ) is a discrete topological space, then (X, τ_{tgr}) is also discrete.

Proof. Let A be arbitrary subset of X , then $cl(int(A)) = cl(A) = A$ (since X has the discrete space), so A is regular closed set, hence A is a tgr -closed set, therefore $A \in \tau_{tgr}$ and τ_{tgr} is the discrete space.

Remark 3.17 The converse of the above theorem need not be true. Consider the real line with the cofinite topology, in example (3.2) we have shown that τ_{tgr} is the discrete topology. However, the cofinite space is not discrete.

Theorem 3.18 If (X, τ) is the trivial topology, then (X, τ_{tgr}) is the discrete topology.

Proof. We will show that any subset of $(X, \tau_{trivial})$ is a tgr -closed set. First, we have ϕ is a tgr -closed set. Now for any nonempty subset A of X , $rcl(A) = X$ since the only regular closed sets in the trivial topology are X, ϕ . Also the only t -sets in the trivial topology are X, ϕ . Hence for any nonempty set A , the only t -set that contains A is X , this means that A is tgr -closed set. Therefore, we have for any $A \subseteq X$, $A \in \tau_{tgr}$ and τ_{tgr} is the discrete space.

Remark 3.19 If $(X, \tau), (X, \sigma)$ are two topological spaces such that $\tau \subseteq \sigma$, it is need not be true that $\tau_{tgr} \subseteq \sigma_{tgr}$. Consider $X = \{a, b\}$, let τ be the trivial topology and σ be the Sierpinski space $\sigma = \{X, \{a\}, \phi\}$, we have $\tau \subseteq \sigma$ and τ_{tgr} is the discrete space but $\sigma_{tgr} = \sigma$ (since the only tgr-closed sets in the Sierpinski space are $X, \{a\}$ and ϕ). Therefor, we have σ contains τ but σ_{tgr} does not contain τ_{tgr} .

4 Functions On the Space τ_{tgr}

If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a continuous function, does f still continuous if we consider X with the generated topology τ_{tgr} or if we consider Y with σ_{tgr} ? if no, under what conditions does f still continuous?

In this this section we will give some results about this questions and other related questions.

Theorem 4.1 Let $(X, \tau), (Y, \sigma)$ be topological spaces and $f : (X, \tau) \rightarrow (Y, \sigma)$ be a continuous open surjection function, then the following are equivalent:

- (1) (Y, σ) is tgr-locally indiscrete.
- (2) $f : (X, \tau) \rightarrow (Y, \sigma_{tgr})$ is continuous.

Proof. Suppose (1) holds, then we get that $\sigma_{tgr} \subseteq \sigma$ (from theorem (3.7)). Now, let $U \in \sigma_{tgr} \subseteq \sigma$, so $U \in \sigma$ and we have $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous, we get that $f^{-1}(U)$ is open in (X, τ) , hence $f : (X, \tau) \rightarrow (Y, \sigma_{tgr})$ is continuous.

Conversely, suppose that (2) holds, and let V be a tgr-closed set (Y, σ) , so $V \in \sigma_{tgr}$, hence $f^{-1}(V)$ is open in (X, τ) (since $f : (X, \tau) \rightarrow (Y, \sigma_{tgr})$ is continuous), but since f is open surjection, we have $f(f^{-1}(V)) = V$ is open in (X, τ) . Hence, (Y, σ) is tgr-locally indiscrete.

Corollary 4.2 Let (X, τ) be a topological space, then the identity function $I : (X, \tau) \rightarrow (X, \tau_{tgr})$ is continuous if and only if (X, τ) is tgr-locally indiscrete.

Proof. Consider the identity function $I : (X, \tau) \rightarrow (X, \tau)$, then I is a continuous open surjection function, and the result follows directly from the above theorem.

Theorem 4.3 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous and (X, τ) is locally indiscrete, then $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is continuous.

Proof. Let V be open set in (Y, σ) , then $f^{-1}(V)$ is open in (X, τ) since $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous, but $\tau \subseteq \tau_{tgr}$ since (X, τ) is locally indiscrete, hence $f^{-1}(V)$ is open in (X, τ_{tgr}) and this means that $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is continuous.

Theorem 4.4 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous and (X, τ) is hyperconnected, then $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is continuous.

Proof. Recall that if (X, τ) is hyperconnected, then $\tau \subseteq \tau_{tgr}$ and then the proof is similar to the proof of the previous theorem.

Theorem 4.5 *Let $(X, \tau), (Y, \sigma)$ be topological spaces and $f : (X, \tau) \rightarrow (Y, \sigma)$ be a contra continuous, contra closed surjection function, then the following are equivalent:*

- (1) (Y, σ) is *tgr*-locally indiscrete.
- (2) $f : (X, \tau) \rightarrow (Y, \sigma_{tgr})$ is contra continuous.

Proof. Suppose (1) holds, then we get that $\sigma_{tgr} \subseteq \sigma$ (from theorem (3.7)). Now, let $U \in \sigma_{tgr} \subseteq \sigma$, so $U \in \sigma$ and we have $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra continuous, we get that $f^{-1}(U)$ is closed in (X, τ) , hence $f : (X, \tau) \rightarrow (Y, \sigma_{tgr})$ is contra continuous.

Conversely, suppose that (2) holds and let V be a *tgr*-closed set (Y, σ) , so $V \in \sigma_{tgr}$, hence $f^{-1}(V)$ is closed set in (X, τ) (since $f : (X, \tau) \rightarrow (Y, \sigma_{tgr})$ is contra continuous), but since f is contra closed surjection, we have $f(f^{-1}(V)) = V$ is open in (X, τ) . Hence, (Y, σ) is *tgr*-locally indiscrete.

Theorem 4.6 *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra continuous and (X, τ) is locally indiscrete, then $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is contra continuous.*

Proof. Let V be any closed set in (Y, σ) , then $f^{-1}(V)$ is open in (X, τ) since $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra continuous, but $\tau \subseteq \tau_{tgr}$ since (X, τ) is locally indiscrete, hence $f^{-1}(V)$ is open in (X, τ_{tgr}) and this means that $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is contra continuous.

Theorem 4.7 *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra continuous and (X, τ) is hyperconnected, then $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is contra continuous.*

Proof. The proof is similar to the previous theorem.

Theorem 4.8 *Let $(X, \tau), (Y, \sigma)$ be topological spaces, if $f : (X, \tau) \rightarrow (Y, \sigma)$ is RC-continuous, then $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is continuous.*

Proof. Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is RC-continuous. Let $V \subseteq Y$ be open set, then $f^{-1}(V)$ is regular closed set in (X, τ) , so $f^{-1}(V)$ is *tgr*-closed set in (X, τ) , so $f^{-1}(V)$ is a basic open set in (X, τ_{tgr}) . Hence, $f : (X, \tau_{tgr}) \rightarrow (Y, \sigma)$ is continuous.

5 On the topology generated by t^* gr-closed sets

In section 3 we have defined τ_{tgr} and it was generated by *tgr*-closed sets, in this section we define τ_{t^*gr} in the same way, we will give some examples and discuss some properties.

Definition 5.1 *Let (X, τ) be a topological space, the topology generated by the t^* gr-closed sets is the topology which have the collection of t^* gr-closed sets as a subbase and it is denoted by τ_{t^*gr} .*

Example 5.2 *Consider the real line with the cofinite topology. Recall that the t^* gr-closed sets in this space are only ϕ and X (see [9]), therefor τ_{t^*gr} is the trivial topology.*

Theorem 5.3 *If (X, τ) is a discrete topological space, then (X, τ_{t^*gr}) is also discrete.*

Proof. Let A be arbitrary subset of X , then $cl(int(A)) = cl(A) = A$ (since X has the discrete space), so A is regular closed set, hence A is a t^*gr -closed set, therefore $A \in \tau_{t^*gr}$ and τ_{t^*gr} is the discrete space.

Theorem 5.4 *If (X, τ) is the trivial topology, then (X, τ_{t^*gr}) is the discrete topology.*

Proof. We will show that any subset of $(X, \tau_{trivial})$ is a t^*gr -closed set. First, we have ϕ is a t^*gr -closed set. It is clear that X, ϕ are t^*gr -closed sets, now if A is a t^* -set such that $A \notin \{\phi, X\}$, hence $cl(int(A)) = cl(A)$. Now $int(A) = \phi$ (since the only open set that contained in A is ϕ), $socl(int(A)) = \phi$ but $cl(A) = X$ (since the only closed set that contains A is X), therefore $cl(int(A)) \neq cl(A)$ this is a contradiction, hence the only t^* -sets in the trivial topology are ϕ, X . Therefore, any subset of X is t^*gr -closed.

Example 5.5 *Let (X, τ) be the Serpinski space where $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$. The t^*gr -closed sets are $X, \phi, \{b\}$ and $\tau_{t^*gr} = \{X, \phi, \{b\}\}$.*

Recall from section 2 that in the Serpinski space where $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$ we have: $\tau_{tgr} = \{X, \phi, \{a\}\}$, so in this example $\tau_{tgr} \neq \tau_{t^*gr}$, hence the question that we must deal with is: under what conditions on (X, τ) we get the equality $\tau_{tgr} = \tau_{t^*gr}$. The following theorem states a fundamental result about this.

Theorem 5.6 *If (X, τ) is a locally indiscrete space, then $\tau_{tgr} = \tau_{t^*gr}$.*

Proof. Suppose that (X, τ) is a locally indiscrete, we will show that the t -sets and the t^* -sets are the same. First, suppose that A is a t -set, so $int(cl(A)) = int(A)$, but since (X, τ) is locally indiscrete, any closed set is open, so $cl(A)$ is open set and $int(cl(A)) = cl(A) = int(A)$, hence $cl(cl(A)) = cl(A) = cl(int(A))$ and A is a t^* -set.

Conversely, suppose that A is a t^* -set, so $cl(int(A)) = cl(A)$, but $int(A)$ is open set, so it is closed (since (X, τ) is locally indiscrete), hence $cl(int(A)) = int(A) = cl(A)$ and this implies that $int(int(A)) = int(A) = int(cl(A))$, therefore A is a t -set. So, we proved that in a locally indiscrete space any t -set is t^* -set which implies that any t^*gr -closed set is tgr -closed set, also we proved that any t^* -set is t -set which implies that any tgr -closed set is t^*gr -closed set, hence $\tau_{tgr} = \tau_{t^*gr}$.

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