Backward Alpha Difference Operator with Real Variable and its Finite Series

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Abstract

In this paper, we derive Multi-series of Generalized alpha difference equation with suitable example. Also we find the closed form solution which is coinciding with the infinite summation form solution of the higher order generalized $\alpha_i -$difference equation.

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1 INTRODUCTION

In 1984, Jerzy Popenda [5] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, Miller and Rose [9] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The general fractional h-difference Riemann-Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in [1, 2]. In 2011, M.Maria Susai Manuel, et.al, [8, 11] extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ defined on $u(k)$ as $\Delta_{\alpha(\ell)} v(k) = v(k+\ell) - \alpha v(k)$, where $\alpha \neq 0$, $\ell > 0$ are fixed and $k \in [0,\infty)$ is variable. The results derived in [11] are coincide with the results in [7] when $\alpha = 1$.

An equation involving both Δ and Δ_α is called mixed difference equation. Oscillatory behaviour of solutions certain types of mixed difference equations have been dicussed in [3, 4, 6, 12]. An equation involving Δ_ℓ and $\Delta_{\alpha(\ell)}$ is called as generalized mixed difference equation.

The higher order generalized $\alpha_i -$ difference equation

$$\begin{aligned}\Delta_{\alpha_1(\ell_1)}(\Delta_{\alpha_2(\ell_2)}(\cdots\Delta_{\alpha_n(\ell_n)}(v(k))\cdots)) \\ = u(k), k \in [0,\infty), \ell_i > 0\, \alpha_i \neq 0\end{aligned} \quad (1)$$

becomes generalized mixed difference equation if $\alpha_i = 1$ for some i and $n \geq 2$. The equation (1) has three types of solutions which are closed, finite and infinite multi-series forms. Equation (1) becomes backward alpha difference equation when ℓ_i is replaced by $-\ell_i$.

2 PRELIMINARIES

In this section, we define the generalized backward alpha difference operator and we presents certain results on its inverse alpha difference operator with polynomial and polynomial factorials for positive and negative variable k.

Definition 2.1 *If $v(k)$ is a real valued function on $(-\infty,\infty)$, then the generalized $\alpha -$difference operator for negative ℓ denoted by $\Delta_{\alpha(-\ell)}$ is defined as*

$$\Delta_{\alpha(-\ell)} v(k) = v(k-\ell) - \alpha v(k), \ell \in (0,\infty) \quad (2)$$

The inverse generalized backward $\alpha -$difference operator is defined as

$$\text{if } \Delta_{\alpha(-\ell)} v(k) = u(k), \text{ then } v(k) = \Delta^{-1}_{\alpha(-\ell)} u(k) \quad (3)$$

Definition 2.2 *The higher order generalized backward $\alpha_i -$difference equation is defined as*

$$\begin{aligned}\Delta_{\alpha_1(-\ell_1)}(\Delta_{\alpha_2(-\ell_2)}\cdots\Delta_{\alpha_n(-\ell_n)}(v(k)))) \\ = u(k), k \in [0,\infty), -\ell_i > 0\end{aligned} \quad (4)$$

Lemma 2.3 *If $a^{-s\ell_i} - \alpha_i \neq 0$ for $i = 1,2,\cdots,n$, then we have*

$$\prod_{i=1}^{n} \Delta^{-1}_{\alpha_i(-\ell_i)} a^{sk} = \frac{a^{sk}}{\prod_{i=1}^{n}(a^{-s\ell_i} - \alpha_i)} \quad (5)$$

is a closed form solution of the equation (4) when $u(k) = a^{sk}$.

Proof: Since

$$\Delta_{\alpha_i(-\ell_i)}a^{sk} = a^{s(k-\ell_i)} - \alpha_i a^{sk} = a^{sk}(a^{-s\ell_i} - \alpha_i)$$

,

from (3) we get $\Delta^{-1}_{\alpha_i(-\ell_i)}a^{sk} = \dfrac{a^{sk}}{a^{-s\ell_i} - \alpha_i}$ which yields (5).

3 Finite α summation for positive variable k

Lemma 3.1 *(Finite α -summation formula for $k>0$) For $\ell>0$, we have*

$$\Delta^{-1}_{\alpha(-\ell)}u(k) - \alpha^{[\frac{k}{\ell}]}\Delta^{-1}_{\alpha(-\ell)}u(k+[\frac{k}{\ell}]\ell) = \sum_{r=1}^{[\frac{k}{\ell}]}\alpha^{r-1}u(k+r\ell) \quad (6)$$

Proof: By taking $\Delta^{-1}_{\alpha(-\ell)}u(k) = v(k)$, we get $\Delta_{\alpha(-\ell)}v(k) = u(k)$ and

$$v(k-\ell) = u(k) + \alpha v(k) \quad (7)$$

Replacing k by $k+\ell$ in (7), we get

$$v(k) = u(k+\ell) + \alpha v(k+\ell) \quad (8)$$

Substituting (8) in (7), we get

$$v(k-\ell) = u(k) + \alpha u(k+\ell) + \alpha^2 v(k+\ell) \quad (9)$$

Replacing k by $k+\ell$ in (9), we obtain

$$v(k) = u(k+\ell) + \alpha u(k+2\ell) + +\alpha^2 v(k+2\ell) \quad (10)$$

Substituting (10) in (7), we get

$$v(k-\ell) = u(k) + \alpha u(k+\ell) + \alpha^2 u(k+2\ell) + \alpha^3 v(k+2\ell) \quad (11)$$

Replacing k by $k+\ell$ in (11), we get

$$v(k) = u(k+\ell) + \alpha u(k+2\ell) + \alpha^2 u(k+3\ell) + \alpha^3 v(k+3\ell) \quad (12)$$

Proceeding like this we get

$$v(k) = u(k+\ell) + \cdots + \alpha^{[\frac{k}{\ell}]-1}u(k+[\frac{k}{\ell}]\ell) + \alpha^{[\frac{k}{\ell}]}v(k+[\frac{k}{\ell}]\ell),$$

which gives (6).

Theorem 3.2 *If $a^{-s\ell} \neq \alpha, k>0, \ell>0$, then we have*

$$\sum_{r=1}^{[\frac{k}{\ell}]}\alpha^{r-1}a^{s(k+r\ell)} = \frac{a^{sk}}{a^{-s\ell}-\alpha} - \alpha^{[\frac{k}{\ell}]}\left\{\frac{a^{sj}}{a^{-s\ell}-\alpha}\right\} \quad (13)$$

Proof: The proof follows by taking $u(k) = a^{sk}$ in (6) and putting $n=1$ in (5).

Example 3.3 *Let* $s=1, a=2, k=7, \ell=3, \alpha=2$ *in* (13)

$$\sum_{r=1}^{2}2^{r-1}2^{7+3r} = \frac{2^7}{2^{-3}-2} - 2^2\left\{\frac{2^{13}}{2^{-3}-2}\right\}$$

$$\Rightarrow 17408 = 17408$$

Theorem 3.4 *If $\alpha \neq 1, k>0, \ell>0$, then*

$$\sum_{r=1}^{[\frac{k}{\ell}]}\alpha^{r-1}(k+r\ell)^0 = \frac{1}{(1-\alpha)} - \alpha^{[\frac{k}{\ell}]}\left\{\frac{1}{(1-\alpha)}\right\} \quad (14)$$

Proof: Since $\Delta^{-1}_{\alpha(-\ell)}(1) = \dfrac{1}{1-\alpha}$, the proof follows by taking $u(k) = k^0$ in (6).

Theorem 3.5 *If $\alpha \neq 1, k \in [\ell, 0)$, then we have*

$$\sum_{r=1}^{[\frac{k}{\ell}]}\alpha^{r-1}(k+r\ell) = \frac{k}{1-\alpha} + \frac{\ell}{(1-\alpha)^2} - \alpha^{[\frac{k}{\ell}]}\left\{\frac{j}{(1-\alpha)} + \frac{\ell}{(1-\alpha)^2}\right\} \quad (15)$$

Proof: From (2), we have

$$\Delta_{\alpha(-\ell)}k = (k-\ell) - \alpha k = (1-\alpha)k - \ell$$

$$\Rightarrow k = (1-\alpha)\Delta^{-1}_{\alpha(-\ell)}k - \ell\Delta_{\alpha(-\ell)}$$

$$(1-\alpha)\Delta^{-1}_{\alpha(-\ell)}k = k + \frac{\ell}{1-\alpha}$$

$$\Rightarrow \Delta^{-1}_{\alpha(-\ell)}k\,|_j^k = \frac{k}{1-\alpha} + \frac{\ell}{(1-\alpha)^2}\,|_j^k$$

The proof follows from (6).

Example 3.6 *Let* $k=71, \ell=10, \alpha=7, j=141$ *in* (15)

$$\sum_{r=1}^{7}7^{r-1}(71+10r) = \frac{71}{-6} + \frac{10}{36} - 7^7\left\{\frac{141}{(-6)} + \frac{10}{(36)}\right\}$$

$$\Rightarrow 19124487 = 19124487$$

Theorem 3.7 *If $\alpha \neq 1, k \in [\ell, 0)$, then we have*

$$\sum_{r=1}^{[\frac{k}{\ell}]}\alpha^{r-1}(k+r\ell)^2 = \frac{k^2}{1-\alpha} + \frac{2\ell k - \ell^2}{(1-\alpha)^2} + \frac{2\ell^2}{(1-\alpha)^3} - \alpha^{[\frac{k}{\ell}]}\left\{\frac{j^2}{1-\alpha} + \frac{2\ell j - \ell^2}{(1-\alpha)^2} + \frac{2\ell^2}{(1-\alpha)^3}\right\} \quad (16)$$

Proof: From (2), we have

$$\Delta_{\alpha(-\ell)}k^2 = (k-\ell)^2 - \alpha k^2$$
$$= (1-\alpha)k^2 - 2\ell k + \ell^2$$
$$\Rightarrow k^2 = (1-\alpha)k^2 - 2\ell\Delta^{-1}_{\alpha(-\ell)}(k) + \ell\Delta^{-1}_{\alpha(-\ell)}(1)$$
$$\Rightarrow (1-\alpha)k^2 - 2\ell\Delta^{-1}_{\alpha(-\ell)}(k) = k^2 - \ell\Delta^{-1}_{\alpha(-\ell)}(1)$$
$$\Rightarrow k^2 \,|_j^k = \left\{\frac{k^2}{1-\alpha} + \frac{2\ell k - \ell^2}{(1-\alpha)^2} + \frac{2\ell^2}{(1-\alpha)^3}\right\}|_j^k$$

The proof follows from (6).

Example 3.8 *Let* $k = 31, \ell = 7, \alpha = 2, j = 59$ *in (16)*

$$\sum_{r=1}^{4} 2^{r-1}(31+7r)^2 = \frac{31^2}{1-2} + \frac{2.7.31-7^2}{(1-2)^2} + \frac{2.7^2}{(1-2)^3}$$
$$\Rightarrow 44158 = 44158$$

Theorem 3.9 *If* $\alpha \neq 1, k \in [\ell, 0)$ *, then we have*

$$\sum_{r=1}^{[\frac{k}{\ell}]} \alpha^{r-1}(k+r\ell)^3 = \frac{k^3}{1-\alpha} + \frac{3\ell k^2 - 3\ell^2 k + \ell^3}{(1-\alpha)^2} + \frac{6\ell^2 k - 6\ell^3}{(1-\alpha)^3} + \frac{6\ell^3}{(1-\alpha)^4} \tag{17}$$
$$-\alpha^{[\frac{k}{\ell}]}\left\{\frac{j^3}{1-\alpha} + \frac{3\ell j^2 - 3\ell^2 j + \ell^3}{(1-\alpha)^2} + \frac{6\ell^2 j - 6\ell^3}{(1-\alpha)^3} + \frac{6\ell^3}{(1-\alpha)^4}\right\}$$

Proof: From (2), $\Delta_{\alpha(-\ell)}k^3 = (k-\ell)^3 - \alpha k^3$

$$= k^3(1-\alpha) - 3k^2\ell + 3k\ell^2 - \ell^3$$
$$k^3 = (1-\alpha)\Delta^{-1}_{\alpha(-\ell)}k^3 - 3\ell\Delta^{-1}_{\alpha(-\ell)}k^2 + 3\ell^2\Delta^{-1}_{\alpha(-\ell)}k - \ell^3\Delta^{-1}_{\alpha(-\ell)}(1)$$

$$\Delta^{-1}_{\alpha(-\ell)}k^3 = \frac{k^3}{1-\alpha} + \frac{3\ell}{1-\alpha}\Delta^{-1}_{\alpha(-\ell)}k^2 k^2 - \frac{3\ell^2}{1-\alpha}\Delta^{-1}_{\alpha(-\ell)}k + \frac{\ell^3}{1-\alpha}\Delta^{-1}_{\alpha(-\ell)}(1)$$

$$\Delta^{-1}_{\alpha(-\ell)}k^3 \,|_j^k = \frac{k^3}{1-\alpha} + \frac{3\ell k^2 - 3\ell^2 k + \ell^3}{(1-\alpha)^2} + \frac{6\ell^2 k - 6\ell^3}{(1-\alpha)^3} + \frac{6\ell^3}{(1-\alpha)^4}\,|_j^k$$

The proof follows from (6).

Theorem 3.10 *If* $\alpha \neq 1, k \in [\ell, 0)$ *, then we have*

$$\sum_{r=1}^{[\frac{k}{\ell}]} \alpha^{r-1}(k+r\ell)^n = \frac{1}{1-\alpha}\left\{k^n - \sum_{r=1}^{n}(-1)^r nc_r \ell^r \Delta^{-1}_{\alpha(-\ell)}k^{n-r}\right\} \tag{18}$$

Proof: The proof follows by continuing the process of Theorem 3.5, 3.7 and 3.9.

Theorem 3.11 *If* $\alpha_1, \alpha_2 \neq 1, k \in (0, \infty)$*, then we have*

$$\sum_{r_1=1}^{[\frac{k}{\ell_1}]}\sum_{r_2=1}^{[\frac{k+r_1\ell_1}{\ell_2}]} \alpha_1^{r_1-1}\alpha_2^{r_2-1}u(k+r_1\ell_1+r_2\ell_2) = \Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k)$$
$$-\alpha_1^{[\frac{k}{\ell_1}]}\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k+[\frac{k}{\ell_1}]\ell_1)$$
$$-\sum_{r_1=1}^{[\frac{k}{\ell_1}]}\alpha_2^{[\frac{k+r_1\ell_1}{\ell_2}]}\alpha_1^{r_1-1}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k+r_1\ell_1+[\frac{k+r_1\ell_1}{\ell_2}]\ell_2) \tag{19}$$

Proof: Replacing $\ell = \ell_2, \alpha = \alpha_2$ Lemma(3.1), we get

$$\Delta^{-1}_{\alpha_2(-\ell_2)}u(k) - \alpha_2^{[\frac{k}{\ell_2}]}\Delta^{-1}_{\alpha_{\alpha_2(-\ell_2)}}u(k+[\frac{k}{\ell_2}]\ell_2)$$
$$= u(k+\ell_2) + \alpha_2 u(k+2\ell_2) + \cdots + \alpha_2^{[\frac{k}{\ell_2}]-1}u(k+[\frac{k}{\ell_2}]\ell_2)$$

Replacing k by $k+\ell_1$

$$\Delta^{-1}_{\alpha_{2(-\ell_2)}}u(k+\ell_1) - \alpha_2^{[\frac{k+\ell_1}{\ell_2}]}\Delta^{-1}_{\alpha_{\alpha_2(-\ell_2)}}u(k+\ell_1+[\frac{k+\ell_1}{\ell_2}]\ell_2)$$
$$= u(k+\ell_1+\ell_2) + \alpha_2 u(k+\ell_1+2\ell_2) + \cdots + \alpha_2^{[\frac{k+\ell_1}{\ell_2}]-1}u(k+\ell_1+[\frac{k+\ell_1}{\ell_2}]\ell_2)$$

Replacing k by $k+2\ell_1$ and multiplying α_1 on both sides, we get

$$\Delta^{-1}_{\alpha_2(-\ell_2)}u\Delta(k+2\ell_1)-\alpha_2^{[\frac{k+2\ell_1}{\ell_2}]}\Delta^{1-}_{\alpha_2(-\ell_1)}u(k+2\ell_1+[\frac{k+2\ell_1}{\ell_2}]$$

$$=u(k+2\ell_1+\ell_2)+\alpha_2 u(k+2\ell_1+2\ell_2)+\cdots$$

$$+\alpha_2^{[\frac{k+2\ell_1}{\ell_2}]-1}u(k+2\ell_1+[\frac{k+2\ell_1}{\ell_2}]$$

Replacing k by $k+3\ell_1$ and multiplying α_1^2 on both sides, we get

$$\Delta^{-1}_{\alpha_2(-\ell_2)}u\Delta(k+3\ell_1)-\alpha_2^{[\frac{k+3\ell_1}{\ell_2}]}\Delta^{1-}_{\alpha_2(-\ell_1)}u(k+3\ell_1+[\frac{k+3\ell_1}{\ell_2}]$$

$$=u(k+3\ell_1+\ell_2)+\alpha_2 u(k+3\ell_1+2\ell_2)+\cdots$$

$$+\alpha_2^{[\frac{k+3\ell_1}{\ell_2}]-1}u(k+2\ell_1+[\frac{k+3\ell_1}{\ell_2}]$$

Proceeding like this which gives (18).

Corollary 3.12 *If* $\alpha_1=\alpha_2=\alpha$ *in Theorem 3.11, then we have*

$$\sum_{r_1=1}^{[\frac{k}{\ell_1}]}\sum_{r_2=1}^{[\frac{k+r_1\ell_1}{\ell_2}]}\alpha^{r_1+r_2-2}u(k+r_1\ell_1+r_2\ell_2)=\Delta^{-1}_{\alpha(-\ell_1)}\Delta^{-1}_{\alpha(-\ell_2)}u(k)$$

$$-\alpha^{[\frac{k}{\ell_1}]}\Delta^{-1}_{\alpha(-\ell_1)}\Delta^{-1}_{\alpha(-\ell_2)}u(k+[\frac{k}{\ell_1}]\ell_1)$$

$$-\sum_{r_1=1}^{[\frac{k}{\ell_1}]}\alpha^{[\frac{k+r_1\ell_1}{\ell_2}]+r_1-1}\Delta^{-1}_{\alpha(-\ell_2)}u(k+r_1\ell_1+[\frac{k+r_1\ell_1}{\ell_2}]\ell_2)$$

Example 3.13 *Putting* $k=7,\ell_1=2,\ell_2=3,\alpha_1=\alpha_2=2$ *in corollary 3.12*

$$\sum_{r_1=1}^{3}\sum_{r_2=1}^{[\frac{7+2r_1}{3}]}2^{r_1+r_2-2}(7+2r_1+3r_2)=\frac{7}{(1-2)^2}+\frac{5}{(1-2)^3}$$

$$-2^3\left\{\frac{13}{(1-2)^2}+\frac{5}{(1-2)^3}\right\}$$

$$-\sum_{r_1=1}^{3}\left\{2^{[\frac{7+2r_1}{3}]+r_1-1}+[\frac{7+2r_1}{3}]3+\frac{3}{(1-2)^2}\right\}$$

$$\Rightarrow 1738=1738$$

Theorem 3.14 *If* $\alpha,\ell>0,k\in(0,\infty)$ *, then we have*

$$\sum_{r_1=1}^{[\frac{k}{\ell_1}]}\sum_{r_2=1}^{[\frac{k+r_1\ell_1}{\ell_2}]}\sum_{r_3=1}^{[\frac{k+r_1\ell_1+r_2\ell_2}{\ell_3}]}\alpha_1^{r_1-1}\alpha_2^{r_1-1}\alpha_3^{r_3-1}u(k+r_1\ell_1+r_2\ell_2+r_3\ell_3)$$

$$=\Delta^{-1}_{\alpha(-\ell_1)}\Delta^{-1}_{\alpha(-\ell_2)}\Delta^{-1}_{\alpha_3(-\ell_3)}u(k)$$

$$-\alpha_1^{[\frac{k}{\ell_1}]}\Delta^{-1}_{\alpha(-\ell_1)}\Delta^{-1}_{\alpha(-\ell_2)}\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+[\frac{k}{\ell_1}]\ell_1)$$

$$-\sum_{r_1=1}^{[\frac{k}{\ell_1}]}\alpha_2^{[\frac{k+r_1\ell_1}{\ell_2}]}\alpha_1^{r_1-1}\Delta^{-1}_{\alpha(-\ell_2)}\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+r_1\ell_1+[\frac{k+r_1\ell_1}{\ell_2}]\ell_2)$$

$$-\sum_{r_1=1}^{[\frac{k}{\ell_1}]}\sum_{r_2=1}^{[\frac{k+r_1\ell_1}{\ell_2}]}\alpha_3^{[\frac{k+r_1\ell_1+r_2\ell_2}{\ell_3}]}\alpha_2^{[\frac{k+r_1\ell_1}{\ell_2}]}\alpha_1^{r_1-1}$$

$$\times\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+r_1\ell_1+r_2\ell_2+[\frac{k+r_1\ell_1+r_2\ell_2}{\ell_3}]\ell_3) \qquad (20)$$

Proof: Replacing r_1,r_2 and ℓ_1,ℓ_2 by r_2,r_3 and ℓ_2,ℓ_3 and α_1,α_2 by α_2,α_3 in (19), we get

$$\sum_{r_2=1}^{[\frac{k}{\ell_2}]}\sum_{r_3=1}^{[\frac{k+r_2\ell_2}{\ell_2}]}\alpha_2^{r_2-1}\alpha_3^{r_3-1}u(k+r_2\ell_2+r_3\ell_3)=\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k)$$

$$-\alpha_2^{[\frac{k}{\ell_2}]}\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+[\frac{k}{\ell_2}]\ell_2)$$

$$-\sum_{r_2=1}^{[\frac{k}{\ell_2}]}\alpha_3^{[\frac{k+r_2\ell_2}{\ell_3}]}\alpha_2^{r_2-1}\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+r_2\ell_2+[\frac{k+r_2\ell_2}{\ell_3}]\ell_3)$$

Replacing k by $k+\ell_1$ and multiplying α_1 on both sides, we get

$$\sum_{r_2=1}^{[\frac{k+\ell_1}{\ell_2}]}\sum_{r_3=1}^{[\frac{k+\ell_1+r_2\ell_2}{\ell_2}]}\alpha_2^{r_2-1}\alpha_3^{r_3-1}u(k+\ell_1+r_2\ell_2+r_3\ell_3)$$

$$=\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k+\ell_1)$$

$$-\alpha_2^{[\frac{k+\ell_1}{\ell_2}]}\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+\ell_1+[\frac{k+\ell_1}{\ell_2}]\ell_2)$$

$$-\sum_{r_2=1}^{[\frac{k+\ell_1}{\ell_2}]}\alpha_3^{[\frac{k+\ell_1+r_2\ell_2}{\ell_3}]}\alpha_2^{r_2-1}$$

$$\times\,\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+\ell_1+r_2\ell_2+[\frac{k+\ell_1+r_2\ell_2}{\ell_3}]\ell_3)$$

Replacing k by $k+2\ell_1$ and multiplying α_1^2 on both sides

$$\sum_{r_2=1}^{[\frac{k+2\ell_1}{\ell_2}]}\sum_{r_3=1}^{[\frac{k+2\ell_1+r_2\ell_2}{\ell_2}]}\alpha_2^{r_2-1}\alpha_3^{r_3-1}u(k+2\ell_1+r_2\ell_2+r_3\ell_3)$$

$$=\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k+2\ell_1)$$

$$-\alpha_2^{[\frac{k+2\ell_1}{\ell_2}]}\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+2\ell_1+[\frac{k+2\ell_1}{\ell_2}]\ell_2)$$

$$-\sum_{r_2=1}^{[\frac{k+2\ell_1}{\ell_2}]}\alpha_3^{[\frac{k+2\ell_1+r_2\ell_2}{\ell_3}]}\alpha_2^{r_2-1}$$

$$\times\Delta^{-1}_{\alpha_3(-\ell_3)}u(k+2\ell_1+r_2\ell_2+[\frac{k+2\ell_1+r_2\ell_2}{\ell_3}]\ell_3)$$

Proceeding like this we get (20).

Theorem 3.15 *(α_i- higher order summation formula) If $k\in[0,\infty)$ and $\ell_i>0$ and*

$$\sum_{(r\ell)_{1\to i}}^{[k]}=\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]}\cdots\sum_{r_i=0}^{\left[\frac{k-r_1\ell_1-r_2\ell_2-\cdots-r_{i-1}\ell_{i-1}}{\ell_1}\right]},$$ *then*

we have

$$\sum_{i=1}^{n}\sum_{(r\ell)_{1\to i}}^{[k]}\prod_{t=1}^{i}\alpha_{[1\to i]}^{\sum_{t=1}^{i}r_t-1}u(k+\sum_{t=1}^{i}r_t\ell_t)=\Delta^{-1}_{(\alpha(-\ell))_{[n]}}u(k)$$

$$-\alpha^{[\frac{k}{\ell_1}]}\Delta^{-1}_{(\alpha(-\ell))_{[n]}}\hat{\ell}_1(k)$$

$$-\sum_{(r\ell)_{[1\to m]}}^{[k]}\alpha_{n+1}^{[\frac{k+(r\ell)_{[1\to m]}}{\ell_{n+1}}]}\alpha_n^{[\frac{k+(r\ell)_{[1\to(m-1)]}}{\ell_n}]}$$

$$\times\alpha_{n-1}^{[\frac{k+(r\ell)}{\ell_{n-1}}]}\alpha_{n-2}^{r-1}\Delta^{-1}_{(\alpha(-\ell))_{[n-m]}}\hat{\ell}_m(k)$$

Proof: The proof follows by Theorem 3.11 Theorem 3.14 and Lemma 3.1

4. FINITE α SUMMATION FOR NEGATIVE VARIABLE k

Lemma 4.1 *(Finite $(\frac{1}{\alpha})$-summation formula for $k<0$) For $\ell>0$, we have*

$$\Delta^{-1}_{\alpha(-\ell)}u(k)-\frac{1}{\alpha^{[-\frac{k}{\ell}]+1}}\Delta^{-1}_{\alpha(-\ell)}u(k-([-\frac{k}{\ell}]+1)\ell)$$

$$=\sum_{r=0}^{[-\frac{k}{\ell}]}\frac{-1}{\alpha^{r+1}}u(k-r\ell) \tag{21}$$

Proof: By taking $\Delta^{-1}_{-\ell}u(k)=v(k)$, we get $\Delta_{-\ell}v(k)=u(k)$, which gives

$$v(k)=\frac{-1}{\alpha}u(k)+\frac{1}{\alpha}v(k-\ell) \tag{22}$$

Replacing k by $k-\ell$ in (22), we get

$$v(k-\ell)=\frac{-1}{\alpha}u(k-\ell)+\frac{1}{\alpha}v(k-2\ell) \tag{23}$$

Substituting (24) in (22), we get

$$v(k)=\frac{-1}{\alpha}u(k)-\frac{1}{\alpha^2}u(k-\ell)+\frac{1}{\alpha^2}v(k-2\ell) \tag{24}$$

Replacing k by $k-\ell$ in (24), we obtain

$$v(k-\ell)=\frac{-1}{\alpha}u(k-\ell)-\frac{1}{\alpha^2}u(k-2\ell)+\frac{1}{\alpha^2}v(k-3\ell) \tag{25}$$

Substituting (25) in (22), we get

$$v(k)=\frac{-1}{\alpha}u(k)-\frac{1}{\alpha^2}u(k-\ell)-\frac{1}{\alpha^3}u(k-2\ell)$$

$$+\frac{1}{\alpha^3}v(k-3\ell)$$

Proceeding like this we get (21).

Theorem 4.2 *If $\alpha_1,\alpha_2\neq 1, k\in(0,\infty)$, then we have*

$$\sum_{r_1=0}^{[\frac{-k}{\ell_1}]}\sum_{r_2=0}^{[-(\frac{k+r_1\ell_1}{\ell_2})]}(\frac{-1}{\alpha_1^{r_1+1}})(\frac{-1}{\alpha_2^{r_2+1}})u(k-r_1\ell_1-r_2\ell_2)$$

$$= \Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k) - (\frac{-1}{\alpha_1^{[\frac{-k}{\ell_1}]+1}})$$

$$\times \Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k-([\frac{-k}{\ell_1}]+1)\ell_1)$$

$$+\sum_{r_1=0}^{[\frac{-k}{\ell_1}]}(\frac{1}{\alpha_1^{r_1+1}})(\frac{1}{\alpha_2^{[-(\frac{k-r_1\ell_1}{\ell_2})]+1}}) \qquad (26)$$

$$\times \Delta^{-1}_{\alpha_2(-\ell_2)}u(k-r_1\ell_1-([\frac{-(k-r_1\ell_1)}{\ell_2}]+1)\ell_2)$$

Proof: Replacing $\ell = \ell_1$ and $\alpha = \alpha_1$ in (21), we get

$$\Delta^{-1}_{\alpha_1(-\ell_1)}u(k) - \frac{1}{\alpha_1^{[\frac{-k}{\ell_1}]+1}}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k-([\frac{-k}{\ell_1}]+1)\ell_1)$$

$$= (\frac{-1}{\alpha_1})u(k) - (\frac{-1}{\alpha_1^2})u(k-\ell_1) - (\frac{1}{\alpha_1^3})u(k-2\ell_1)\cdots$$

$$-(\frac{1}{\alpha_1^{[\frac{-k}{\ell_1}]+1}})u(k-[\frac{-k}{\ell}]\ell_1)$$

Replacing ℓ_1 by ℓ_2 and α_1 by α_2 and multiplying $(\frac{-1}{\alpha_1})$ on both sides, we get

$$\Delta^{-1}_{\alpha_2(-\ell_2)}u(k) - \frac{1}{\alpha_2^{[\frac{-k}{\ell_2}]+1}}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-([\frac{-k}{\ell_2}]+1)\ell_2)$$

$$= (\frac{-1}{\alpha_2})u(k) - (\frac{-1}{\alpha_2^2})u(k-\ell_2) - (\frac{1}{\alpha_2^3})u(k-2\ell_2)\cdots$$

$$-(\frac{1}{\alpha_2^{[\frac{-k}{\ell_2}]+1}})u(k-[\frac{-k}{\ell}]$$

Replacing k by $k-\ell_1$ and multiplying $(\frac{-1}{\alpha_1^2})$ both sides, we get

$$\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-\ell_1)$$

$$-\frac{1}{\alpha_2^{[\frac{-(k-\ell_1)}{\ell_2}]+1}}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-\ell_1-([\frac{-(k-\ell_1)}{\ell_2}]+1)\ell_2)$$

$$= (\frac{-1}{\alpha_2})u(k-\ell_1) - (\frac{1}{\alpha_2^2})u(k-\ell_1-\ell_2)$$

$$-(\frac{1}{\alpha_2^3})u(k-\ell_1-2\ell_2)\cdots$$

$$-(\frac{1}{\alpha_2^{[\frac{-(k-\ell_1)}{\ell_2}]+1}})u(k-\ell_1-[\frac{-(k-\ell_1)}{\ell}]\ell_2)$$

Replacing k by $k-2\ell_1$ and multiplying $(\frac{-1}{\alpha_1^3})$ both sides

$$\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-2\ell_1)$$

$$-\frac{1}{\alpha_2^{[\frac{-(k-2\ell_1)}{\ell_2}]+1}}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-2\ell_1-([\frac{-(k-2\ell_1)}{\ell_2}]+1)\ell_2)$$

$$= (\frac{-1}{\alpha_2})u(k-2\ell_1) - (\frac{1}{\alpha_2^2})u(k-2\ell_1-\ell_2)$$

$$-(\frac{1}{\alpha_2^3})u(k-2\ell_1-2\ell_2)\ldots$$

$$-(\frac{1}{\alpha_2^{[\frac{-(k-2\ell_1)}{\ell_2}]+1}})u(k-2\ell_1-[\frac{-(k-2\ell_1)}{\ell}]\ell_2)$$

Proceeding like this we get (26).

Theorem 4.3 *If* $\frac{1}{\alpha} > 0, k \in (0,\infty)$, *then we have*

$$\sum_{r_1=0}^{[\frac{-k}{\ell_1}]}\ \sum_{r_2=0}^{[-(\frac{k+r_1\ell_1}{\ell_2})]}\ \sum_{r_3=0}^{[-(\frac{k-r_1\ell_1-r_2\ell_2}{\ell_3})]}(\frac{-1}{\alpha_1^{r_1+1}})(\frac{-1}{\alpha_2^{r_2+1}})(\frac{-1}{\alpha_3^{r_3+1}})$$

$$\times u(k-r_1\ell_1-r_2\ell_2-r_3\ell_3)$$

$$= \Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k)$$

$$+(\frac{1}{\alpha_1^{[\frac{-k}{\ell_1}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k-([\frac{-k}{\ell_1}]+1)\ell_1)$$

$$+\sum_{r_1=0}^{[\frac{-k}{\ell_1}]}\frac{1}{\alpha_1^{r_1+1}}\alpha_2^{-[-(\frac{k-r_1\ell_1}{\ell_2})]+1}$$

$$\times\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-r_1\ell_1-([\frac{-(k-r_1\ell_1)}{\ell_2}]+1)\ell_2)$$

$$+\sum_{r_1=0}^{[-\frac{k}{\ell}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{-\ell_2}]}\alpha_3^{-[\frac{k-r-1\ell-1+r_2\ell_2}{-\ell_3}]}\alpha_2^{-[\frac{k-r-1\ell-1}{-\ell_2}]}\alpha_1^{-[\frac{k-r\ell}{-\ell_1}]}$$

$$\Delta^{-1}_{\alpha_3(-\ell_3)}u(k-r_1\ell_1-r_2\ell_2-[-(\frac{k-r_1\ell_1-r_2\ell_2}{\ell_3})]\ell_3)\ (27)$$

Proof: Replacing r_1,r_2 by r_2,r_3 and ℓ_1,ℓ_2 by ℓ_2,ℓ_3 and α_1,α_2 by α_2,α_3 in 26

$$\sum_{r_2=0}^{[\frac{-k}{\ell_2}]}\sum_{r_3=0}^{[-(\frac{k+r_2\ell_2}{\ell_3})]}(\frac{-1}{\alpha_2^{r_2+1}})(\frac{-1}{\alpha_3^{r_3+1}})u(k-r_2\ell_2-r_3\ell_3)$$

$$=\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k)$$

$$-(\frac{-1}{\alpha_2^{[\frac{-k}{\ell_2}]+1}})\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-([\frac{-k}{\ell_2}]+1)\ell_2)$$

$$+\sum_{r_2=0}^{[\frac{-k}{\ell_2}]}(\frac{1}{\alpha_2^{r_2+1}})(\frac{1}{\alpha_3^{[-(\frac{k-r_2\ell_2}{\ell_3})]+1}})$$

$$\times\Delta^{-1}_{\alpha_3(-\ell_3)}u(k-r_2\ell_2-([\frac{-(k-r_2\ell_2)}{\ell_3}]+1)\ell_3)$$

Replacing k by $k-\ell_1$ and multiplying $\frac{1}{\alpha_1}$ on both sides

$$\sum_{r_2=0}^{[\frac{-(k+\ell_1)}{\ell_2}]}\sum_{r_3=0}^{[-(\frac{k-\ell_1+r_2\ell_2}{\ell_3})]}(\frac{-1}{\alpha_2^{r_2+1}})(\frac{-1}{\alpha_3^{r_3+1}})u(k-\ell_1-r_2\ell_2-r_3\ell_3)$$

$$=\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-\ell_1)$$

$$-(\frac{-1}{\alpha_2^{[\frac{-k-\ell_1}{\ell_2}]+1}})\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-\ell_1-([\frac{-k-\ell_1}{\ell_2}]+1)\ell_2)$$

$$+\sum_{r_2=0}^{[\frac{-k-\ell_1}{\ell_2}]}(\frac{1}{\alpha_2^{r_2+1}})(\frac{1}{\alpha_3^{[-(\frac{k-\ell_1-r_2\ell_2}{\ell_3})]+1}})$$

$$\times\Delta^{-1}_{\alpha_3(-\ell_3)}u(k-\ell_1-r_2\ell_2-([\frac{-(k-\ell_1-r_2\ell_2)}{\ell_3}]+1)\ell_3)$$

Proceeding like this we get 27.

Theorem 4.4 *(α_i- higher order summation formula) If $k\in(0,\infty)$ and $\ell_i>0$ and*

$$\sum_{(r\ell)_{1\to i}}^{[-k]}=\sum_{r_1=0}^{\left[\frac{-k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{-\ell_2}\right]}\cdots\sum_{r_i=0}^{\left[\frac{k-r_1\ell_1-r_2\ell_2-\cdots-r_{i-1}\ell_{i-1}}{\ell_i}\right]}$$ *, then we have*

$$\sum_{i=1}^{n}\sum_{(r\ell)_{1\to i}}^{[-k]}\prod_{t=1}^{i}(\frac{-1}{\alpha_{[1\to i]}^{\sum_{t=1}^{i}r_t-1}})u(k-\sum_{t=1}^{i}r_t\ell_t)$$

$$=\Delta^{-1}_{(\alpha(-\ell))_{[n]}}u(k)-\frac{1}{\alpha^{\frac{k}{\ell_1}}}\Delta^{-1}_{(\alpha(-\ell))_{[n]}}\hat{\ell}_1(k)$$

$$-\sum_{(r\ell)_{[1\to m]}}^{[-k]}\frac{1}{\alpha_{n+1}^{(-[\frac{k+(r\ell)_{[1\to m]}}{\ell_{n+1}}])}}\frac{1}{\alpha_n^{(-[\frac{k+(r\ell)_{[1\to(m-1)]}}{\ell_n}])}}\frac{1}{\alpha_{n-1}^{(-[\frac{k+(r\ell)}{\ell_{n-1}}])}}$$

$$\times(-\frac{1}{\alpha_{n-2}^{r-1}})\Delta^{-1}_{(\alpha(-\ell))_{[n-m]}}\hat{\ell}_m(k)$$

Proof: The proof follows by Lemma 4.1,Theorem 4.2 and Theorem 4.3.

Corollary 4.5 *(α - higher order summation formula) If $k\in[0,\infty)$ and $\ell_i>0$, then we have*

$$\sum_{i=1}^{n}\sum_{(r\ell)_{1\to i}}^{[-k]}\prod_{t=1}^{i}(\frac{-1}{\alpha^{\sum_{t=1}^{i}r_t-1}})u(k-\sum_{t=1}^{i}r_t\ell_t)=\Delta^{-1}_{(\alpha(-\ell)_{[n]})}u(k)$$

$$-\frac{1}{\alpha^{\frac{k}{\ell_1}}}\Delta^{-1}_{(\alpha(-\ell)_{[n]})}\hat{\ell}_1(k)$$

$$-\sum_{(r\ell)_{[1\to m]}}^{[-k]} \frac{1}{\alpha^{(-[\frac{k+(r\ell)_{[1\to m]}}{\ell_{n+1}}])}} \frac{1}{\alpha^{(-[\frac{k+(r\ell)_{[1\to(m-1)]}}{\ell_n}])}} \frac{1}{\alpha^{(-[\frac{k+(r\ell)}{\ell_{n-1}}])}} \times (-\frac{1}{\alpha^{r-1}})\Delta^{-1}_{(\alpha(-\ell)_{[n-m]})}\hat{\ell}_m(k)$$

Proof: The proof follows by Theorem 4.4 when $\alpha_i = \alpha$.

Corollary 4.6 *(Higher order summation formula)* *If* $k \in [0,\infty)$ *and* $\ell_i > 0$, *then we have*

$$\Delta^{-1}_{((-\ell)_{[n]})}u(k) - \Delta^{-1}_{((-\ell)_{[n]})}\hat{\ell}_1(k) - \sum_{(r\ell)_{[1\to m]}}^{[-k]} \Delta^{-1}_{((-\ell)_{[n-m]})}\hat{\ell}_m(k)$$

$$= -\sum_{i=1}^{n} \sum_{(r\ell)_{1\to i}}^{[-k]} u(k - \sum_{t=1}^{i} r_t \ell_t)$$

Proof: The proof follows by corollary 4.5 when $\alpha = 1$.

5 CONCLUSION

We have obtained formulas for several finite $\alpha-$ series on polynomial using inverse of generalized alpha difference operator.

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