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ON NEIGHBORHOODS OF SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED USING DIFFERENTIAL OPERATOR

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ABSTRACT. In the present work we introduce new subclasses $\mathcal{S}_m^{\lambda;\alpha}(b,l,\beta)$, $\mathcal{R}_m^{\lambda;\alpha}(b,l,\beta;\mu)$, $\mathcal{S}_m^{\lambda;\alpha}(b,l,\sigma,\beta)$ and $\mathcal{R}_m^{\lambda;\alpha}(b,l,\sigma,\beta;\mu)$ of analytic functions defined using differential operator and discuss certain neighborhood properties.

1. Introduction

Let $\mathcal{T}(n)$ denote the family of analytic functions defined in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ which are of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0,\, k \in \mathbb{N}\backslash\{1\},\, n \in \mathbb{N}). \tag{1.1}$$

For any function $f \in \mathcal{T}(n)$ and $\delta \geq 0$, the (n,δ)-neighborhood of f is defined as,

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{T}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \; and \; \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\}. \tag{1.2}$$

For the identity function $e(z) = z$, we see that,

$$\mathcal{N}_{n,\delta}(e) = \left\{ g \in \mathcal{T}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \; and \; \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}. \tag{1.3}$$

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The concept of neighborhoods was first introduced by Goodman and then generalized by Ruscheweyh [7].

Now consider the following differential operator [5].

$$\mathrm{D}_{\lambda,\alpha}^{m,l} f(z) = z - \sum_{k=n+1}^{\infty} \left[\frac{(\lambda - \alpha)(k-1) + l + k}{l+1} \right]^m a_k z^k. \tag{1.4}$$

where $\alpha \geq 0,\ \lambda \geq 0,\ m \in \mathbb{N}_0 = \mathcal{N} \cup \{0\},\ l \geq 0$.

This operator generalizes various operators studied earlier by Sălăgean [8], Al - Oboudi [1], Cho and Kim [3], Cho and Srivastava [4], Catas [2] and Uralegaddi and Somanath [9], Maslina Darus and Rabha Ibrahim [6].

By using the generalized differential operator $\mathrm{D}_{\lambda,\alpha}^{m,l} f(z)$ we define the following subclasses:

Definition 1.1. *A function $f \in \mathcal{T}(n)$ is said to be in the class $\mathcal{S}_m^{\lambda,\alpha}(b, l, \beta)$ if*

$$\left| \frac{1}{b} \left(\frac{z(\mathrm{D}_{\lambda,\alpha}^{m,l} f(z))'}{\mathrm{D}_{\lambda,\alpha}^{m,l} f(z)} - 1 \right) \right| < \beta, \tag{1.5}$$

where $b \in \mathbb{C} \backslash \{0\}$, $\lambda \geq 0$, $0 < \beta \leq 1$, and $z \in \mathcal{U}$.

Definition 1.2. *A function $f \in \mathcal{T}(n)$ is said to be in the class $\mathcal{R}_m^{\lambda,\alpha}(b, l, \beta; \mu)$ if*

$$\left| \frac{1}{b} \left((1-\mu) \frac{\mathrm{D}_{\lambda,\alpha}^{m,l} f(z)}{z} + \mu (\mathrm{D}_{\lambda,\alpha}^{m,l} f(z))' - 1 \right) \right| < \beta, \tag{1.6}$$

where $b \in \mathbb{C} \backslash \{0\}$, $\lambda \geq 0$, $0 < \beta \leq 1, 0 \leq \mu \leq 1$ and $z \in \mathcal{U}$.

2. Neighborhoods for classes $\mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$ and $\mathcal{R}_m^{\lambda,\alpha}(b,l,\beta;\mu)$

In this section, we obtain inclusion relations involving $\mathcal{N}_{n,\delta}$ for functions in the classes $\mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$ and $\mathcal{R}_m^{\lambda,\alpha}(b,l,\beta;\mu)$.

Lemma 2.1. *A function $f \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left[\frac{(\lambda-\alpha)(k-1)+l+k}{l+1}\right]^m(\beta|b|+k-1)a_k \leq \beta|b|. \tag{2.1}$$

Proof. Suppose $f \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$. Then by using (1.5) we can write,

$$\Re\left\{\frac{z(\mathrm{D}_{\lambda,\alpha}^{m,l}f(z))'}{\mathrm{D}_{\lambda,\alpha}^{m,l}f(z)}-1\right\} > -\beta|b|, \quad (z\in\mathcal{U}). \tag{2.2}$$

Equivalently,

$$\Re\left\{\frac{-\sum_{k=n+1}^{\infty}\left[\frac{(\lambda-\alpha)(k-1)+l+k}{l+1}\right]^m a_k z^k(k-1)}{z-\sum_{k=n+1}^{\infty}\left[\frac{(\lambda-\alpha)(k-1)+l+k}{l+1}\right]^m a_k z^k}\right\} > -\beta|b|, \quad (z\in\mathcal{U}). \tag{2.3}$$

Letting $z \to 1$, through the real values, the inequality (2.3) yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z|=1$, we obtain,

$$\left|\frac{z(\mathrm{D}_{\lambda,\alpha}^{m,l}f(z))'}{\mathrm{D}_{\lambda,\alpha}^{m,l}f(z)}-1\right| = \left|\frac{\sum_{k=n+1}^{\infty}\left[\frac{(\lambda-\alpha)(k-1)+l+k}{l+1}\right]^m a_k z^k(k-1)}{z-\sum_{k=n+1}^{\infty}\left[\frac{(\lambda-\alpha)(k-1)+l+k}{l+1}\right]^m a_k z^k}\right|$$

$$\leq \frac{\beta|b|\left[1-\sum_{k=n+1}^{\infty}\left[\frac{(\lambda-\alpha)(k-1)+l+k}{l+1}\right]^m a_k\right]}{1-\sum_{k=n+1}^{\infty}\left[\frac{(\lambda-\alpha)(k-1)+l+k}{l+1}\right]^m a_k} \leq \beta|b|.$$

Hence, by the maximum modulus theorem, we have $f \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$. Thus the proof is complete. □

In the same way, we can prove the following lemma.

Lemma 2.2. *A function* $f \in \mathcal{R}_m^{\lambda,\alpha}(b,l,\beta;\mu)$ *if and only if*

$$\sum_{k=n+1}^{\infty}\left[\frac{(\lambda-\alpha)(k-1)+l+k}{l+1}\right]^m [\mu(k-1)+1]a_k \leq \beta|b|. \tag{2.4}$$

Theorem 2.3. *If*

$$\delta = \frac{(n+1)\beta|b|}{(\beta|b|+n)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m}, \quad (|b|<1), \tag{2.5}$$

then $\mathcal{S}_m^{\lambda,\alpha}(b,l,\beta) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. Let $f \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$. By Lemma 2.1, we have,

$$\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m (\beta|b|+n)\sum_{k=n+1}^{\infty} a_k \leq \beta|b|$$

so,

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|b|}{\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m(\beta|b|+n)}. \tag{2.6}$$

Using (2.1) and (2.6), we have,

$$\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m \sum_{k=n+1}^{\infty} k a_k$$

$$\leq \beta|b| + (1-\beta|b|)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m \sum_{k=n+1}^{\infty} a_k$$

$$\leq \beta|b| + (1-\beta|b|)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m \frac{\beta|b|}{\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m (n+\beta|b|)}$$

$$\leq \frac{(n+1)\beta|b|}{\beta|b|+n}$$

That is,

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)\beta|b|}{(\beta|b|+n)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m} = \delta$$

Thus, by the definition given by (1.3), $f \in \mathcal{N}_{n,\delta}(e)$. This completes the proof. □

Theorem 2.4. *If*

$$\delta = \frac{(n+1)\beta|b|}{(\mu n+1)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m}, \quad (|b|<1), \tag{2.7}$$

then $\mathcal{R}_m^{\lambda,\alpha}(b,l,\beta;\mu) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. Let $f \in \mathcal{R}_m^{\lambda,\alpha}(b,l,\beta;\mu)$. By Lemma 2.2, we have,

$$\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m (\mu n+1) \sum_{k=n+1}^{\infty} a_k \leq \beta|b|,$$

which yields the following inequality

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|b|}{\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m (\mu n+1)}. \tag{2.8}$$

Using (2.4) and (2.8), we also have,

$$\mu\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m \sum_{k=n+1}^{\infty} k a_k$$

$$\leq \beta|b| - (1-\mu)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m \sum_{k=n+1}^{\infty} a_k$$

$$\leq \beta|b| - (1-\mu)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m \frac{\beta|b|}{\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m (\mu n+1)}$$

$$\leq \frac{(n+1)\mu\beta|b|}{(\mu n+1)}$$

That is,

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)\beta|b|}{(\mu n+1)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m} = \delta.$$

Thus, by the definition given by (1.3), $f \in \mathcal{N}_{n,\delta}(e)$. This completes the proof. □

3. Neighborhoods for classes $\mathcal{S}_m^{\lambda,\alpha}(b,l,\sigma,\beta)$ and $\mathcal{R}_m^{\lambda,\alpha}(b,l,\sigma,\beta;\mu)$

In this section, we define the subclasses $\mathcal{S}_m^{\lambda,\alpha}(b,l,\sigma,\beta)$ and $\mathcal{R}_m^{\lambda,\alpha}(b,l,\sigma,\beta;\mu)$ of $\mathcal{T}(n)$ and certain neighborhood properties for functions belonging to these classes are obtained.

For $0 \leq \sigma < 1$ and $z \in \mathcal{U}$, a function $f \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\sigma,\beta)$ if there exists a function $g \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \sigma. \tag{3.1}$$

For $0 \leq \sigma < 1$ and $z \in \mathcal{U}$, a function $f \in \mathcal{R}_m^{\lambda,\alpha}(b,l,\sigma,\beta;\mu)$ if there exists a function $g \in \mathcal{R}_m^{\lambda,\alpha}(b,l,\beta;\mu)$ such that the inequality (3.1) holds true.

Theorem 3.1. *If $g \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$ and*

$$\sigma = 1 - \frac{\delta(\beta|b|+n)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m}{(n+1)\left[(\beta|b|+n)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m - \beta|b|\right]} \tag{3.2}$$

then, $\mathcal{N}_{n,\delta}(g) \subset \mathcal{S}_m^{\lambda,\alpha}(b,l,\sigma,\beta)$.

Proof. Let $f \in \mathcal{N}_{n,\delta}(g)$. Then,

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta, \tag{3.3}$$

which yields the coefficient inequality,

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1}, \quad (n \in \mathbb{N}). \tag{3.4}$$

Since $g \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\beta)$ by (2.6), we have,

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|b|}{(\beta|b|+n)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m} \tag{3.5}$$

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so that,

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty}b_k}.$$

$$\leq \frac{\delta}{n+1}\frac{(\beta|b|+n)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m}{\left[(\beta|b|+n)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m-\beta|b|\right]}$$

$$=1-\sigma$$

Thus, by definition, $f \in \mathcal{S}_m^{\lambda,\alpha}(b,l,\sigma,\beta)$ for σ given by (3.2). Thus the proof is complete. □

On similar lines, we can prove the following theorem.

Theorem 3.2. *If* $g \in \mathcal{R}_m^{\lambda,\alpha}(b,l,\beta;\mu)$ *and*

$$\sigma = 1-\frac{\delta(\mu n+1)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m}{(n+1)\left[(\mu n+1)\left[\frac{(\lambda-\alpha)n+l+n+1}{l+1}\right]^m-\beta|b|\right]} \tag{3.6}$$

then, $\mathcal{N}_{n,\delta}(g) \subset \mathcal{R}_m^{\lambda,\alpha}(b,l,\sigma,\beta;\mu)$.

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