Sum of Finite and Infinite Series Derived by Generalized Q-Alpha Derivative Operator

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ABSTRACT: *In this paper, by defining the generalized q-derivative operator of first kind and its inverse, we obtain some identities and formulas on finite and infinite series in the field of finite difference methods. Suitable numerical examples verified by MATLAB are provided to illustrate the main results.*

KEY WORDS: *Generalized q-alpha derivative operator, geometric progression and polynomial factorial.*

AMS Subject Classification: 39A10, 39A11, 39A13.

I. INTRODUCTION

The modern theory of differential or integral calculus began in the 17^{th} century with the works of Newton and Leibnitz [1]. In 1989, K.S.Miller and Ross [2] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional derivative operator. Several groups have intensified their research on the amazing mathematics world featuring q-calculus. The theory of q-derivative equations of q-calculus or quantum calculus is based on the definition of the q-derivative operator, which was introduced by Jackson [3],[4]. This q-derivative operator, sometimes called Jackson q-derivative operator or Euler Jackson q-derivative operator, is defined by

$$D_q y_k = \frac{y_{qk} - y_k}{(q-1)k}, \qquad q \neq 1,$$

Where y_k is a sequence of real numbers. In [5], the authors introduced the derivative operator on two variables which turned out to be suitable for dealing with the Cauchy polynomials and also derived a binomial identity which unifies the two identities of Rota and Goldman, as well as, the q-vandermonde identity. Using this operator, the q-Leibnitz formula and the generating function of the homogeneous Rogers-Szego polynomials are derived in [6].

In 2014, G.Britto Antony Xavier, et al. [7],[8] proved several interesting results of geometric progression using q-differencee operator Δ_q. Hence in this paper, we define the Generalized q-derivative operator $\underset{q(\ell)\alpha}{\mathrm{D}}$ and we develop the basic theory for $\underset{q(\ell)\alpha}{\mathrm{D}}$, relations connecting $\underset{q(\ell)\alpha}{\mathrm{D}}$, the shift operator E_q, q-difference operator Δ_q and the q-derivative operator $\underset{q(\ell)}{\mathrm{D}}$. Also we obtain formula for finding the sum of higher powers of geometric progression by using Generalized inverse q-alpha derivative operator.

II. PRELIMINARIES

Before stating and proving our results, we present basic definitions and preliminary results which will be used for the subsequent discussions. Let u(k) be a real valued function on $(-\infty,\infty), \ell \neq q, \alpha$ is a non-zero real and m is a positive integer.

Definition 2.1 *[7] Let $u(k)$ be a real valued function on $(-\infty,\infty)$ and $1 \neq q$ be a fixed real number. Then the q-difference operator, denoted by Δ_q, on $u(k)$ is defined as*

$$\Delta_q u(k) = u(qk) - u(k).$$

(1)

Definition 2.2 *[7] Let n be a positive integer, $k \neq 0$ be real and $q \in (0,\infty)$. Then the generalized positive reciprocal polynomial factorial is defined by*

$$\left(\frac{1}{k}\right)_q^{(n)} = \frac{1}{k}\left(\frac{1}{k} - q\right)\left(\frac{1}{k} - 2q\right)\ldots\left(\frac{1}{k} - (n-1)q\right)$$

(2)

and

$$k_q^{(-n)} = \frac{1}{k(k+q)(k+2q)\cdots(k+(n-1)q)},$$

(3)

where s_r^n are stirling numbers of first kind.

Definition 2.3 *[9] Let $u(k)$ be a real valued function on $(-\infty,\infty)$, then the* generalized q-derivative operator on $u(k)$ is defined as

$$\underset{q(\ell)}{\mathrm{D}} u(k) = \frac{u(qk) - \ell u(k)}{(q-\ell)k}, \qquad q \neq \ell.$$

(4)

Lemma 2.4 *[10] Let* $n \in N(1)$, $q \neq 0$, $s_0^0 = 1$ *and* $s_r^0 = s_0^r = 0$ *if* $r \neq 0$. *If* $k_q^{(n)} = \prod_{i=0}^{n-1}(k - iq)$, *then we have*

$$k_q^{(n)} = \sum_{r=1}^{n} s_r^n q^{n-r} k^r \tag{5}$$

and

$$\left(\frac{1}{k}\right)_q^{(n)} = \sum_{r=1}^{n} s_r^n q^{n-r} \left(\frac{1}{k}\right)^r. \tag{6}$$

Definition 2.5 *Let* $u(k)$ *be a real valued function on* $(-\infty, \infty)$ *and* $\ell \neq q$ *be fixed real. Then the* $q-\alpha$ *derivative operator, denoted by* $\underset{q(\ell)\alpha}{\mathrm{D}}$, *on* $u(k)$ *is defined as*

$$\underset{q(\ell)\alpha}{\mathrm{D}} u(k) = \frac{u(qk) - \alpha\ell u(k)}{(q-\ell)k}, \tag{7}$$

$$\underset{q(\ell)\alpha}{\mathrm{D}^m} u(k) = \underset{q(\ell)\alpha}{\mathrm{D}}\left(\underset{q(\ell)\alpha}{\mathrm{D}^{m-1}} u(k)\right)$$

and q-shift operator is defined as

$$E_q u(k) = u(qk). \tag{8}$$

Remark 2.6 *From (1), (7), (8) and by taking* $\ell = 1$, $\alpha = 1$, *we can easily arrive*

$$\underset{q(\ell)\alpha}{\mathrm{D}} (c_1 u(k) + c_2 v(k)) = c_1 \underset{q(\ell)\alpha}{\mathrm{D}} u(k) + c_2 \underset{q(\ell)\alpha}{\mathrm{D}} v(k), \tag{9}$$

$$\underset{q(\ell)\alpha}{\mathrm{D}} u(k) = \frac{(E_q - \alpha\ell)u(k)}{(q-\ell)k}, \qquad q \neq \ell \tag{10}$$

and

$$\underset{q(1)}{\mathrm{D}} u(k) = \frac{\Delta_q u(k)}{(q-1)k}, \qquad q \neq 1. \tag{11}$$

Theorem 2.7 *If* m *is any positive integer,* $q \neq \ell$ *and* $k \neq 0$, *then*

$$\underset{q(\ell)\alpha}{\mathrm{D}^m} u(k) = \frac{1}{(q-\ell)^m k^m} \sum_{r=0}^{m} (-1)^r mc_r (\alpha\ell)^r u(q^{m-r}k). \tag{12}$$

Proof: The proof follows from (8) and by operating $q-\alpha$ derivative operator (m-1) times on equation (10).

III. MAIN RESULTS

We are in a position to state and prove our main results. The purpose of this section is for obtaining sum of finite and infinite series of geometric progression using inverse $q-\alpha$ derivative operator.

Definition 3.1 *A function* $v(k)$ *satisfying* m^{th} *order q-alpha difference equation* $\underset{q(\ell)\alpha}{\mathrm{D}^m} v(k) = u(k)$ *is called solution of that equation and it is denoted as* $v(k) = D_{q(\ell)\alpha}^{-m} u(k)$ *and in particular,*

$$\text{if} \quad \underset{q(\ell)\alpha}{\mathrm{D}} v(k) = u(k), \quad \text{then} \quad v(k) = \underset{q(\ell)\alpha}{\mathrm{D}^{-1}} u(k). \tag{13}$$

Theorem 3.2 *If* n *is any positive integer, then we have*

$$E_q^n u(k) = \sum_{r=0}^{n} nc_r (\alpha\ell)^{n-r} (q-\ell)^r k^r \underset{q(\ell)\alpha}{\mathrm{D}^r} u(k).$$

Proof: From equation (10), we have $E_q u(k) = (\alpha\ell + (q-\ell)k \underset{q(\ell)\alpha}{\mathrm{D}})u(k)$.

Raising the power $'n'$ on both sides and using Binomial theorem, we get the proof of the theorem.

Lemma 3.3 *If* m *and* n *are positive integers, then we can write*

$$\underset{q(\ell)\alpha}{\mathrm{D}^m} k^n = \frac{\prod_{i=0}^{m-1}(q^{n-i} - \alpha\ell) k^{n-m}}{(q-\ell)^m}. \tag{14}$$

Proof: From equation (7), we have

$$\underset{q(\ell)\alpha}{\mathrm{D}} k^n = \frac{(q^n - \alpha\ell)k^{n-1}}{q-\ell}.$$

Operating $\underset{q(\ell)\alpha}{\mathrm{D}}$ on both sides, we obtain

$$\underset{q(\ell)\alpha}{\mathrm{D}^2} k^n = \frac{(q^n - \alpha\ell)}{q-\ell} \frac{(q^{n-1} - \alpha\ell)k^{n-2}}{q-\ell}.$$

Repeat the process $(m-2)$ times to complete the proof.

Corollary 3.4 *For any positive integer* n *and* $q \neq \ell$ *we have*

$$\underset{q(\ell)\alpha}{\mathrm{D}^n} k^n = \frac{\prod_{i=0}^{n-1}(q^{n-i} - \alpha\ell)}{(q-\ell)^n}.$$

Proof: The proof completes by putting $m = n$ in

(14).

Corollary 3.5 *If* $q^{n+1} \neq \alpha\ell$ *and* n *is any positive integer, then*

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1} k^n = \frac{(q-\ell)k^{n+1}}{q^{n+1}-\alpha\ell}.$$

(15)

Proof: Taking $u(k) = k^{n+1}$ in (7), we get

$$\underset{q(\ell)\alpha}{\mathrm{D}} k^{n+1} = \frac{(q^{n+1}-\alpha\ell)k^{n+1}}{(q-\ell)k},$$

which completes the proof of the corollary.

Theorem 3.6 *Let* $u(k)$ *be a real valued function on* $(-\infty,\infty)$ *and* $\ell \neq q$. *Then*

$$\sum_{r=0}^{m}(q-\ell)(\alpha\ell)^r \left(\frac{k}{q^r}\right) u\left(\frac{k}{q^r}\right) = \underset{q(\ell)\alpha}{\mathrm{D}}^{-1} u(qk)$$

$$-(\alpha\ell)^{m+1} \underset{q(\ell)\alpha}{\mathrm{D}}^{-1} u\left(\frac{k}{q^m}\right)$$

(16)

is a solution of the $q-\alpha$ difference equation $\underset{q(\ell)\alpha}{\mathrm{D}} v(k) = u(k)$ and hence

$$\sum_{r=m+1}^{n}(q-\ell)(\alpha\ell)^r \left(\frac{k}{q^r}\right) u\left(\frac{k}{q^r}\right) = (\alpha\ell)^{t+1} \underset{q(\ell)\alpha}{\mathrm{D}}^{-1} u\left(\frac{k}{q^t}\right)\Bigg|_{t=n}^{m}$$

$$\text{for} \quad m < n.$$

(17)

Proof: Taking $\underset{q(\ell)\alpha}{\mathrm{D}}^{-1} u(k) = v(k)$ and by using (7), we get

$$v(qk) = (q-\ell)ku(k) + \alpha\ell v(k).$$

(18)

Replacing k by $\frac{k}{q}$, we get

$$v(k) = (q-\ell)\frac{k}{q} u\left(\frac{k}{q}\right) + \alpha\ell v\left(\frac{k}{q}\right).$$

(19)

Substituting (19) in (18) gives

$$v(qk) = (q-\ell)ku(k) + \alpha\ell(q-\ell)\frac{k}{q} u\left(\frac{k}{q}\right) + (\alpha\ell)^2 v\left(\frac{k}{q}\right).$$

(20)

Replacing k by $\frac{k}{q}$ in (19) gives

$$v\left(\frac{k}{q}\right) = (q-\ell)\frac{k}{q^2} u\left(\frac{k}{q^2}\right) + \alpha\ell v\left(\frac{k}{q^2}\right).$$

(21)

Substituting (21) in (20) gives

$$v(qk) = (q-\ell)ku(k) + \alpha\ell(q-\ell)\frac{k}{q} u\left(\frac{k}{q}\right)$$

$$+(\alpha\ell)^2(q-\ell)\frac{k}{q^2} u\left(\frac{k}{q^2}\right) + (\alpha\ell)^3 v\left(\frac{k}{q^2}\right).$$

(22)

Again replacing k by $k/q, k/q^2, k/q^3, \ldots, k/q^{m-1}$ in (21) repeatedly and substituting the resultant expressions in (18), we arrive

$$\sum_{r=0}^{m}(q-\ell)(\alpha\ell)^r \frac{k}{q^r} u\left(\frac{k}{q^r}\right) = v(qk) - (\alpha\ell)^{m+1} v\left(\frac{k}{q^m}\right),$$

which yields (16).

Replacing m by n in (16), where $m < n$, we get

$$\sum_{r=0}^{n}(q-\ell)(\alpha\ell)^r \frac{k}{q^r} u\left(\frac{k}{q^r}\right) = \underset{q(\ell)\alpha}{\mathrm{D}}^{-1} u(qk)$$

$$-(\alpha\ell)^{n+1} \underset{q(\ell)\alpha}{\mathrm{D}}^{-1} u\left(\frac{k}{q^n}\right).$$

(23)

Hence (17) follows by subtracting (16) from (23).

The following example illustrates Theorem 3.6:

Example 3.7 *Consider* $u(k) = k^3$ *in (15). Then (17) becomes*

$$\sum_{r=m+1}^{n}(q-\ell)(\alpha\ell)^r \frac{k}{q^r}\left(\frac{k}{q^r}\right)^3 = (\alpha\ell)^{t+1}\frac{(q-\ell)}{q^4-\alpha\ell}\left(\frac{k}{q^t}\right)^4\Bigg|_{t=n}^{m}.$$

(24)

Taking $m = 3, n = 4, \alpha = 2, q = 5, \ell = 6$ and $k = 15$ in (24), we get

L.H.S $= \dfrac{-12^4 \times 3^4}{5^{12}} = -6.879707136 \times 10^{-03}$

and

R.H.S =

$$(12)^{t+1}\frac{(-1)}{5^4-12}\left(\frac{15}{5^t}\right)^4\Bigg|_{t=4}^{3}=-6.879707136\times10^{-03}.$$

The following corollary gives formula for infinite series.

Corollary 3.8 *If* $\ell<q$, $u(k)$ *is bounded and* $\lim_{n\to\infty}(\alpha\ell)^{n+1}\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}u\left(\frac{k}{q^n}\right)=0$, *then*

$$\sum_{r=m+1}^{\infty}(q-\ell)(\alpha\ell)^r\frac{k}{q^r}u\left(\frac{k}{q^r}\right)=(\alpha\ell)^{m+1}\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}u\left(\frac{k}{q^m}\right).$$

(25)

In particular, we have

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}u(k)=\frac{(q-\ell)k}{\alpha\ell}\sum_{r=0}^{\infty}\left(\frac{\alpha\ell}{q}\right)^{r+1}u\left(\frac{k}{q^{r+1}}\right).$$

(26)

Proof. The proof of (25) follows by taking $n\to\infty$ in (17) and the proof of (26) follows by putting m=0 in (25).

The example given below illustrates Corollary 3.8.

Example 3.9 *Taking* $u(k)=k^2$ *in (15) and using (26), we get*

$$\frac{(q-\ell)k^3}{q^3-\alpha\ell}=\frac{(q-\ell)k}{\alpha\ell}\sum_{r=0}^{\infty}\left(\frac{\alpha\ell}{q}\right)^{r+1}\left(\frac{k}{q^{r+1}}\right)^2.$$

(27)

Putting $q=5,\ell=4,k=9,\alpha=3$ in (27), we obtain

$$L.H.S=\frac{(5-4)9^3}{5^3-12}=6.451327434$$

and

$$R.H.S=\frac{(5-4)9}{12}\sum_{r=0}^{\infty}\left(\frac{12}{5}\right)^{r+1}\left(\frac{9}{5^{r+1}}\right)^2=6.451327434.$$

Theorem 3.10 *If* p *is a positive integer,* $k>0$ *and* $1-\alpha\ell q^{p-1}\neq0$, *then*

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\left(\frac{1}{k}\right)^p=\frac{(q-\ell)q^{p-1}}{(1-\alpha\ell q^{p-1})k^{p-1}}$$

(28)

and hence

$$\sum_{r=m+1}^{n}(\alpha\ell)^r q^{r(p-1)}=\frac{(\alpha\ell)^{t+1}q^{(p-1)(t+1)}}{1-\alpha\ell q^{p-1}}\Bigg|_{t=n}^{m}.$$

(29)

Proof: Taking $u(k)=\left(\frac{1}{k}\right)^{p-1}$ in (7), we get

$$\underset{q(\ell)\alpha}{\mathrm{D}}\left(\frac{1}{k}\right)^{p-1}=\frac{1-\alpha\ell q^{p-1}}{(q-\ell)q^{p-1}k^p},$$

(30)

which yields (28).

Using (28) in (17) gives

$$\sum_{r=m+1}^{n}(q-\ell)(\alpha\ell)^r\left(\frac{q^r}{k}\right)^{p-1}=(\alpha\ell)^{t+1}\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\left(\frac{q^t}{k}\right)^p\Bigg|_{t=n}^{m}.$$

(31)

The relation (29) follows from (15) and (31).

The following example is a verification of (29).

Example3.11 *Putting* $\alpha=q=2,\ell=p=3,m=4$ *and* $n=5$ *in (29), we obtain*

$$\sum_{r=m+1}^{n}(\alpha\ell)^r q^{r(p-1)}=(\alpha\ell)^5q^{10}=6^5\times2^{10}=7962624$$

and

$$\frac{(\alpha\ell)^{t+1}q^{(p-1)(t+1)}}{1-\alpha\ell q^{p-1}}\Bigg|_{t=n}^{m}=$$

$$\frac{6^{t+1}2^{2(t+1)}}{1-6\times4}\Bigg|_{t=5}^{4}=7962624.$$

Corollary 3.12 *Let* $\left(\frac{1}{k}\right)_\ell^{(p)}$ *be the generalized polynomial factorial for* $\frac{1}{k}$. *Then*

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\left(\frac{1}{k}\right)_\ell^{(p)}=\sum_{r=1}^{p}\frac{s_r^p\ell^{p-r}q^{r-1}(q-\ell)}{(1-\alpha\ell q^{r-1})k^{r-1}}$$

(32)

and

$$(\alpha\ell)^{t+1}\sum_{r=1}^{p}\frac{s_r^p\ell^{p-r}q^{(r-1)(t+1)}}{1-\alpha\ell q^{r-1}\ k^r}\Bigg|_{t=n}^{m}=\sum_{r=m+1}^{n}\frac{(\alpha\ell)^r}{q^r}\left(\frac{q^r}{k}\right)_\ell^{(p)}.$$

(33)

Proof: Using (2), we can write

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\left(\frac{1}{k}\right)_\ell^{(p)}=\sum_{r=1}^{p}s_r^p\ell^{p-r}\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\left(\frac{1}{k}\right)^r,$$

(34)

which yields (32) and the proof of (33) completes by applying (32) in (17).

Theorem 3.13 *Let* $q,k>0, 1-\alpha\ell\neq0$ *and* m,n *are positive integers. Then*

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\frac{\log k}{k}=\frac{q-\ell}{1-\alpha\ell}\left\{\log k-\frac{\log q}{1-\alpha\ell}\right\}$$
(35)

and hence

$$\sum_{r=m+1}^{n}(q-\ell)(\alpha\ell)^{r}\log\left(\frac{k}{q^{r}}\right)=(\alpha\ell)^{t+1}\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\frac{\log(\frac{k}{q^{t}})}{(\frac{k}{q^{t}})}\Bigg|_{t=n}^{m}.$$
(36)

Proof: By taking $u(k)=\log k$ in (7), we get

$$\underset{q(\ell)\alpha}{\mathrm{D}}\log k=\frac{\log(qk)-\alpha\ell\log k}{(q-\ell)k}=\frac{\log q+(1-\alpha\ell)\log k}{(q-\ell)k}.$$
(37)

Since $\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}$ is linear, we have

$$\log k=\left(\frac{1-\alpha\ell}{q-\ell}\right)\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\left(\frac{\log k}{k}\right)+\frac{\log q}{(q-\ell)}\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\left(\frac{1}{k}\right).$$
(38)

The proof of (35) completes by applying (28) in (38) and (36) follows from (17) and (35).
An example given below illustrates the Theorem 3.13.

Example 3.14 *Putting* $\alpha=3, m=4, n=5, \ell=10,$ $q=6$ *and* $k=18$ *in equation (36), we get*

L.H.S $=$ $-4\times 30^{5}\log(\frac{3}{6^{4}})=256169020.2$

and

$$\text{R.H.S}=30^{t+1}\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}\frac{\log(\frac{k}{6^{t}})}{(\frac{k}{6^{t}})}\Bigg|_{t=5}^{4}$$

$$=30^{t+1}\frac{q-\ell}{1-\alpha\ell}\left\{\log(\frac{k}{6^{t}})-\frac{\log q}{1-\alpha\ell}\right\}\Bigg|_{t=5}^{4}=256169020.2.$$

Theorem 3.15 *If* $u(k)$ *and* $v(k)$ *are real valued functions on* $(-\infty,\infty)$, *then*

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}(u(k)v(k))=\frac{q-\ell}{\alpha\ell(q-1)}\left\{u(k)\underset{q(1)}{\mathrm{D}}^{-1}v(k)-\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}(\underset{q(1)}{\mathrm{D}}^{-1}v(qk)\underset{q(\ell)\alpha}{\mathrm{D}}u(k))\right\}.$$
(39)

Proof: By (7), we find that

$$\underset{q(\ell)\alpha}{\mathrm{D}}(u(k)w(k))=(\underset{q(\ell)\alpha}{\mathrm{D}}u(k))w(qk)+\frac{\alpha\ell(q-1)}{q-\ell}u(k)\underset{q(1)}{\mathrm{D}}w(k).$$
(40)

Hence the proof follows by applying equation (13) in (40) and by taking $v(k)=\underset{q(1)}{\mathrm{D}}w(k)$ and $w(k)=\underset{q(1)}{\mathrm{D}}^{-1}v(k).$

The following corollary gives formula for finding the sum of finite series involving polynomial and logarithmic function.

Corollary 3.16 *Let* $k>0$ *and* p *be any integer. Then we have*

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}(\log k)k^{p}=\frac{q-\ell}{q^{p+1}-\alpha\ell}\left\{(\log k)k^{p+1}-\frac{(qk)^{p+1}\log q}{q^{p+1}-\alpha\ell}\right\},$$
(41)

which yields

$$\sum_{r=m+1}^{n}(\alpha\ell)^{r}\left(\frac{k}{q^{r}}\right)^{p+1}\log\left(\frac{k}{q^{r}}\right)=\frac{(\alpha\ell)^{t+1}}{q^{p+1}-\alpha\ell}\left(\frac{k}{q^{t}}\right)^{p+1}\left\{\log\left(\frac{k}{q^{t}}\right)-\frac{q^{p+1}\log q}{q^{p+1}-\alpha\ell}\right\}\Bigg|_{t=n}^{m}. \quad (42)$$

Proof: Taking $u(k)=\log k$ and $v(k)=k^{p}$ in (39), we get

$$\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}(\log k)k^{p}=\frac{q-\ell}{\alpha\ell(q-1)}\left\{\log k\underset{q(1)}{\mathrm{D}}^{-1}k^{p}-\underset{q(\ell)\alpha}{\mathrm{D}}^{-1}(\underset{q(\ell)\alpha}{\mathrm{D}}\log k\underset{q(1)}{\mathrm{D}}^{-1}(qk)^{p})\right\}.$$
(43)

Taking $\ell=\alpha=1$ and $n=p$ in (15), we have

$$\underset{q(1)}{\mathrm{D}}^{-1}k^{p}=\frac{(q-1)k^{p+1}}{q^{p+1}-1}$$
(44)

and

$$\underset{q(1)}{\mathrm{D}}^{-1}(qk)^{p}=\frac{(q-1)(qk)^{p+1}}{q^{p+1}-1}.$$
(45)

The proof of (41) completes by using (37), (44) and (45) in (43). And (42) follows by applying (41) in (17).
Similarly the Theorem 3.15 can be verified for the

rational function $\frac{1}{k^p}$ and the polynomial factorials $k_q^{(p)}$ and $\left(\frac{1}{k}\right)_q^{(p)}$ We present an example below to illustrate Corollary 3.16.

Example 3.17 *Taking* $p=\alpha=2, q=m=3, n=4, \ell=5$ *and* $k=17$ *in (42), we obtain*

$$L.H.S=10^4\left(\frac{17}{3^4}\right)^3\log\left(\frac{17}{3^4}\right)=-62.68224219$$

and

$$R.H.S=\frac{(10)^{t+1}}{3^3-10}\left(\frac{17}{3^t}\right)^3\left\{\log\left(\frac{17}{3^t}\right)-\frac{3^3\log 3}{3^3-10}\right\}\Bigg|_{t=4}^{3}$$

$$=-62.68224219.$$

IV. CONCLUSION

In this paper, we have derived some identities and formulas for sum of finite and infinite series of terms of product of geometric function and logarithmic functions. By using the results obtained in this paper, one can find the sum of certain generation population of animal propagation problems in Mathematical Biology.

ACKNOWEDGEMENT

One of the authors (Dr.G.Britto Antony Xavier) acknowledges University Grants Commission, SERO, Hyderabad, India for financial assistance (MRP-4021/11, SERO,UGC).

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