Solution of Impulsive differential equations by using Improved Euler & Runge Kutta Method

Sanjay Kumar Srivastava[1], Jyoti Sharma[2]

[1]Department: Applied Sciences, Beant college of Engineering and Technology,

Gurdaspur-143521,(Punjab), India

[2]Department: Applied Sciences, BKSJ Engineering College, Amritsar-

143107,(Punjab), India

Abstract *-The theory of impulsive differential equation is a natural frame work for mathematical modeling of several real phenomena. Many impulsive differential equations cannot be solved analytically or their solving is complicated. In this paper, the algorithm for solving impulsive differential equations is presented. A new type of impulsive differential equations can be solved with the implementation of Improved Euler and Runge kutta method. At last better result can be obtained by solving the numerical example.*

Keywords: *Differential equations, Impulsive Differential equations, Fixed impulse, impulse jump, Improved Euler Method, Runge Kutta Method*

I. Introduction

Presently, impulsive differential equations solution was searched in form of analytical expression. Significantly results are presented by V.Lakshmikantham, D.Bainov, P.Simeonov, S.Kostadinov and N.Van.Minch. However, many impulsive differential equations remained unsolved as their solution are complicated. Numerous problems can't be solved in analytical form but there is need of their numerical values of solution. Several real world problems can be solved with the help of impulsive differential equations. For example, mechanical system with impact, biological phenomena involving thresholds , bursting rhythm models in medicine and biology, industrial robotics, optimal control models in economics, pharmaco kinetics and many more, do exhibit impulsive effect. In this paper, the numerical solutions of impulsive differential equations are sought by improved Euler method and Runge Kutta method. The algorithm is interpreted according to the theory of impulsive differential equations written by V.Lakshmikantham et.al [8].Based on the theory , the better numerical solution of the problem is illustrated in example.

II. Impulsive Differential Equations

A Continuous time differential equation ,which governs the state of the system between impulses an impulse equation which models an impulsive jump defined by a jump function at the instant an impulse occurs, and a jump criterion, which define a set of jump events. Mathematically, the equation takes the from

$$x'(t) = f(t,x)\,,\quad t \neq t_K\,,\quad t \in Z$$

$$\Delta x(t_K) = I_K(x(t_K))\quad ,K = 1,2,3 \ldots\ldots\ldots\ldots\ldots\ldots, m \quad (2.1)$$

Where Z is any real interval , $f: Z \times R^n \to R^n$ is a given function.

$I_k : R^n \to R^n, k = 1, 2,\ldots, m$ and $\Delta x(t_k) = x(t_k^{\,+}) - x(t_k^{\,-}), k = 1, 2,\ldots, m.$

The numbers t_K are called instants (or moments) of impulse, I_K represent the jump of state at each t_K ,where $x(t_K^+)$ and $x(t_K^-)$ *represent* the right limit and left limit, respectively ,of the state at $t = t_K$. The moments of impulse may be fixed or depended on the state of the system. In this paper we will concerned with fixed moments only. Moreover, impulsive differential equations can be classified according to three components.

1. Systems with impulse at fixed moments. The equations have the following form

$$x'(t) = f(t,x)\,, t \neq t_K$$

$$\Delta x = I_K(x), t = t_K \quad (2.2)$$

Where $t_0 < t_1 < --< t_K < t_{K+1} < ---, K \in Z \; and \; for \; t = t_K$

$$\Delta x(t_k) = x(t_k^{\,+}) - x(t_k^{\,-}) \text{ where } x(t_k^{\,+}) = \lim_{h \to 0^+} x(t_k + h).$$

We surely see that any solution, $x(t)$ of (2.2) satisfy

$$(i)\; x'(t) = f\big(t, x(t)\big), t \in (t_K, t_{K+1}) and$$

$$(ii) \Delta x(t_K) = I_K(x(t_K)), t = t_K \;, K = 1,2,3 \ldots\ldots$$

2. Systems with impulse at variable time. The equations have the following form

$$x'(t) = f(t,x) \;, t \neq \tau_K(x)$$

$$\Delta x = I_K(x), t = \tau_K(x), K = 1,2,3 \ldots \ldots \quad (2.3)$$

Where $\tau_K : \Omega \to R$, Ω is the phase space and $\tau_K(x) < \tau_{K+1}(x), K \in Z, x \in \Omega$.

Systems with variable moments of impulsive effect involve more difficult problems than systems with fixed (instants) or moments of impulsive effect. This is the fact that the instant of impulsive effect of (2.3) depend upon the solution, i.e $t = \tau_K(x)$,for each K. Therefore, solutions at different starting points will have different points of discontinuity.

3. Autonomous systems with impulse. The equations take the form $x'(t) = f(x), x \notin M$

$$\Delta x = I(x), x \in M \quad (2.4)$$

Let the sets M(t)=M,N(t)=N and the operator A(t)=A be independent of t and let $A: M \to N$ be defined by $Ax = x + I(x)$, where $I: \Omega \to \Omega$. Whenever, any solution $x(t) = x(t, o, x_0)$ hits the set M at some time t, the operator A instantly transfers the point $x(t) \in M$ in to the point $y(t) = x(t) + I(x(t)) \in N$.

Let it be $X = R^n$ and $S = \{ t_K | K \epsilon Z \} \subset R$ Where is $t_K < t_{K+1}$ for all $K \in Z, t_K \to +\infty$ when $K \to +\infty$ and $t_K \to -\infty$ when $K \to -\infty$. Also, let $t_K^+ = t_K + 0$ and $t_K^- = t_K - 0$.If Ω c R is any real interval, we suppose that $x(t) = [\,x_1(t)\;\; x_2(t) \ldots \ldots \ldots \ldots . x_n(t)]^T$, is a vector of unknown functions and

$$f(t,x): \Omega \times X \to X.$$

$$f(t,x) = \begin{bmatrix} f_1(t, x_1(t), x_2(t), \ldots x_n(t)) \\ f_2(t, x_1(t), x_2(t), \ldots x_n(t)) \\ \vdots \\ f_n(t, x_1(t), x_2(t), \ldots x_n(t)) \end{bmatrix}$$

is Continuous operator on every set $[t_K \,,\; t_{K+1}] \times X$.

A. Definition 2.1

A System of differential equation of the form

$$\frac{dx}{dt} = f(t,x) \qquad (t = t_K) \qquad (2.5)$$

With conditions $\Delta x|_{t=t_K} = x(t_K^+) - x(t_K^-) = I_K(x(t_K))$

Where $I_K : X \to X$ continuous operator, K= $0, \pm 1, \pm 2, \pm 3, ------$, is called impulsive differential equation (in further IDE) at fixed impulse. A state of the process, $x_0 = x(t_0)$ is taken as the start condition to solving (2.5)

III Properties of solution of IDE

The problem of existence and uniqueness of the solutions of impulsive differential is similar to that of corresponding ordinary differential equations .The continuability of solutions is affected by the nature of the impulsive action.

A. Definition 3.1

A solution of the IDE (2.5) means a piecewise continuous $X: J \to R$ with piecewise continuous first derivative such that

$$\frac{dx(t)}{dt} = f(t, x(t)) \qquad , t = \tau_K$$

$$x(\tau_K^+) - x(\tau_K^-) = I_K(x(\tau_K));\; K = 0\,, \pm 1, \pm 2, \pm 3,\text{--}$$

B. Theorem 3.1

Let the function $f: X \times \Omega \to X^n$ be continuous on the sets $(\tau_K, \tau_{K+1}] \times \Omega$,$K \in Z$ for each $K \in Z$ and $x \in \Omega$. Suppose $\exists$ the finite limit of $f(t,x)$ as $(t,y) \to (\tau_K, x), t > \tau_K$. Then, for each $(t_0, x_0) \in X \times \Omega$, there exist $T > t_0$ and a solution $x: (\,t_0, T) \to X^n$ of the problem(2.5) with the initial condition $x(t_0^+) = x_0$. Furthermore, if the function f is locally lipschitz continuous with respect to x in $X \times \Omega$,then this solution is unique.

Let $x(t)$ be the solution of IDE(2.5) with initial condition $x(t_0^+) = x_0$.Then $x(t)$ can be represented as

$$x(t) = \left\{ x_0 + \int_{t_0}^{t} f(s, x(s))ds + \sum_{t_0 < t_k < t} I_K(x(t_K)) \;; t \in \Omega^+ \right\}$$

$$x(t) = \left\{ x_0 + \int_{t_0}^{t} f(s, x(s))ds - \sum_{t_0 < t_K < t} I_K(x(t_K)) \;\; ; t \in \Omega^- \right\}$$

Where Ω^+ and Ω^- are the maximal intervals on which the solution can be continued to the right or to the left of the point $t = t_0$ respectively.

C. Theorem 3.2

Assume that

(A_0) The sequence $\{t_K\}$ satisfies $0 \le t_0 < t_1 < t_2 < \cdots < t_K < \cdots$ with $t_0 \to \infty$ as $K \to \infty$

(A_1) $n \in PC'[R_+, R]$ and $n(t)$ is left Continuous at $t_K, K = 1,2 \ldots \ldots \ldots$

$(A_2) \forall\; K = 1,2, \ldots \ldots .$ and $t \ge t_0$

$$\begin{cases} D^{+}n(t) \leq g\big(t, n(t)\big), t \neq t_K \\ n(t_K^{+}) \leq \gamma_K(n(t_K)) \\ n(t_0) \leq u_0 \end{cases} \quad (3.1)$$

where $g: R_+^2 \rightarrow R$ *is continuous in* $(t_{K-1}, t_K] \times R_+$ *and for each* $u \in R_+$

$\lim_{(t,Z)\rightarrow(t_K^+,u)} g(t,Z) = g(t_K^+, u)$ *exist* $\gamma_K: R_+ \rightarrow R$ *is non − decreasing*

(A_3) $r(t) = r(t, t_0, u_0)$ is the maximal solution of

$$\begin{cases} u' = g(t,u) \qquad t \neq t_K \\ u(t_K^{+}) = \gamma_K(u(t_K)) \\ u(t_0) = u_o \geq 0 \end{cases} \quad (3.2)$$

Existing on $[t_0, \infty)$. *Then* $n(t) \leq r(t), t \geq t_0$ (3.3)

We recall that the maximal solution r(t) of (3.2) means the following

$$r(t) = r_0(t, t_0, u_0)$$

$$= \begin{cases} r_1\big(t, t_1, r_0(t_1^{+})\big) & t \in (t_1, t_2] \\ r_2(t, t_2, r_1(t_2^{+})) & t \in (t_2, t_1] \\ \vdots & \\ r_K\big(t, t_K, r_{K-1}(t_K^{+})\big) & t \in (t_K, t_{K+1}] \\ \vdots & \end{cases} \quad (3.4)$$

Where each $r_i(t, t_i, r_{i-1}(t_i^{+}))$ is the maximal solution of (2.3) on the interval $(t_i, t_{i+1}]$ for each

i=1,2,……….. and $r_{i-}(t_i^{+}) = \gamma_i(r_{i-1}(t_i, t_{i-1}, r_{i-2}(t_{i-1}^{+}))$

IV Algorithm

Suppose the IDE (2.5) with start condition $x = x(t_0)$ and the impulsive operators $I_K, K \in Z$ is given. The impulsive operators work at the moments of jump happen, t_K for all $K \in Z$. Which is depicted by the quadrate matrices of dimensions $n \times n$ .The numerical algorithm differs only at the jump point. Where we have to implement the operators concern with the particular point. Other than that, we apply regular kinds to solve the IDE using numerical method chosen.

1. At the instant $t = t_0$, we implement the numerical method to the function with the initial values $x = x_0$ .The algorithm applies until the first jump point , by now we will get
2. At the jump point, $t = t_K$, we implement the operators to find the values of the right limit.
3. The first step is repeated until the next jump point.
4. Then, we implement the operators concern with the particular jump point.
5. The above steps are repeated and the iteration stop until we reach to the desired values that has to be found, assume that, t_s where $t_s > t_0$. Note that we only have the approximate values of the function at $t = t_s$.

Using algorithm for solving impulsive differential equation, we practically, do not obtain analytical expressions for functions $x_1(t), x_2(t), - - - - - - - -, x_n(t)$ at all, for generating the sequence $x_0, x_1, x_2, - - - - - - - -$. We choose some numerical method for solving the system of differential equations(Improved Euler method , some multi-step method or some of Runge-Kutta method). Whose Characteristics are well known (see, for example 11).We can pick the number of nodes and the sizes of sub-segments on which we divide half segments $(t_K, t_{K+1}]$. It is the easiest, from the aspect of programming, to choose equal number of the equidistant nodes on each half segment. But a way picking nodes can affect the exactness of the result. A similar procedure can be applied in the case $t_s < t_0$, *i.e* $t \in \Omega^{-}$. Only condition is that impulsive operator I_K *for* $K \leq 0$ have their inverse operators I_K^{-1}.The value of the argument t decreases and at the instant $t = t_K$ .when impulsive operator apply an inverse mapping must be $\Delta x|_{t=t_K} = I_K^{-1}.x(t_K)$.

Example1:

We solved an impulsive differential equation

$$\frac{dx}{dt} = f(t,x) \qquad (t \neq t_K)$$
$$\Delta x|_{t=t_K} = I_K\big(x(t_K)\big) \quad (t = t_K) \quad K = 1,2, - - - -$$

$$x_0 = x(t_0) \quad where \ \ t_0 = 0.0 \qquad (5.1)$$

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad x_0 = \begin{bmatrix} -1.0 \\ 0.0 \end{bmatrix}$$

$$f(t,x) = \begin{bmatrix} 1666666x_1 + .1666666x_2 + .1666666 \\ -.1666666x_1 - .1666666x_2 + .5833333 \end{bmatrix}$$

The impulsive operators act at moments (instant) $t_1 = 1.0$ *and* $t_2 = 2.0$

$$I_1 = \begin{bmatrix} 0.25 & 0.25 \\ 0.0 & -1.0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 3.0 & 4.0 \\ 0.0 & -1.0 \end{bmatrix} \qquad (5.2)$$

Here, we found the approximate value of $t_K = 2.3$

We applied the algorithm using one of the simplest methods for numerical solving of differential equations, so came it as Improved Euler's method

Table1

t_k	$x_1(t_k)$ (5.3)	$x_1(t_K)$ (5.4)	$x_1(t_K)$ (5.5)	$x_2(t_K)$ (5.3)	$x_2(t_K)$ (5.4)	$x_2(t_K)$ (5.5)
0.0	-1	-1	-1	0.04124994	0	0
0.1	-1.0522763	-0.9987496	-0.999375	0.08493511	0.07436853	0.074375
0.2	-1.1040172	-0.9962496	-0.997500	0.13118751	0.14747645	0.147500
0.3	-1.1158160	-0.9924993	-0.994375	0.18014810	0.21932476	0.219375
0.4	-1.2148599	-0.9874994	-0.990000	0.26828122	0.28990279	0.290000
0.5	-1.2744374	-0.9812491	-0.984375	0.32526151	0.35923172	0.359375
0.6	-1.3367563	-0.9737493	-0.977500	0.38554990	0.42730070	0.427500
0.7	-1.4021499	-0.9650007	-0.969375	0.44933109	0.49411072	0.494375
0.8	-1.4708762	-0.9550010	-0.960000	0.51679984	0.55966110	0..560000
0.9	-1.5430127	-0.9438013	-0.949375	0.58816332	0.62395199	0.624375
1.0	-1.6188396	-0.9313024	-0.937500	0.66364006	0.68698435	0.687500
1.0	-1.05	-1	-1.0	0.041128994	0	0
1.1	-1.1029583	-0.9989916	-0.986875	0.084692370	0.07416192	0.0618750
1.2	-1.1579271	-0.9969158	-0.972500	0.130701246	0.14686711	0.1225000
1.3	-1.2156037	-0.9938069	-0.956875	0.179289313	0.21810568	0.1818750
1.4	-1.2760372	-0.9895551	-0.940000	0.230642174	0.28796843	0.2400000
1.5	-1.3397121	-0.9843897	-0.921875	0.323593759	0.35629630	0.2968750
1.6	-1.4065805	-0.9782031	-0.902500	0.383095966	0.42317416	0.3525000
1.7	-1.4766430	-0.9709983	-0.881875	0.445929887	0.48860614	0.4068750
1.8	-1.5503504	-0.9627800	-0.860000	0.512280732	0.55259464	0.4600000
1.9	-1.6282203	-0.9535513	-0.836875	0.582345585	0.61514370	0.5118750
2.0	-1.7102338	-0.9433171	-0.812500	0.656332750	0.67625562	0.5625000
2.0	-1.05	-1	-1.0	0	0	0
2.1	-1.1024550	-0.9990773	-0.974375	0.084705946	0.073955317	0.049375
2.2	-1.1576232	-0.9974232	-0.947500	0.130486458	0.146243643	0.097500
2.3	-1.2155264	-0.9949530	-0.919375	0.178717151	0.216871044	0.144375
Δx (t_k)	0.29615146	0.0755780		0.034342151	0.072496044	

$$x_{j+1} = x_j + \frac{h}{2}\left[f(x_j, t) + f(x^*_{j+1}, \quad t)\right]$$

Where $x^*_{j+1} = x_j + hf(x_j, t)$ (5.3)

Where $j \in Z$ is index of iteration and h is a distance between a neighboring nodes.

Then , we applied the algorithm on the same I D E .But instead of Improved Euler's method (5.3)

Of the first order, we used Runge Kutta M ethod of fourth order

$$x_{j+1} = x_j + \frac{h}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$k_1 = f(t_j, x_j)$$

$$K_2 = f\left(t_j + \frac{h}{2}, x_j + \frac{h}{2}K_1\right) \quad (5.4)$$

$$K_3 = f\left(t_j + \frac{h}{2}, x_j + \frac{h}{2}K_2\right)$$

$$K_4 = f(t_j + h, x_j + hK_3)$$

Where $j \in Z$ is index of iteration , and h is the step size of each iteration . Here the step size h=0.1 . on comparing obtained results by using the analytical expression that is the solution of I D E (5.1)

$$x = \begin{cases} x_1(t) = 0.0625t^2 - 1.0 \quad for\ t \in (-\infty, 1) \\ x_2(t) = -0.0625t^2 + 0.75t \quad for\ t \in (-\infty, 1) \\ x_1(t) = 0.0625t^2 - 1.0625 \quad for\ t \in [1, 2) \\ x_2(t) = -0.0625t^2 + 0.75t - 0.6875 \quad for\ t \in [1, 2) \\ x_1(t) = 0.0625t^2 - 1.25 \quad for\ t \in [2, \infty) \\ x_2(t) = -0.0625t^2 + 0.75t - 1.25 \quad for\ t \in [2, \infty) \end{cases}$$

The numerical values of the solution are obtained by using the Matlab programming and the results of Improved Euler method and Runge Kutta method as well as the analytical expression are compared in Table 1.

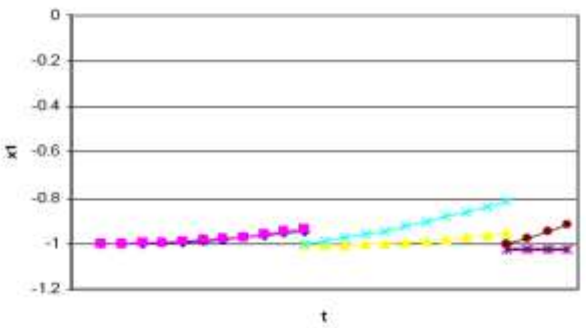

Fig 1 The approximate values of $x_1$versus time t between the Improved Euler also Runge Kutta methods and analytical method for Example 1 .

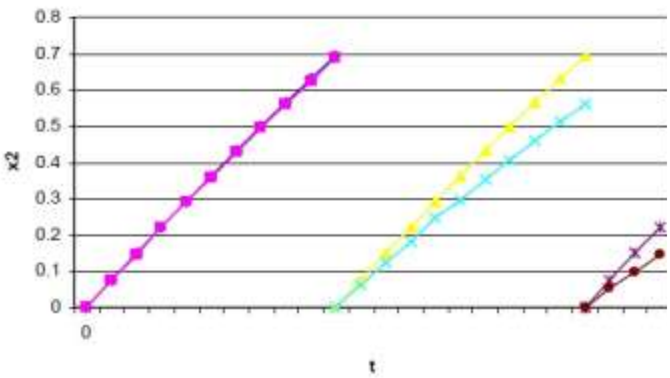

Fig 2 The approximate values of $x_2$versus time t between the Improved Euler also Runge Kutta methods and analytical method for Example 1.

V Conclude

The truthfulness of the results can be enhanced by under seeking the solution of the other numerical methods. We proposed a general numerical procedure for treating the impulsive differential equations at fixed moments. We translate numerical algorithm following the theory of impulsive differential equations and started with Improved Euler and Runge Kutta methods. Although it is the efficient method, We will study, it is by far the simplest, and analyzing Improved Euler and Runge Kutta methods. Only the few researchers worked on this method, so there is still need to improve it more. Therefore, many studies have to be done in order to improve and verify the existing results. In this paper is an effort to show better results with diagram to the convergence and the behavior of the solutions.

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