Numerical Solutions of the Harmonic Oscillators using Adomian Decomposition Method

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Abstract — *In this article, the Adomian Decomposition Method (ADM) is used to study the harmonic oscillators [6]. The obtained discrete solutions using ADM are compared with the Runge-Kutta (RK) method, Single-term Haar wavelet series (STHW) and ODE45 solutions of the harmonic oscillators and are found to be very accurate. Solution and Error graphs for discrete and exact solutions are presented in a graphical form to show efficiency of this method. This ADM can be easily implemented in a digital computer and the solution can be obtained for any length of time.*

Keywords — *Harmonic oscillators, ODE45 in Matlab, Runge-Kutta method, Single-term Haar wavelet series.*

I. Introduction

Most of the realistic singular non Most of the realistic singular non-linear systems do not admit any analytical solution and hence a numerical procedure has to be used. In the last few years substantial progress has been made in finding the numerical solution of special classes of nonlinear singular systems of differential equations. A general numerical procedure for their solution has not previously existed. Hence it is important to understand the structure of such systems and develop efficient methods for solving them. The conventional methods such as Euler, Runge-Kutta and Adams-Moulton are restricted to very small step size in order that the solution is stable. [3]

In this paper we developed numerical methods for addressing harmonic oscillators by an application of the Adomian Decomposition Method which was studied by Sekar and team of his researchers [4-5,7-9]. Recently, Sekar *et al*. [6] discussed the harmonic oscillators using STHW. In this paper, the same harmonic oscillators problem was considered (discussed by Sekar *et al.* [6]) but present a different approach using the Adomian Decomposition Method with more accuracy for harmonic oscillators problems.

II. Adomian Decomposition Method

Suppose k is a positive integer and $f_1, f_2, \ldots, f_k$ are k real continuous functions defined on some domain G. To obtain k differentiable functions $y_1, y_2, \ldots, y_k$ defined on the interval I such that $(t, y_1(t), y_2(t), \ldots, y_k(t)) \in G$ for $t \in I$.

Let us consider the problems in the following system of ordinary differential equations:

$$\frac{dy_i(t)}{dt} = f_i(t, y_1(t), y_2(t), \ldots, y_k(t)) \ , \quad y_i(t)|_{t=0} = \beta_i \quad (1)$$

where β_i is a specified constant vector, $y_i(t)$ is the solution vector for $i = 1, 2, \ldots, k$. In the decomposition method, (1) is approximated by the operators in the form: $Ly_i(t) = f_i(t, y_1(t), y_2(t), \ldots, y_k(t))$ where L is the first order operator defined by $L = d/dt$ and $i = 1, 2, \ldots, k$.

Assuming the inverse operator of L is L^{-1} which is invertible and denoted by $L^{-1}(.) = \int_{t_0}^{t} (.)\, dt$, then applying L^{-1} to $L\,y_i(t)$ yields

$$L^{-1}Ly_i(t) = L^{-1} f_i(t, y_1(t), y_2(t), \ldots, y_k(t))$$

where $i = 1, 2, \ldots, k$. Thus

$$y_i(t) = y_i(t_0) + L^{-1} f_i(t, y_1(t), y_2(t), \ldots, y_k(t)).$$

Hence the decomposition method consists of representing $y_i(t)$ in the decomposition series form given by

$$y_i(t) = \sum_{n=0}^{\infty} f_{i,n}(t, y_1(t), y_2(t), \ldots, y_k(t)) \quad (2)$$

where the components $y_{i,n}$, $n \geq 1$ and $i = 1, 2, \ldots, k$ can be computed readily in a recursive manner. Then the series solution is obtained as

$$y_i(t) = y_{i,0}(t) + \sum_{n=1}^{\infty} \{L^{-1} f_{i,n}(t, y_1(t), y_2(t), \ldots, y_k(t))\} \ . \quad (3)$$

For a detailed explanation of decomposition method and a general formula of Adomian polynomials, we refer reader to [Adomian 1].

III. HARMONIC OSCILLATORS

Unforced harmonic oscillators can be modelled by the following second order homogeneous differential equation Blanchard *et al.* [2],

$$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=0$$
(4)

where $m,k>0$ and $b\geq 0$. If b = 0, then the system is undamped. However, if $b>0$, then different types of behavior are possible. For the above harmonic oscillatory equation, the characteristic equation is,

$$\frac{-b\pm\sqrt{b^2-4mk}}{2m}$$

The three different possibilities for the roots of the characteristic equation are,

- If $b^2-4mk<0$, then we have complex roots and the harmonic oscillator is said to be under damped. In this case, we expect the system to oscillate about its equilibrium position.
- If $b^2-4mk=0$, then we have repeated roots and the oscillator is critically damped.
- IF $b^2-4mk>0$, then the roots are real and distinct, and the oscillator is said to be over-damped, and the system will move to its equilibrium position without any oscillations.

We now consider the second order homogenous differential equation given as an initial value problem, with m = 1, k = 1 and b = 0.01t,

$$\frac{d^2y}{dt^2}+0.01t\frac{dy}{dt}+y=0,\ \ y(0)=-1, \dot{y}(0)=2$$
(5)

We note that for Equation (5), at time t = 0 the system is undamped, and the critical damping value is given by t^* = 200. The interval of time $(0,200)$ and $(200,\infty)$ corresponds to the system being under damped and over damped respectively.

To solve Equation (5) with numerical integrators, we can rewrite this second order homogeneous differential equation as a system of first order differential equations, by using the substitution $dy/dt = v$, and a vector $Y(t) = [y(t),v(t)]$,

$$\dot{Y}(t)=A(t)Y(t) \qquad (6)$$

where the matrix $A(t)$ and the initial condition $Y(0)$ are given by,

$$A(t)=\begin{bmatrix}0 & 1\\ -1 & -0.01t\end{bmatrix}, Y(0)=\begin{bmatrix}-1\\ 2\end{bmatrix}$$

IV. NUMERICAL SOLUTION

Firstly, we solved the matrix differential equation with the classical methods like the Runge-Kutta method (RK) and STHW. Then, we solved the same differential equation with ADM method. All 3 integrators are highly stable, so when step size h is share equally, their global error rations should change roughly by a factor of 16. This can be observed from the numerical results, for the integration period from $t_0=0$ to $t_f=0.1$. Also, it turns out that if we plot such results on a log-log scale of global error against step size h, then solutions from the ADM method have a better accuracy in comparison with RK or STHW methods. This can be observed from Figure 1.

Next, we turn our attention to the numerical solutions from integrating Equation (6) along a certain interval with fixed step size h = 1/20. We integrated the differential equation firstly over the integration period t = [0, 40], with the 3 integrators RK, STHW, and the ADM, all methods are stable. Then we also integrate over the same period with the highly accurate ODE45 in Matlab, with a specified tolerance of 10^{-14}. Comparisons can be then be done between numerical solutions from the 3 integrators under consideration, and that of Matlab's ODE45 integrator. Note that the interval t = [0.40] is below the critical damping value of t^* = 200 in Equations (6). This implies that the dynamical system is under damped, and we should expect oscillations to occur. Similarly, we can integrate over the interval t = [220,230], which is now above t^* = 200. So we would not expect any oscillations in the numerical solutions since the system is over damped in this case.

Figure 2 and Figure 3 contain the results from the experiments mentioned above. As seen from the top 3 graphs in Figure 2, solutions $y(t)$ are oscillating against time t throughout the integration period in the under damped system, while the results from the over damped system in the top 3 graphs of Figure 3 move toward the equilibrium at $y(t)$ = 0, with no oscillations against t in the integration interval of t = [220,230].

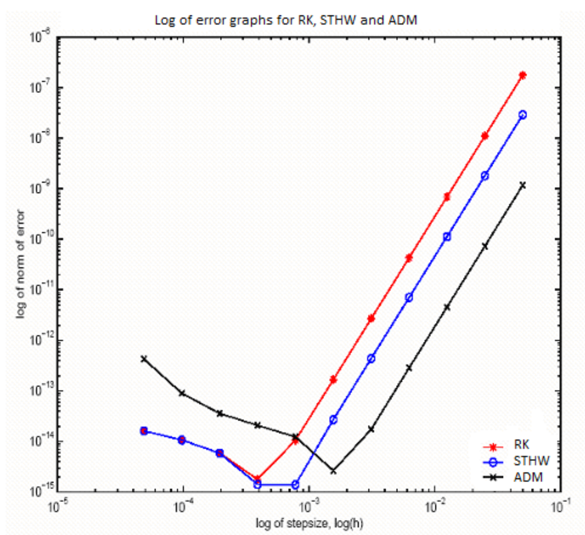

Fig. 1 Log-log plot of global error against step size for results from RK, STWS and STHW. They solved the initial value problem in Equation (6), from t_0 = 0, t_f = 0.1.

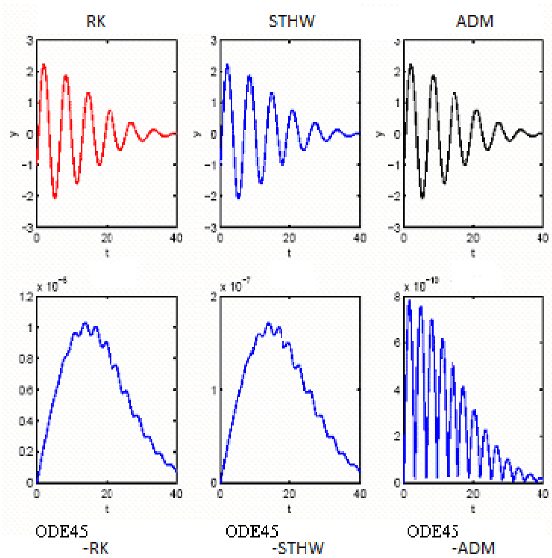

Figure 2 Solving the under damped system in Equation (6) for the interval $t \in [0,40]$, for which $t < t^* = 200$.

The top 3 graphs are plots of numerical solutions *y(t)* against time *t*, with h = 1/20. The bottom 3 graphs are the differences between numerical solutions from each of the 3 integrators and the highly accurate ODE45.

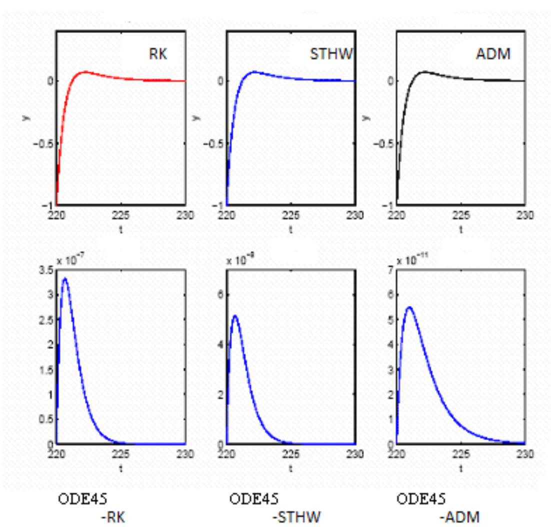

Figure 3 Solving the over damped system in Equation (6) for the interval $t \in [220,230]$, for which $t < t^* = 200$.

The top 3 graphs are plots of numerical solutions *y(t)* against time *t*, with h = 1/20. The bottom 3 graphs are the differences between numerical solutions from each of the 3 integrators and the highly accurate ODE45.

V. Conclusions

The accuracy achieved from the ADM method is higher than that of the RK and the STHW methods. This can be observed if we compare results from the 3 integrators and that ODE45. The differences in this comparison are also plotted in the bottom 3 graphs of Figure 2 and Figure 3. For both the under damped and over damped systems, the difference between ODE45 and ADM are several degrees smaller in magnitude than the differences between smaller in magnitude than the differences between numerical solutions from RK and STHW methods against ODE45's. With a relatively low computational cost, and a relatively good accuracy for fixed step size *h*, these brief experiments suggest the suitability of using the ADM to integrate systems of first order differential equations describing the dynamics of an unforced harmonic oscillator. Hence the ADM method is more suitable for studying the harmonic oscillators.

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