Construction New Types of Matrices

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Abstract— *In this paper, we introduced new types of matrices. We called them h-orthogonal matrix and h-unitary matrix depend on h-transpose. We discussed the properties of these matrices such as, their eigenvalues and determinants. These matrices preserve the length and the inner product.*

Keywords— Orthogonal matrix, unitary matrix, eigenvalues.

I. INTRODUCTION

Orthogonal matrices and unitary matrices are important types of matrices. These matrices have important applications in many fields of sciences.

This paper introduce new types of matrices: h-orthogonal and h-unitary. These matrices have many properties.

In this paper, $\langle,\rangle, \|\ \ \|, |\ \ |, and$ mean the inner product, norm, and determinant, respectively.

II. FUNDAMENTAL CONCEPTS

2.1 Definition [2]

Let V be a complex vector. An inner product on V is a function that assigns to each ordered pair of vectors u, v in V , a complex number $\langle u,v\rangle$ satisfying the following conditions:

(i) $\langle u,v\rangle \geq 0; \langle u,u\rangle = 0\ iff\ u = 0_v$

(ii) $\overline{\langle u,v\rangle} = \langle v,u\rangle, \forall\ u,v\ in\ V$

(iii) $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle, \forall\ u,v,w\ \in V$

(iv) $\langle cu,v\rangle = c\langle u,v\rangle, \forall\ u,v \in V, and\ c \in$

2.2 Example

Let u, v $\in^n$, $u = \begin{bmatrix}2\\3\\1\end{bmatrix}, v = \begin{bmatrix}-1\\1\\2\end{bmatrix}$, then $\langle u,v\rangle = 3$

2.3 Definition [1]

A matrix A ∈ $M_{n\times n}$(□),is called orthogonal matrix iff AA^T = I.

This means that $A^T = A^{-1}$.

2.4 Definition [3]

A matrix A ∈ $M_{n\times n}$(□),is called unitary matrix iff AA^θ = I.

This means that $A^\theta = A^{-1}$.

III. MAIN RESULTS

3.1 Definition

Let A= $[a_{ij}]$ is an $m \times n$ matrix. We define the h-transpose of A, denoted by$\mathrm{A^h}$, as the $n \times m$matrix where

$\mathrm{A^h} = [a^h_{ij}] = \left[a_{(m+1-j)(n+1-i)}\right]_{n\times m}$, i=1, 2… m, j=1, 2… n

3.2 Example

(1) A=$\begin{bmatrix}8 & 6 & 7\\4 & -2 & 5\end{bmatrix}$, (2) B =$\begin{bmatrix}2i & 1+i & 3\\ i & 2 & 2-i\\ 3i & 4 & 1\end{bmatrix}$

(1) $\mathrm{A^h} = \begin{bmatrix}5 & 7\\-2 & 6\\4 & 8\end{bmatrix}$, (2) $\mathrm{B^h} = \begin{bmatrix}1 & 2-i & 3\\4 & 2 & 1+i\\3i & i & 2i\end{bmatrix}$.

3.3 Theorem

Properties of h-Transpose:

If r is a scalar and A and B are matrices of the appropriate size, then.

(a) $(A^h)^h = A$.

(b) $(A^h)^T = (A^T)^h$.

(c) $(\bar{A})^h = \overline{(A^h)}$.

(d) $(A+B)^h = A^h + B^h$

(e) $(AB)^h = B^h A^h$.

(f) $(rA)^h = rA^h$.

(g) $(A^h)^{-1} = (A^{-1})^h\ , if\ A \neq 0$.

Proof:

(a) Let A = $[a_{ij}]_{m\times n}$, then $\mathrm{A^h}$ $=\left[a_{(m+1-j)(n+1-i)}\right]_{n\times m}$.

So $(A^h)^h = [a_{(n+1-(n+1-i))(m+1-(m+1-j))}]_{m\times n}$.

$= [a_{(n+1-n-1+i)(m+1-m-1+j)}]_{m\times n}$

$= [a_{ij}]_{m\times n}$.

= A.

(b) Let A = $[a_{ij}]_{m\times n}$, then A□= $[a_{ji}]_{n\times m}$

So $(A^T)^h = [a_{(n+1-i)(m+1-j)}]_{m\times n}$

$= [a_{(m+1-j)(n+1-i)}]_{m\times n}$

$= ([a_{(m+1-j)(n+1-i)}]_{n\times m})$ □

$= (A^h)^T$.

(c)Let A = $[a_{ij}]_{m\times n}$, then A□= $[\bar{a}_{ij}]_{m\times n}$

So $(\bar{A})^h = [\bar{a}_{(m+1-j)(n+1-i)}]_{n\times m}$

$= \overline{([a_{(m+1-j)(n+1-i)}])}_{n\times m}$

$=\overline{(A^h)}$.

(d) Let A = $[a_{ij}]_{m\times n}$, B = $[b_{ij}]_{m\times n}$, then

A+B = $[a_{ij} + b_{ij}]_{m\times n}$ = C = $[c_{ij}]_{m\times n}$; $c_{ij} = a_{ij} + b_{ij}, \forall\ i,j$.

So $(A+B)^h = [c^h_{ij}]_{n\times m}$

$=[c_{(m+1-j)(n+1-i)}]_{n\times m}$

$=[a_{(m+1-j)(n+1-i)} + b_{(m+1-j)(n+1-i)}]_{n\times m}$

$=[a_{(m+1-j)(n+1-i)}]_{n\times m} + [b_{(m+1-j)(n+1-i)}]_{n\times m}$

$= A^h + B^h$.

(e) Let A = $[a_{ij}]_{m\times p}$, B = $[b_{ij}]_{p\times n}$, then

AB = C = $[c_{ij}]_{m\times n}$; $c_{ij} = \sum_{r=1}^{p} a_{ir}b_{rj}$

So $(AB)^h$ = $\mathrm{C^h}$ = $[c_{(n+1-j)(m+1-i)}]_{n\times m}$

Let $c^h_{ij} \in (AB)^h$, then

$c^h_{ij} = c_{(m+1-j)(n+1-i)}$

$= \sum_{r=1}^{p} a_{(m+1-j)r} b_{r(n+1-i)}$

$= \sum_{r=1}^{p} a^T_{r(m+1-j)}\, b^T_{(n+1-i)r}$
$= \sum_{r=1}^{p} b^T_{(n+1-i)r} a^T_{r(m+1-j)}$
= the (i,j) entry in $B^h A^h$.
(f) It is clear $(rA)^h = rA^h$.
(g)Let A ∈ $M_{m\times n}$(□) and| A| ≠ 0, then $AA^{-1} = I$
So $(AA^{-1})^h = I^h$
$\Rightarrow (A^{-1})^h A^h = I$
Thus $(A^h)^{-1} = (A^h)^{-1}$

3.4 Theorem

Let A ∈ $M_{n\times n}$(□), then
(a) | A | = | A^h| = | A^T| = | $(A^T)^h$| .
(b) tr(A) = tr(A^h) = tr(A□) = tr($(A^T)^h$).
(c) Let A ∈ $M_{n\times n}$(□), then A and A^h have the same eigenvalues.

Proof:
(a) Let $A \in M_{n\times n}$(□), then
$A^h = [a_{(n+1-j)(n+1-i)}]_{n\times n}$
$= [a_{(n+1-i)(n+1-j)}]_{n\times n}$
$= ([a_{(n+1-i)(n+1-j)}]_{n\times n})$ □
So $(A^h)^T = [a_{(n+1-i)(n+1-j)}]_{n\times n}$.

$$= \begin{pmatrix} a_{nn} & a_{n(n-1)} & a_{n(n-2)} & \cdots & a_{n1} \\ a_{(n-1)n} & a_{(n-1)(n-1)} & a_{(n-1)(n-2)} & \cdots & a_{(n-1)1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{1(n-1)} & a_{1(n-2)} & \cdots & a_{11} \end{pmatrix}$$

We have that:

$$|(A^h)^T| = \begin{vmatrix} a_{nn} & a_{n(n-1)} & a_{n(n-2)} & \cdots & a_{n1} \\ a_{(n-1)n} & a_{(n-1)(n-1)} & a_{(n-1)(n-2)} & \cdots & a_{(n-1)1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{1(n-1)} & a_{1(n-2)} & \cdots & a_{11} \end{vmatrix}$$

$$= \begin{cases} (-1)^{n/2} \begin{vmatrix} a_{1n} & a_{1(n-1)} & a_{1(n-2)} & \cdots & a_{11} \\ a_{2n} & a_{2(n-1)} & a_{2(n-2)} & \cdots & a_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{nn} & a_{n(n-1)} & a_{n(n-2)} & \cdots & a_{n1} \end{vmatrix} & , if\ n\ is\ even \\ (-1)^{(n-1)/2} \begin{vmatrix} a_{1n} & a_{1(n-1)} & a_{1(n-2)} & \cdots & a_{11} \\ a_{2n} & a_{2(n-1)} & a_{2(n-2)} & \cdots & a_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{nn} & a_{n(n-1)} & a_{n(n-2)} & \cdots & a_{n1} \end{vmatrix} & , if\ n\ is\ odd \end{cases}$$

Since (^{n}□$_2$) and ($^{n-1}$□$_2$) are even numbers

$$\text{So } |(A^h)^T| = \begin{vmatrix} a_{1n} & a_{1(n-1)} & a_{1(n-2)} & \cdots & a_{11} \\ a_{2n} & a_{2(n-1)} & a_{2(n-2)} & \cdots & a_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{nn} & a_{n(n-1)} & a_{n(n-2)} & \cdots & a_{n1} \end{vmatrix}$$

By the same way, we are interchanging the columns
Therefore, we have that

$$|(A^h)^T| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = |A|$$

Since |A| = |A□| and $|A^h| = |(A^h)^T|$
So |A| = |A□| = $|A^h| = |(A^h)^T|$
b) Let A ∈ $M_{n\times n}$(□), then $A^h = [a_{(n+1-j)(n+1-i)}]_{n\times n}$
Let $A^h = D_{n\times n}$, then
tr (A^h) $= d_{11} + d_{22} + \cdots + d_{nn}$
Since $d_{ij} = a^h_{ij} = a_{(n+1-j)(n+1-i)}$
So tr (A^h) $= a_{nn} + a_{(n-1)(n-1)} + a_{(n-2)(n-2)} + \cdots + a_{11}$
$= a_{11} + a_{22} + \cdots + a_{nn}$
= tr (A) = tr(A^h) = tr(A□) = tr($(A^h)^T$).
Since | A| = | A^h| c)
| A^h-λI| = | $(A^h - \lambda I)^h$ | =| A-λI| . So

Note

(1) We define A□ as A□ = $(\bar{A})^h = \overline{(A^h)}$.
(2) We define A^* as $A^* = (\bar{A})^T = \overline{(A^T)}$.

3.5Theorem

Let x,y∈ ^{n}, then $x^* y = x^\theta (y^h)^T$.

Proof:

Let $x \in$ □^{n}, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

So $x^\theta (y^h)^T = [\bar{x}_n \quad \bar{x}_{n-1} \quad \cdots \quad \bar{x}_1] \begin{bmatrix} y_n \\ y_{n-1} \\ \vdots \\ y_1 \end{bmatrix}$
$= \bar{x}_n y_n + \bar{x}_{n-1} y_{n-1} + \cdots + \bar{x}_1 y_1$
$= \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$
$= x^* y$.

Note

Weshalldenotethe matrix $\begin{pmatrix} 0 & & 1 \\ & 1 & \\ \iddots & & \\ 1 & & 0 \end{pmatrix}_{n\times n}$,by S_n.

3.6 Theorem (Properties of S_n):

(1) S = $S^h = S^T = (S^h)$ □ = $\bar{S}$
(2) | S| = -1
(3) S^{-1} = S
(4) 4) tr(S) = $\begin{cases} 1 & , if\ n\ is\ odd \\ 0 & , if\ n\ is\ even \end{cases}$
(5) Let $x \in$ □^{m}, $y \in {}^n$ *and* $A \in M_{m\times n}()$, then
 (a) $x = S_m (x^h)^T = ((S_m x)^h)^T$.
 (b) $A = S_m (A^h)^T S_n = ((SAS)^h)^T$.
 (c) $x_1^T x_2 = x_1^h (x_2^h)^T, \forall\, x_1, x_2 \in$ □^{m}.
 (d) $x_1^h x_2 = x_1^T (x_2^h)^T, \forall\, x_1, x_2 \in$ □^{m}.
 (e) $x_1 x_2^T = x_1 x_2^h S = (x_2 x_1^h S)^T, \forall\, x_1, x_2 \in$ □^{m}.
 (f) $x_1 x_2^h = x_1 x_2^T S = (x_2 x_1^T S)^h, \forall\, x_1, x_2 \in$ □^{m}.

Proof:
(5) Let $x \in$ □^{m}, $y \in {}^n$ *and* $A \in M_{m\times n}()$, then

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ and

A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

(a) $S_n(x^h)^T = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}_{n\times n} \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix} = x$

(b)

$S_m(A^h)^T = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}_{m\times m} \begin{pmatrix} a_{mn} & a_{m(n-1)} & a_{m(n-2)} & \cdots & a_{m1} \\ a_{(m-1)n} & a_{(m-1)(n-1)} & a_{(m-1)(n-2)} & \cdots & a_{(m-1)1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{1(n-1)} & a_{1(n-2)} & \cdots & a_{11} \end{pmatrix}$

$= \begin{pmatrix} a_{1n} & a_{1(n-1)} & a_{1(n-2)} & \cdots & a_{11} \\ a_{2n} & a_{2(n-1)} & a_{2(n-2)} & \cdots & a_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{mn} & a_{m(n-1)} & a_{m(n-2)} & \cdots & a_{m1} \end{pmatrix}$

$S_m(A^h)^T S_n$

$= \begin{pmatrix} a_{1n} & a_{1(n-1)} & a_{1(n-2)} & \cdots & a_{11} \\ a_{2n} & a_{2(n-1)} & a_{2(n-2)} & \cdots & a_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{mn} & a_{m(n-1)} & a_{m(n-2)} & \cdots & a_{m1} \end{pmatrix} \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}_{n\times n}$

$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$

(c) Let $x_1, x_2 \in$ □^{m}

Then $x_1^T x_2 = (x_1^h S)x_2$

$= x_1^h(Sx_2)$

$= x_1^h(x_2^h)^T$

(d) Let $x_1, x_2 \in$ □^{m}

Then $x_1^h x_2 = (x_1^T S)x_2$

$= x_1^T(Sx_2)$

$= x_1^T(x_2^h)^T$

(e) Let $x_1, x_2 \in$ □^{m}

Then $x_1 x_2^T = x_1(x_2^h S)$

$= (x_1 x_2^h)S$

$= S((x_1 x_2^h)^T \quad)^h$

$= S(x_1^h)^T \quad x_2{}^T$

$= (x_2 \quad x_1^h \quad S)^T$

(f) **Let** $x_1, x_2 \in$ □^{m}

Then $x_1 x_2^h = x_1(x_2^T S)$

$= (x_1 x_2^T)S$

$= S((x_1 x_2^T)^T \quad)^h$

$= S(x_1^h)^T \quad x_2{}^h$

$= (x_2 \quad x_1^T \quad S)^h$

Note

(1) $(A^\theta)^\theta = A$.

(2) $(A + B)^\theta = A^\theta + B^\theta$.

(3) $(AB)^\theta = B^\theta A^\theta$.

(4) $(kA)^\theta = \bar{k}A^\theta, k \in$.

3.7 Definition

A matrix A ∈ $M_{n\times n}$(□)is called h-orthogonal matrix *iff* AA^h = I.

This means that $A^h = A^{-1}$.

1) A=$\begin{pmatrix} Z & 0 \\ 0 & 1/z \end{pmatrix}$, 0≠ z ϵ □, 2) B=$\begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}$.

A and B are h-orthogonal matrices.

3) C=$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is not h-orthogonal matrix.

3.8 Theorem

Let A ∈ $M_{n\times n}$(□) then the following statements are equivalent :

(1) A is h-orthogonal matrix.
(2) A^{-1} is h-orthogonal matrix.
(3) A□ is h-orthogonal matrix.
(4) A* is h-orthogonal matrix.
(5) A□ is h-orthogonal matrix.
(6) A^h is h-orthogonal matrix.
(7) A□ is h-orthogonal matrix.
(8) (A^h) □ is h-orthogonal matrix.
(9) (A□) □ is h-orthogonal matrix.

Proof:

(1) 1⇒2:

Suppose A is h-orthogonal matrix

So $AA^h = I$

$\Rightarrow (AA^h)^{-1} = I^{-1}$

Thus $(A^{-1})^h A^{-1} = I$

Hence A^{-1} is h-orthogonal matrix.

(2) 2⇒3:

Suppose $A^{-1}$is h-orthogonal matrix.

So $A^{-1}(A^{-1})^h = I$

$\Rightarrow (A^h A)^{-1} = I$

$\Rightarrow (A^h A) \quad = I$

$\Rightarrow (A^h A)^T = I \quad ^T$

$\Rightarrow A^T(A^h)^T = I$

$\Rightarrow A^T(A^T)^h = I$

Hence A^T is h-orthogonal matrix.

(3) 3⇒4:

Suppose $A^T$is h-orthogonal matrix.

So $A^T(A^T)^h = I$

$\Rightarrow \overline{(A^T(A^h)^T)} = \bar{I}$

$\Rightarrow \overline{(A^T)((A^T)^h)} = I$

$\Rightarrow A^*(A^*)^h = I$

Hence A^* is h-orthogonal matrix.

(4) 4⇒5:

Suppose $A^*$is h-orthogonal matrix.

So $A^*(A^*)^h = I$

$\Rightarrow (A^*(A^*)^h)^T = I^T$

$\Rightarrow ((A^*)^h)^T(A^*)^T = I$

$\Rightarrow (\bar{A})^h \quad \bar{A} \quad = I$

Hence $\underline{A}$ is h-orthogonal matrix.

(5) 5⇒6:

Suppose $\bar{A}$ is h-orthogonal matrix.

So $(\bar{A})\ (\bar{A})^h = I$

$\Rightarrow \overline{(\bar{A}\ (\bar{A})^h)} = \bar{I}$

$\Rightarrow \overline{(\bar{A})}\ (\overline{(\bar{A})})^h = I$

$\Rightarrow AA^h = I$

$\Rightarrow (A^h)^h A^h = I$

Hence A^h is h-orthogonal matrix.

(6) 6⇒7:

Suppose $A^h$is h-orthogonal matrix.

So $A^h\ (A^h)^h = I$

$\Rightarrow A^h\ A = I$

$\Rightarrow (A^h\ A)^\theta = I^\theta$

$\Rightarrow A^\theta (A^h)^\theta = I$

$\Rightarrow A^\theta (A^\theta)^h = I$

Hence A^θ is h-orthogonal matrix.

(7)7⇒8:

Suppose $A^\theta$is h-orthogonal matrix.

So $A^\theta\ (A^\theta)^h = I$

$\Rightarrow (A^\theta\ (A^\theta)^h)^* = I^*$

$\Rightarrow ((A^\theta)^*)^h (A^\theta)^* = I$

$\Rightarrow ((A^h)^T)^h (A^h)^T = I$

Hence $(A^h)^T$ is h-orthogonal matrix.

(8)8⇒9:

Suppose $(A^h)^T$is h-orthogonal matrix.

So $((A^h)^T)^h (A^h)^T = I$

$\Rightarrow \overline{[((A^h)^T)^h (A^h)^T]} = \bar{I}$

$\Rightarrow \overline{[((A^h)^T)^h]}\overline{[(A^h)^T]} = I$

$\Rightarrow ((A^\theta)^T)^h (A^\theta)^T = I$

Hence $(A^\theta)^T$ is h-orthogonal matrix.

(9)9⇒1:

Suppose $(A^\theta)^T$is h-orthogonal matrix.

So $((A^\theta)^T)^h (A^\theta)^T = I$

$\Rightarrow ((A^\theta)^h)^T (A^*)^h = I$

$\Rightarrow A^*(A^*)^h = I$

$\Rightarrow (A^*(A^*)^h)^* = I^*$

$\Rightarrow A^h A = I$

Hence A is h-orthogonal matrix.

3.9 Theorem

If A is h-orthogonal matrix, then A^n is h-orthogonal matrix,n=2, 3…

Proof:

Suppose A is h-orthogonal matrix

So $A^n (A^n)^h = \underbrace{(AA\cdots A)}_{n\_times}\underbrace{(AA\cdots A)^h}_{n\_times}$

$= (AA\cdots A)\ \ (A^h A^h \cdots A^h)$

$= \underbrace{(AA\cdots A)}_{(n-1)\_times}\ \ (AA^h)\underbrace{(A^h A^h \cdots A^h)}_{(n-1)\_times}$

$= \underbrace{(AA\cdots A)}_{(n-1)\_times}\ I\ \underbrace{(A^h A^h \cdots A^h)}_{(n-1)\_times}$

⋮

$= AA^h$

$= I$

Hence A^n is h-orthogonal matrix.

3.10 Theorem

Let A ∈ $M_{n\times n}$(□), then the following statements areequivalent.

(1) A is h-orthogonal matrix.
(2) (A□) ⁻¹ =A□.
(3) (A□) ⁻¹= (A^h) □.
(4) (A*) ⁻¹ = (A□) □ =$(A^*)^h$.

Proof:

(1) 1⇒2:

Suppose A is h-orthogonal matrix, then

$AA^h = A^h A = I$

So $\overline{(AA^h)} = \bar{I}$

$\bar{A}A^\theta = I$

Hence $(\bar{A})^{-1}$ =A□.

(2) 2⇒3:

Suppose $(\bar{A})^{-1}$ =A,

So $\overline{((\bar{A})^{-1})} = \overline{(A)}$

$\Rightarrow A^{-1} = A^h$

$\Rightarrow (A^{-1})^T = (A^h)^T$

(3) 3⇒4:

Suppose$(A^{-1})^T = (A^h)^T$,

So $\overline{(A^T)^{-1}} = \overline{(A^h)^T}$

$\Rightarrow (A^*)^{-1} = (A^*)^h = (A^\theta)^T$

(4) 4⇒1:

Suppose$(A^*)^{-1} = (A^*)^h$,

So $((A^*)^{-1})^* = ((A^*)^h)^*$

$\Rightarrow A^{-1} = A^h$

Hence A is h-orthogonal matrix.

3.11 Theorem

If A is h-orthogonal matrix, then

(1) If A□A^h = A^hA□ then A^hA□ is Orthogonal matrix.
(2) If A*A^h=A^hA* then A^hA* is Unitary matrix.

Proof:

(1) Suppose A is h-orthogonal matrix and A□A^h = A^hA□

So $(A^h A)^T (A^h A^T) = (A(A^h)^T)(A^h A^T)$

$= A((A^h)^T)A^T)A^h$ (A□A^h = A^hA□)

$= A(I\ \ )A^h$ (Theorem 3.8)

$= I$

Hence$A^h A^T$ *is orthoganl matrix*.

(2) Suppose A is h-orthogonal matrix and $A^* A^h = A^h A^*$

So $(A^h A^*)^*(A^h A^*) = (A(A^h)^*)(A^h A^*)$

$= A((A^h)^*)A^*)A^h\ \ (A^* A^h = A^h A^*)$

$= A((AA^h)^*\ \ )A^h$

$= A(I^*)A^h$

$= I$

Hence$A^h A^*$is unitary matrix.

3.12 Theorem

If $A_1, A_2,\ A_3 \ldots\ A_n$are h-orthogonal matrices, and $\acute{1}, \acute{2} \ldots \acute{n}$ be any rearrangement of the indices $1, 2 \ldots n$, then $A_{\acute{1}} A_{\acute{2}}\ A_{\acute{3}} \ldots\ A_{\acute{n}}$ is h-orthogonal matrix.

Proof:

Let $A_1, A_2, A_3 \dots A_n$ be h-orthogonal matrices and $\acute{1}, \acute{2} \dots \acute{n}$
be any rearrangement of the indices $1, 2 \dots n$, then
$(A_{\acute{1}}A_{\acute{2}} \dots A_{\acute{n}})(A_{\acute{1}}A_{\acute{2}} \dots A_{\acute{n}})^h$
$= (A_{\acute{1}}A_{\acute{2}} \dots A_{\acute{n}})(A_{\acute{n}}^h A_{(\acute{n-1})}^h \cdots A_{\acute{1}}^h)$
$= (A_{\acute{1}}A_{\acute{2}} \dots A_{(\acute{n-1})})(A_{\acute{n}}A_{\acute{n}}^h)(A_{(\acute{n-1})}^h \cdots A_{\acute{1}}^h)$
$= (A_{\acute{1}}A_{\acute{2}} \dots A_{(\acute{n-1})})I(A_{(\acute{n-1})}^h \cdots A_{\acute{1}}^h)$
⋮
$= A_{\acute{1}}A_{\acute{1}}^h$
$= I$
Hence, $A_{\acute{1}}A_{\acute{2}} A_{\acute{3}} \dots A_{\acute{n}}$ is h-orthogonal matrix.

3.13 Theorem

If AB is h-orthogonal matrix, then A is h-orthogonal matrix ⇔
B is h-orthogonal matrix.

Proof:

⇒

Suppose AB and A are h-orthogonal matrices
So $(AB)^h(AB) = I$
$\Rightarrow (B^hA^h)(AB) = I$
$\Rightarrow B^h(A^hA)B = I$
$\Rightarrow B^h(I) \quad B = I \Rightarrow B^hB = I$
Hence, B is h-orthogonal matrix.

⇐

Suppose AB and B are h-orthogonal matrices
So $(AB)(AB)^h = I$
$\Rightarrow (AB)(B^hA^h) = I$
$\Rightarrow A(BB^h)A^h = I$
$\Rightarrow A(I) \quad A^h = I$
$\Rightarrow AA^h = I$
Hence, A is h-orthogonal matrix.

3.14 Theorem

Let A∈$M_{n\times n}$(□) be a h-orthogonal matrix, if A $=(A^T)^h$, then

(1) ‖ Ax‖ = ‖ x‖ , $x \in □^n$.

(2) $\langle Ax, Ay\rangle = \langle x, y\rangle$, x, y ∈ $□^n$.

Proof:

(1)Let A∈$M_{n\times n}$(□) be a h-orthogonal matrix and A $=(A^T)^h$, then
$\|Ax\|^2 = \langle Ax. Ax\rangle$,x ∈ $□^n$
$= (Ax)^*Ax$
$= x^*(A^*A)x$
$= x^*(SA^\theta(A^h)^TS)x$ (Theorem 3.5(5, b))
$= x^*S(A^\theta(A^h)^T)Sx$
$= x^\theta(A^\theta A)(x^h)^T$ (Theorem 3.5(5, a) and $A = (A^h)^T$).
$= x^\theta I(x^h)^T$
$= x^*x$ (Theorem 3.5)
$= \|x\|^2$
Hence ‖ Ax‖ = ‖ x‖ .

(2) Let A∈$M_{n\times n}$(□)be a h-orthogonal matrix, x, y ∈ $□^n$
and A $=(A^T)^h$,then
$\langle Ax, Ay\rangle = (Ax)^*Ay$
$= x^*(A^*A)y$
$= x^*(SA^\theta(A^h)^TS)y$ (Theorem 3.5(5, b))
$= x^*S(A^\theta(A^h)^T)Sy$
$= x^\theta(A^\theta A)(y^h)^T$ (Theorem 3.5(5, a) and $A = (A^h)^T$).
$= x^\theta I(y^h)^T$
$= x^*y$ (Theorem 3.5)
$= \langle x, y\rangle$
Hence, $\langle Ax, Ay\rangle = \langle x, y\rangle$.

3.15 Theorem

Let A be h-orthogonal matrix, then

(1) The eigenvalues of A are of modulus 1.
(2) ∣ A∣ = ± 1.

Proof:

(1) Suppose that λ be an eigenvalue of A
So $Ax = \lambda x, x \neq 0$
$\Rightarrow \|Ax\| = \|\lambda x\|$
$\Rightarrow \|x\| = |\lambda|\|x\|$(Theorem 3.14)
Hence, $|\lambda| = 1$.
(2) Let A be h-orthogonal matrix, then
$AA^h = I$
$|AA^h| = |I|$
$|A||A^h| = 1$
$|A||A \quad| = 1$(Theorem 3.4)
$|A|^2 = 1$
Hence,$|A \quad| = \pm 1$

3.16 Definition

A matrix A ∈ $M_{n\times n}$(□),is called h-unitary matrix *iff* AA^θ = I.
This means that $A^\theta = A^{-1}$.

3.17 Example

(1) A $=\begin{pmatrix} i & 0 & i \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}$, (2) B $=\begin{pmatrix} 0 & ai \\ \frac{1}{a}i & 0 \end{pmatrix}$, a ∈ $R\backslash\{0\}$.
A and B are h-unitary matrices.

(3) C $=\begin{pmatrix} 2i & 0 \\ 0 & 3i \end{pmatrix}$ is not h-unitary matrix.

3.18 Theorem

Let A ∈ $M_{n\times n}$(□), then the following statementsare equivalent:

(1) A is h-unitary matrix.
(2) A^{-1} is h-unitary matrix.
(3) A□ is h-unitary matrix.
(4) A* is h-unitary matrix.
(5) $\bar{A}$ is h-unitary matrix.
(6) A^h is h-unitary matrix.
(7) A□ is h-unitary matrix.
(8) (A^h) □ is h-unitary matrix.
(9) (A□) □ is h-unitary matrix.

Proof:

(1) 1⇒2:
Suppose A is h- unitary matrix
So $AA^\theta = I$
$\Rightarrow (AA^\theta)^{-1} = I^{-1}$
Thus $(A^{-1})^\theta A^{-1} = I$
Hence A^{-1} is h-unitary matrix.

(2) 2⇒3:
Suppose $A^{-1}$is h-unitary matrix.

So $A^{-1}(A^{-1})^{\theta} = I$

$\Rightarrow (A^{\theta}A)^{-1} = I$

$\Rightarrow ((A^{\theta}A)^{-1})^{-1} = I^{-1}$

$\Rightarrow A^{\theta}A = I$

$\Rightarrow (A^{\theta}A)^{T} = I^{T}$

$\Rightarrow A^{T}(A^{\theta})^{T} = I$

$\Rightarrow A^{T}(A^{T})^{\theta} = I$

Hence A^{T} is h-unitary matrix.

(3) 3⇒4:

Suppose A^{T} is h-unitary matrix.

So $A^{T}(A^{T})^{\theta} = I$

$\Rightarrow \overline{(A^{T}(A^{\theta})^{T})} = \bar{I}$

$\Rightarrow \overline{(A^{T})}(\overline{(A^{T})^{\theta}}) = I$

$A^{*}(A^{*})^{\theta} = I \Rightarrow$

Hence A^{*} is h-unitary matrix.

(4) 4⇒5:

Suppose A^{*} is h-unitary matrix.

So $A^{*}(A^{*})^{\theta} = I$

$\Rightarrow (A^{*}(A^{*})^{\theta})^{T} = I^{T}$

$\Rightarrow ((A^{*})^{\theta})^{T}(A^{*})^{T} = I$

$\Rightarrow (\bar{A})^{\theta}\ \bar{A} = I$

Hence $\bar{A}$ is h-unitary matrix.

(5) 5⇒6:

Suppose $\bar{A}$ is h-unitary matrix.

So $(\bar{A})\ (\bar{A})^{\theta} = I$

$\Rightarrow \overline{(\bar{A}\,(\bar{A})^{\theta})} = \bar{I}$

$\Rightarrow \overline{(\bar{A})}\ \overline{((\bar{A})}\ )^{\theta} = I$

$\Rightarrow AA^{\theta} = I$

$\Rightarrow (AA^{\theta})^{h} = I^{h}$

$\Rightarrow (A^{h})^{\theta}A^{h} = I$

Hence A^{h} is h-unitary matrix.

(6) 6⇒7:

Suppose A^{h} is h-unitary matrix.

So $A^{h}\,(A^{h})^{\theta} = I$

$\Rightarrow A^{h}\bar{A} = I$

$\Rightarrow \overline{(A^{h}\bar{A})} = \bar{I}$

$\Rightarrow A^{\theta}\ A = I$

$\Rightarrow A^{\theta}(A^{\theta})^{\theta} = I$

Hence A^{θ} is h-unitary matrix.

(7)7⇒8:

Suppose A^{θ} is h-unitary matrix.

So $A^{\theta}\,(A^{\theta})^{\theta} = I$

$\Rightarrow (A^{\theta}\,(A^{\theta})^{\theta})^{*} = I^{*}$

$\Rightarrow (\,(A^{\theta})^{*})^{\theta}(A^{\theta})^{*} = I$

$\Rightarrow (\,(A^{h})^{T})^{\theta}(A^{h})^{T} = I$

Hence $(A^{h})^{T}$ is h-unitary matrix.

(8)8⇒9:

Suppose $(A^{h})^{T}$ is h-unitary matrix.

So $(\,(A^{h})^{T})^{\theta}(A^{h})^{T} = I$

$\Rightarrow \overline{[(\,(A^{h})^{T})^{\theta}(A^{h})^{T}]} = \bar{I}$

$\Rightarrow \overline{[(\,(A^{h})^{T})^{\theta}]}\overline{[(A^{h})^{T}]} = I$

$\Rightarrow (\,(A^{\theta})^{T})^{\theta}(A^{\theta})^{T} = I$

Hence $(A^{\theta})^{T}$ is h-unitary matrix.

(9)9⇒1:

Suppose $(A^{\theta})^{T}$ is h-unitary matrix.

So $(\,(A^{\theta})^{T})^{\theta}(A^{\theta})^{T} = I$

$\Rightarrow (\,(A^{\theta})^{\theta})^{T}(A^{\theta})^{T} = I$

$\Rightarrow A^{T}(A^{\theta})^{T} = I$

$\Rightarrow (A^{T}(A^{\theta})^{T})^{T} = I^{T}$

$\Rightarrow A^{\theta}A = I$

Hence A is h-unitary matrix.

3.19 Theorem

If A is h-unitary matrix, then A^{n} is h-unitary matrix

Proof:

Suppose A is h-unitary matrix

So $A^{n}(A^{n})^{\theta} = \underbrace{(AA\cdots A)}_{n\_times}\underbrace{(AA\cdots A)}_{n\_times}{}^{\theta}$

$= (AA\cdots A)\ \ (A^{\theta}A^{\theta}\cdots A^{\theta})$

$= \underbrace{(AA\cdots A)}_{(n-1)\_times}\ \ (AA^{\theta})\underbrace{(A^{\theta}A^{\theta}\cdots A^{\theta})}_{(n-1)\_times}$

$= \underbrace{(AA\cdots A)}_{(n-1)\_times}\ I\ \underbrace{(A^{\theta}A^{\theta}\cdots A^{\theta})}_{(n-1)\_times}$

⋮

$= AA^{\theta}$

$= I$

Hence A^{n} is h-unitary matrix.

3.20 Theorem

Let $A \in M_{n\times n}$(□), then the following are equivalent

(1) A is h-unitary matrix.

(2) $(\bar{A})^{-1} = A^{h}$.

(3) $(A^{T})^{-1} = (A^{*})^{h} = (A^{\theta})^{T}$.

(4) $(A^{*})^{-1} = (A^{T})^{h}$.

Proof:

(1) 1⇒2:

Suppose A is h- unitary matrix, then

$AA^{\theta} = I$

So $(AA^{\theta})^{h} = I^{h}$

$\bar{A}A^{h} = I$

Hence $(\bar{A})^{-1} = A^{h}$

(2) 2⇒3:

Suppose $(\bar{A})^{-1} = A^{h}$,

So $\overline{((\bar{A})^{-1})} = \overline{(A^{h})}$

$\Rightarrow A^{-1} = A^{\theta}$

$\Rightarrow (A^{-1})^{T} = (A^{\theta})^{T}$

$\Rightarrow (A^{T})^{-1} = (A^{\theta})^{T} = (A^{*})^{h}$

(3) 3⇒4:

Suppose $(A^{T})^{-1} = (A^{\theta})^{T}$

So $\overline{(A^{T})^{-1}} = \overline{(A^{\theta})^{T}}$

$\Rightarrow (A^{*})^{-1} = (A^{T})^{h} = (A^{h})^{T}$

(4) 4⇒1:

Suppose $(A^{*})^{-1} = (A^{T})^{h}$, then

So $((A^{*})^{-1})^{*} = ((A^{T})^{h})^{*}$

$\Rightarrow A^{-1} = A^{\theta}$

Hence A is h-orthogonal matrix.

3.21Corollary

Let A be a real matrix, then A is h-orthogonalmatrix⇔

A is h-unitary matrix

3.22 Theorem

Let A beh-unitary matrix

(1) If (A□) □A = A (A□) □, then A (A□) □ is orthogonal matrix.
(2) If A□A^h= A^hA□, then A^h A□ is unitary matrix.
(3) If AA^h =A^hA, then A^hA is h-unitary matrix.
(4) If AA□ = A□A, then AA□ is h-orthogonal matrix.
(5) If A^hA□ = A□A^h, then A^hA□ is h-orthogonal matrix.

Proof:

(1) Suppose A is h-unitary matrix and $A(A^\theta)^T = A(A^\theta)^T$

So $(A(A^\theta)^T)^T(A(A^\theta)^T) = (A^\theta A^T)(A(A^\theta)^T)$

$= (A^\theta A^T)((A^\theta)^T A) \quad (A(A^\theta)^T = A(A^\theta)^T)$

$= A^\theta(A^T(A^\theta)^T)A)$(Theorem 3.20(3))

$= A^\theta(I)A$

$= I$

Hence$A(A^\theta)^T$ *is orthoganl matrix.*

2) Suppose A is h-unitary matrix and $A^T A^h = A^h A^T$

So $(A^h A^T)^*(A^h A^*) = (\bar{A}(A^h)^*)(A^h A^T)$

$= \bar{A}((A^h)^*)A^T)A^h(A^T A^h = A^h A^T)$

$= \bar{A}(I \quad)A^h$(Theorem 3.20(3))

$= \bar{A}A^h$(Theorem 3.20(2))

$= I$

Hence$A^h A^*$ *is* unitary *matrix*.

Similarly, we can prove 3, 4, 5

3.23 Theorem

If $A_1, A_2, A_3 \ldots A_n$are h-unitary matrices, and $\acute{1}, \acute{2} \ldots \acute{n}$ be any rearrangement of the indices $1, 2 \ldots n$ then $A_{\acute{1}}A_{\acute{2}}A_{\acute{3}} \ldots A_{\acute{n}}$ is h-unitary matrix.

Proof:

Let $A_1, A_2, A_3 \ldots A_n$be h-unitary matrices and $\acute{1}, \acute{2} \ldots \acute{n}$

be any rearrangement of the indices $1, 2 \ldots n$, then

$(A_{\acute{1}}A_{\acute{2}} \ldots A_{\acute{n}})(A_{\acute{1}}A_{\acute{2}} \ldots A_{\acute{n}})^\theta$

$= (A_{\acute{1}}A_{\acute{2}} \ldots A_{\acute{n}})(A_{\acute{n}}^\theta A_{(\acute{n-1})}^\theta \cdots A_{\acute{1}}^\theta)$

$= (A_{\acute{1}}A_{\acute{2}} \ldots A_{(\acute{n-1})})(A_{\acute{n}}A_{\acute{n}}^\theta)(A_{(\acute{n-1})}^\theta \cdots A_{\acute{1}}^\theta)$

$= (A_{\acute{1}}A_{\acute{2}} \ldots A_{(\acute{n-1})})I(A_{(\acute{n-1})}^\theta \cdots A_{\acute{1}}^\theta)$

$\vdots$

$= A_{\acute{1}}A_{\acute{1}}^\theta$

$= I$

Hence, $A_{\acute{1}}A_{\acute{2}}A_{\acute{3}} \ldots A_{\acute{n}}$ is h-unitary matrix.

3.24 Theorem

If AB is h-unitary matrix, then A is h-unitary matrix $\Leftrightarrow$

B is h-unitary matrix.

Proof:

$\Rightarrow$

Suppose AB and A are h-unitary matrices

So $(AB)^\theta(AB) = I$

$\Rightarrow (B^\theta A^\theta)(AB) = I$

$\Rightarrow B^\theta(A^\theta A)B = I$

$\Rightarrow B^\theta(I) \quad B = I \Rightarrow B^\theta B = I$

Hence, B is h-unitary matrix.

$\Leftarrow$

Suppose AB and B are h-unitary matrices

So $(AB)(AB)^\theta = I$

$\Rightarrow (AB)(B^\theta A^\theta) = I$

$\Rightarrow A(BB^\theta)A^\theta = I$

$\Rightarrow A(I) \quad A^\theta = I$

$\Rightarrow AA^\theta = I$

Hence, A is h-unitarymatrix.

3.25 Theorem

Let A ∈ $M_{n\times n}$(□) be a h-Unitary matrix, if A =$(A^T)^h$, then

(1) ‖ Ax‖ = ‖ x‖ ,x ∈ □^{n}.
(2) $\langle Ax, Ay\rangle = \langle x, y\rangle$, x,y∈ □^{n}.

Proof:

(1) Let A ∈ $M_{n\times n}$(□)be an h-unitary matrix and A =$(A^T)^h$,then

$\|Ax\|^2 = \langle Ax, Ax\rangle$x ∈ □^{n}

$= (Ax)^*Ax$

$= x^*(A^*A)x$

$= x^*(SA^\theta(A^h)^T S)x$ (Theorem 3.5(5, b))

$= x^*S(A^\theta(A^h)^T)Sx$

$= x^\theta(A^\theta A)(x^h)^T$ (Theorem 3.5) (5, a) and $A = (A^h)^T$.

$= x^*x$ (Theorem 3.5)

$= \|x\|^2$

Hence ‖ Ax‖ = ‖ x‖ .

(2) Let A ∈ $M_{n\times n}$(□) be an h-unitary matrix, x, y∈ □^{n}

and A =$(A^T)^h$,then

$\langle Ax, Ay\rangle = (Ax)^*Ay$

$= x^*(A^*A)y$

$= x^*(SA^\theta(A^h)^T S)y$ (Theorem 3.5(5, b))

$= x^*S(A^\theta(A^h)^T)Sy$

$= x^\theta(A^\theta A)(y^h)^T$ (Theorem 3.5(5, b))

$= x^\theta I(y^h)^T$

$= x^*y$ (Theorem 3.5)

$= \langle x, y\rangle$

Hence, $\langle Ax, Ay\rangle$ =$\langle x, y\rangle$.

3.26 Theorem

Let A be an h-unitary matrix, then the eigenvalues of A are of modulus 1.

Proof:

Suppose that λ be an eigenvalue of A

So $Ax = \lambda x, x \neq 0$

$\Rightarrow \|Ax\| = \|\lambda x\|$

$\Rightarrow \|x\| = |\lambda|\|x\|$ (Theorem 3.25)

Hence, $|\lambda| = 1$.

IV. CONCLUSION

We have new types of matrices with important properties. They preserve the length and the inner product. Eigenvalues of these matrices are of modulus 1.

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