On *(gr)-Closed Sets in Topological Spaces

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Abstract: *In this paper we introduced and study the notions of *(gr)-closed sets and *(gr)-open sets in topological spaces and also we discussed their properties.*

Keywords: **(gr)-closed set, *(gr)-open set, ĝ-open.*

1. Introduction & Preliminaries

In 1970, Levine [8], introduced and investigated the notion of g-closed sets in topological spaces. In 1987, Bhattacharya, et. al [3], introduced and studied the notion of semi generalized closed sets in topological spaces. In 1990, Arya, et. al [2] introduced the notion of generalized semi closed sets.

In 1993, Maki, et. al [10], introduced generalized α- closed sets in topology. In 1993, Palaniappan, et. al [13], introduced the notion of Regular generalized closed sets and in 1996, Maki, et. al [11], introduced generalized pre closed sets in topological spaces and in 1995, Dontchev [5] introduced and investigated the notion of generalized semi- pre open sets in topological spaces. In 1997, Gnanambal [6] introduced the generalized pre-regular closed sets in topological spaces. In 2011, Bhattacharya [4], introduced generalized regular closed sets in topological spaces.

Let (X, τ) be a topological space with no separation axioms are assumed. If A ⊆ X, cl(A) and int(A) will respectively denote the closure and interior of A in (X, τ).

Definition 1.1 A subset A of a topological space (X, τ) is called

1) Pre- closed set [9], if cl(int(A)) ⊆ A.
2) Semi- closed set [7], if int(cl(A)) ⊆ A.
3) Semi- pre closed set [1], if int(cl(int(A))) ⊆ A.
4) Regular closed set [14], if A = cl(int(A)).
5) α- closed set [12], if cl(int(cl(A))) ⊆ A.

Definition 1.2 [13] For any subset A of (X, τ), rcl(A) = ∩ {B: B ⊇ A, B is a regular closed subsets of X}.

Definition 1.3 A subset A of a topological space (X, τ) is called

1) g- closed set [8], if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
2) gs- closed set [2], if scl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).
3) sg- closed set [3], if scl(A) ⊆ U whenever A ⊆ U and U is semi open in (X, τ).
4) gα - closed set [10], if αcl(A) ⊆ U whenever A ⊆ U and U is α- open in (X, τ).
5) αg- closed set [10], if αcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
6) gp- closed set [11], if pcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
7) gsp- closed set [5], if spcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
8) rg- closed set [13], if cl(A) ⊆ U whenever A ⊆ U and U is regular open in (X, τ).
9) gpr- closed set [6], if pcl(A) ⊆ U whenever A ⊆ U and U is regular open in (X, τ).
10) ĝ - closed set [15], if cl(A) ⊆ U whenever A ⊆ U and U is semiopen in (X, τ).

2. *(gr) - CLOSED SETS

Definition 2.1. A subset A of a topological space (X, τ) is called a star generalized regular closed set [briefly *(gr)-closed set], if rcl(A) ⊆ U whenever A ⊆ U and U is ĝ-open subset of X.

Theorem 2.2 If A is r-closed set in (X, τ). Then A is *(gr)- closed set in (X, τ).

Proof: Let A be an r-closed set in (X, τ) . Let U be a ĝ-open set such that A⊆U. Therefore, we have rcl(A) =A ⊆ U , implies rcl(A) ⊆ U. Hence A is *(gr)- closed in (X, τ).

The converse of the above theorem need not be true as seen from the following example.

Example 2.3 Let X = {a, b, c, d} with τ = {∅, X, {d}, {b, c}, {b, c, d}}. The subset A = {a, c} is *(gr)- closed set but not r-closed set.

Theorem 2.4 For a topological space (X, τ), the following hold.

(i) Every *(gr)- closed set is g-closed.
(ii) Every *(gr)- closed set is rg-closed.
(iii) Every *(gr)- closed set is gs-closed.
(iv) Every *(gr)- closed set is gp-closed.
(v) Every *(gr)- closed set is gsp-closed.
(vi) Every *(gr)- closed set is gpr-closed.
(vii) Every *(gr)- closed set is αg-closed.

Proof:

(i) Let A be *(gr)- closed set in (X, τ).Let U be an open set such that A⊆U. Since every open set is ĝ-open set, cl(A) ⊆ rcl(A) ⊆ U, Therefore cl(A) ⊆ U. Hence A is a g-closed set in X.

(ii) Let A be *(gr)- closed set in (X, τ).Let U be an r-open set such that A⊆U. Since every r-open set is ĝ-open set, cl(A) ⊆ rcl(A) ⊆ U, Therefore cl(A) ⊆ U. Hence A is a rg-closed set in X.

(iii) Let A be *(gr)- closed set in (X, τ).Let U be an open set such that A⊆U. Since every r-open set is ĝ-open set, scl(A) ⊆ rcl(A) ⊆ U, Therefore scl(A) ⊆ U. Hence A is a gs-closed set in X.

(iv) Let A be *(gr)- closed set in (X, τ).Let U be an open set such that A⊆U. Since every open set is ĝ-open set, pcl(A) ⊆ rcl(A) ⊆ U, Therefore pcl(A) ⊆ U. Hence A is a gp-closed set in X.

(v) Let A be *(gr)- closed set in (X, τ).Let U be an open set such that A⊆U. Since every open set is ĝ-open set, spcl(A) ⊆ rcl(A) ⊆ U, Therefore spcl(A) ⊆ U. Hence A is a gsp-closed set in X.

(vi) Let A be *(gr)- closed set in (X, τ).Let U be an r-open set such that A⊆U. Since every r-open set is open set, and every open set is ĝ-open set, pcl(A) ⊆ rcl(A) ⊆ U, Therefore pcl(A) ⊆ U. Hence A is a gpr-closed set in X.

(vii) Let A be *(gr)- closed set in (X, τ).Let U be an open set such that A⊆U. Since every open set is ĝ-open set, αcl(A) ⊆ rcl(A) ⊆ U, Therefore αcl(A) ⊆ U. Hence A is a αg-closed set in X.

The converse of the above theorem need not be true as seen from the following examples.

Example: 2.5

(i) Let X = {a, b, c, d} with τ = {∅, X, {a, b, c}, {d} }.The g-closed sets are {{a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, X, ∅} and *(gr)-closed sets are {{a, b, c}, {d}, X, ∅}. The set A = {a, d} is a g-closed set but not *(gr)-closed set in X.

(ii) Let X = {a, b, c, d} with τ = {∅, X, {a, b, c}, {a, b}, {c}}. The rg-closed sets are {{d}, {a, c}, {b, c}, {a, d}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d},X, ∅} and *(gr)-closed sets are {{d}, {a, d}, {b, d},{c, d}, {a, b, d}, {a, c, d}, {b, c, d}, X, ∅}. Let A = {a, c} is a rg-closed set but not *(gr)-closed set in X.

(iii) Let X = {a, b, c, d} with τ = {∅, X, {a, b, c}, {b, c},{a}}.The gs-closed sets are {{a}, {b}, {c}, {d}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}, X, ∅} and *(gr)-closed sets are {{d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}, X, ∅}. The set A = {a} is a gs-closed set but not *(gr)-closed set in X.

(iv) Let X = {a, b, c, d} with τ = {∅, X, {a, b, c}, {a, b}, {a},{b}}.The gp-closed sets are {{c}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d},{a, c, d}, {b, c, d}, X, ∅} and *(gr)-closed sets are {{d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}, X, ∅}. The set A = {c} is a gp-closed set but not *(gr)-closed set in X.

(v) Let X = {a, b, c, d} with τ = {∅, X, {b, c, d}, {b, c},{b}}.The gsp-closed sets are {{a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, X, ∅} and *(gr)-closed sets are {{a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X, ∅}. The set A = {b} is a gsp-closed set but not *(gr)-closed set in X.

(vi) Let X = {a, b, c, d} with τ = { ∅, X, {a, c, d}, {a, c}, {a},{c}}.The gpr-closed sets are {{b}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d},{a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}, X, ∅} and *(gr)-closed sets are { {b}, {b, c}, {b, d}, {a, b}, {a, b, c}, {a, b, d}, {b, c, d}, X, ∅}. The set A = {d} is a gpr-closed set but not *(gr)-closed set in X.

(vii) Let X = {a, b, c, d} with topology τ = {∅, X, {b, c, d},{b, c},{b},{c} }. Then αg -closed sets are {{a}, {d}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X, ∅ } and *(gr)-closed sets are {{a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X, ∅ }. Then A = {d} is an αg-closed set but not *(gr)-closed set in X.

Remark 2.6

1. Closed set and gr-closed sets are independent of each other.
2. *(gr)-closed set and sg-closed sets are independent of each other.
3. *(gr)-closed set and gr-closed sets are independent of each other. It is shown by the following example.

Example 2.7

1. Let X= {a, b, c, d} with τ= {∅, X, {a}, {d}, {a, d}, {c, d}, {a, c, d}, {b, c, d}}.The subset {c, d} is *(gr)- closed set but not closed set and the subset {a, c, d} is closed set but not *(gr)- closed set in (X, τ).

2. Let X= {a, b, c, d} with τ= {∅, X, {c},{a, b}, {a, b, c}}.The subset {a, b, d} is *(gr)- closed set but not sg-closed set and the subset {a} is sg-closed set but not *(gr)- closed set in (X, τ).

3. Let X={a, b, c, d} with τ= {∅, X, {c}, {b}, {c}, {b, c}, {b, c, d}}. The subset {a, b} in τ= {∅, X, {b}, {c}, {b, c}, {b, c, d}}. The subset {a, b} is *(gr)- closed but not gα-closed set and the subset {d} is gα-closed set but not *(gr)$^-$closed set in (X, τ).

Theorem 2.8 The finite union of the *(gr)-closed sets is a *(gr)-closed set.

Proof: Let A and B be *(gr)-closed set in X. Let U be a ĝ-open in X such that A∪B ⊆ U. Then A ⊆ U and B ⊆ U. Since A and B are *(gr)-closed sets, rcl(A) ⊆ U and rcl(B) ⊆ U. Hence rcl(A∪B) = rcl(A) ∪ rcl(B) ⊆ U. Therefore A∪B is a *(gr)-closed set, whenever A and B be *(gr)-closed.

Remark 2.9 The finite intersection of two *(gr)-closed sets need not be a *(gr)-closed set.

Example 2.10 Let X = {a, b, c} with τ = {∅, {b}, X}. Let A = {a, b} be a *(gr)- closed set and B = {b, c} be also a *(gr)- closed set. But A ∩ B = {b} is not *(gr)- closed set in (X, τ).

Theorem 2.11 If A ⊆ B ⊆ rcl(A) and A is *(gr)-closed subset of (X, τ) then B is also a *(gr)-closed subset of (X, τ).

Proof: Let U be a ĝ-open subset such that A ⊆ B ⊆ U, since A is *(gr)-closed subset of (X, τ), rcl(A) ⊆ U, by hypothesis A ⊆ B ⊆ rcl(A), rcl(A) = rcl(B). Hence rcl(B) ⊆ U whenever B⊆ U. Therefore B is also a *(gr)-closed subset of (X, τ).

Theorem 2.12 Let A be a ĝ-open subset of (X, τ). Then A is r-closed set if A is *(gr)-closed set.

Proof: Since A is ĝ-open and *(gr)-closed set in (X, τ), rcl(A) ⊆ A.Therefore A is r-closed set.

Theorem 2.13 Let A ⊆ B ⊆ X, where B is ĝ - open and A is *(gr)-closed in B then A is a *(gr)- closed in X.

Proof: Let U be a ĝ-open subset of X such that A ⊆ U. Since A ⊆ U ∩ B, where U ∩ B is ĝ-open is B and A is *(gr)-closed in B, rcl(A) ⊆ U ∩ B holds we have rcl(A) ∩ B ⊆ U ∩ B. Since A ⊆ B we have rcl(A) ⊆ rcl(B). Since B is ĝ-open and *(gr)- closed is X, by the above theorem, B is r-closed. Therefore rcl(B) = B. Thus rcl(B) ⊆ B implies rcl(A)=rcl(A) ∩ B ⊆ U ∩ B ⊆ U. Hence A is *(gr)- closed is X.

Theorem 2.14 A subset A of X is a *(gr)-closed set in X if and only if rcl(A) – A contains no non-empty ĝ-closed set in X.

Proof: Suppose that F is a non-empty ĝ-closed subset of rcl(A) – A. Now F ⊆ rcl(A) – A. Then F ⊆ rcl(A) ∩ A^C. Therefore F ⊆ rcl(A) and F ⊆ A^C. Since F^C is ĝ-open such that A ⊆ F^C and A is *(gr)-closed, rcl(A) ⊆ F^C , ie, F ⊆ $rcl(A)^C$. Hence F ⊆ rcl(A) ∩ $[rcl(A)]^C$ = ∅. ie F = ∅. Thus rcl(A) – A contains no non-empty ĝ-closed set.

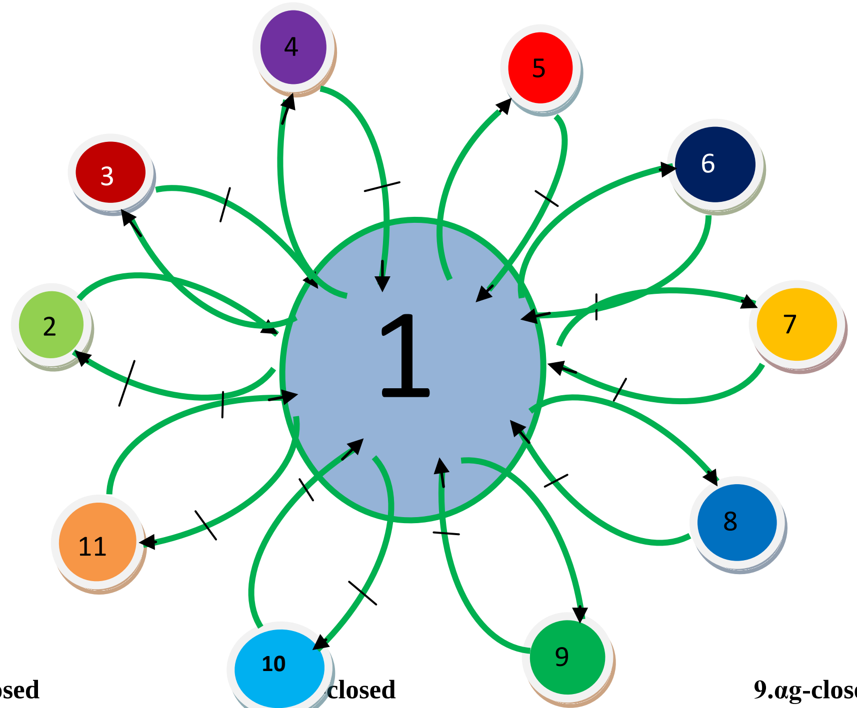
Conversely assume that rcl(A) – A contains no non-empty ĝ-closed set. Let A ⊆ U and U is ĝ-open. Suppose that rcl(A) is not contained in U. Then rcl(A) ∩ U^C is a non-empty ĝ-closed set and contained in rcl(A) – A which is a contradiction. Therefore rcl(A) ⊆ U and hence A is *(gr)-closed set.

Theorem 2.15 For each x ∈ X, either {x} is ĝ-closed or $\{x\}^C$ is *(gr)-closed set in X.

Proof: If {x} is not ĝ-closed, then only ĝ-open set containing $\{x\}^C$ is X. Thus rcl{x^C} is contained in X and hence $\{x\}^C$ is *(gr)-closed set in X.

Remark 2.16

For the subsets defined above, we have the following implications.



1.*(gr)-closed
2.r-closed
3.g-closed
4.rg-closed
closed
6.gp-closed
7.gsp-closed
8.gpr-closed
9.αg-closed
10.gα-closed
11.sg-closed

4. *(gr)-OPEN SETS

Definition 4.1 A subset A of a topological space (X, τ) is called a star generalized regular open set (briefly *(gr)-open) if its complement is *(gr)- closed set.

Theorem 4.2 A subset A of a topological space (X, τ) is *(gr)- open if and only if B ⊆ rint(A) where B is a ĝ-closed subset of X and B ⊆ A.

Proof: Necessity: Let A be *(gr)- open in X with N ⊆ A, where N is ĝ-closed. We have A^C is *(gr)-closed with A^C ⊆ N^C where N^C is ĝ-open. Then rcl(A^C) ⊆ N^C implies N^C ⊆ X-rcl(A^C) = r int(X – A^C) = r int(A).

Sufficiency: Suppose B ⊆ r int (A) where B is ĝ-closed in (X, τ) and B ⊆ A. Let A^C ⊂ M, where M is ĝ-open. Hence M is M^C ⊆ A, where M^C is ĝ-closed. Hence by assumption M^C ⊆ r int(A), which implies (r int(A))^C ⊆ M. Therefore rcl(A^C) ⊆ M. Thus A^C is *(gr)-closed, implies A is *(gr)-open.

Theorem 4.3 Every r-open set is *(gr)-open set.

Proof: Let A be an r-open set. Then X-A is r-closed set. Then by theorem 3.1.2, X-A is *(gr)-closed. Hence A is *(gr)-open set.

Theorem 4.4 If r int(A) ⊆ B ⊆ A and A is a *(gr)-open subset of (X, τ) then B is also a *(gr)-open subset of (X, τ).

Proof: Let r int(A) ⊆ B ⊆ A implies A^C ⊆ B^C ⊆ rcl(A^C). It is known that A^C is *(gr)-closed. Hence by theorem 3.1.13. B^C is *(gr)-closed. Therefore B is *(gr)-open.

Theorem 4.5 If a subset A of a topological space (X, τ) is *(gr)- open in X then F = X, whenever F is ĝ- open and r int(A) ⊆ A^C ⊆ F.

Proof: Let A be *(gr)- open in X and F be ĝ- open, r int(A) ∪ A^C ⊆ F. This gives F^C ⊆ (X-r int(A))∩ A = rcl(A^C) ∩ A = rcl(A^C) – A^C. Since F^C is ĝ-closed and A^C is *(gr)- open by theorem 2.14. we have F^C = ∅. Thus F = X.

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