Isometries of Euler - Lagrange Additive Mapping in Quasi-Banach Spaces

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Abstract: In this paper, we prove the generalized Hyers-Ulam stability of the isometric Euler-Lagrange additive functional equation in quasi-Banach spaces.

$$\sum_{i=1}^{d} r_i\, f\left(\sum_{j=1}^{d} r_j\,(x_i - x_j)\right) + \left(\sum_{i=1}^{d} r_i\right) f\left(\sum_{i=1}^{d} r_i\, x_i\right) = \left(\sum_{i=1}^{d} r_i\right)\sum_{i=1}^{d} r_i\, f(x_i),$$

where $r_i \in (0,\infty)$ in generalized quasi-Banach spaces.

Key words and phrases: Generalized Hyers-Ulam stability, Euler-Lagrange Functional Equations, isometry, quasi- Banach Spaces.

2010 Mathematics Subject classification: 39B72, 46B03, 47B48.

I. INTRODUCTION

A basic question in the theory of Functional Equations is as follows: "When is it true that a function, which approximately satisfies a functional equation must be close to an exact solution of the equation?"

If the problem accepts a solution, we say the equation is stable. The first stability problem concerning Group Homomorphisms was raised by Ulam [20] in 1940 and affirmatively solved by Hyers [8]. The result of Hyers was generalized by Aoki [1] for approximate additive mappings and by Rassias [17] for approxmiate linear mappings by allowing the difference Cauchy Equation $\|f(x+y) - f(x) - f(y)\|$ to be controlled by $\epsilon\,(\|x\|^p + \|y\|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of Functional Equations, the stability phenomenon that was proved by Rassias is called the Hyers-Ulam Rassias stability. In 1994, a generalization of Rassias theorem was obtained by Gavruta [7], who replaced $\epsilon\,(\|x\|^p + \|y\|^p)$ by a general control function $\varphi\,(x,y)$. Since then the stability problems of various functional equations and mappings, such as the Cauchy Equation, the Jenson Equation, the Quadratic Equation, the Cubic Equation, Homomorphisms, Derivations and their Pexiderized versions with more general domains and ranges have been investigated by a number of authors (see [3], [9], [10], [18], [19]).

J.M. Rassias [14], [15] introduced and investigated the stability problem of Ulam for the Euler-Lagrange Quadratic Mappings

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

and

$$f(a_1x_1 + a_2x_2) + f(a_1x_1 - a_2x_2) = (a_1^2 + a_2^2)[f(x_1)\, f(x_2)]$$

(1) Grabiec [5] has generalized these results mentioned above. In addition, J.M. Rassias [16] generalized the Euler-Lagrange Quadratic Mapping (1) and investigated its stability problem. Thus these Euler-Lagrange type equations (mappings) are called as Euler-Lagrange-Rassias Functional Equations (mappings).

Recently, C. Park and J. Park [13] introduced and investigated the following additive functional equation of Euler-Lagrange type

$$\sum_{i=1}^{n} r_i\, L\left(\sum_{j=1}^{n} r_j\,(x_i - x_j)\right) + \left(\sum_{i=1}^{n} r_i\right) L\left(\sum_{i=1}^{n} r_i\, x_i\right) = \left(\sum_{i=1}^{n} r_i\right) \sum_{i=1}^{n} r_i\, L(x_i) \qquad (2)$$

where $r_1 \ldots r_n \in (0,\infty)$, whose solution is said to be a generalized additive mapping of Euler-Lagrange type.

Abbas Najati and Choonkil Park, introduced the following additive functional equation of Euler-Lagrange type which is somewhat different from (2) :

$$\sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j} r_i x_j - \frac{1}{2} r_i x_j\right) + \sum_{i=1}^{n} r_i f(x_i)$$

$$= nf\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_j\right)$$

(3) where $r_1, r_2 \dots r_n \in \mathbb{R}$, and investigated the generalized Hyers-Ulam stability of the functional equation (3) in Banach modules over a C^* - algebra. These results are applied to investigate C^* - algebra Homomorphisms in unital C^* - algebras. The stability of isometries in norm spaces and Banach spaces was investigated in several papers [2], [4], [6],[11]. However C. G. Park and Th. M. Rassias [12] proved the Hyers-Ulam stability of isometric Cauchy additive functional equation in quasi Banach spaces. Zhihua Wang and Wanxiong Zhang proved the Hyers-Ulam-Rassias stability of isometries and Homomorphisms for the following additive functional equations in quasi-Banach Algebra.

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\ne 1}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} x_i\right) \quad (4)$$

In this paper, we prove the generalized Hyers-Ulam stability of the isometric Euler-Lagrange additive functional equation (2) in quasi-Banach spaces.

We now give some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 Quasi-Norm *Let*X be a real linear space. A quasi-norm is a real-valued function on X satisying the following*:*

1. $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$.

2. $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and for all $x \in X$.

3. $\|x + y\| \le k(\|x\| + \|y\|)$ where $k \ge 1$ is constant and for all $x, y \in X$.

The pair $(X, \|.\|)$ is called a quasi- normed space if $\|.\|$ is a quasi-norm on X. A quasi-Banach space is a complete quasi-normed space.

Definition 1.2 Generalized quasi-norm Let X be a linear space. A generalized quasi-norm is a real valued function on X satisfying the following*:*

1. $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$.

2. $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and for all $x \in X$.

3. There is a constant $k \ge 1$ such that $\left\|\sum_{j=1}^{\infty} x_j\right\| = \sum_{j=1}^{\infty} k\left\|x_j\right\|$ for all $x_1, x_2 \dots \in X$.

The pair $(X, \|.\|)$ is called a generalized quasi- normed space if $\|.\|$ is a quasi-norm on X. The smallest possible K is called the modulus of concavity of $\|.\|$. A generalized quasi-Banach space is a complete generalized quasi-normed space.

Definition 1.3 Quasi-normed algebra Let $(X, \|.\|)$ be a quasi-normedspace.The quasi-normed space $(X, \|.\|)$ *is called a quasi-normed algebra* if X *is* an algebra and there is a constant $C > 0$ such that

$\|xy\| \le C\|x\|\|y\|$ for all $x, y \in X$.

A quasi-Banach algebra is a complete quasi-normed algebra.

Definition 1.4 Isometry Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an *isometry* if f satisfies

$$d_Y(f(x), f(y)) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(.,.)$ and $d_Y(.,.)$ denote the metrics in the spaces X and Y respectively. For some fixed number $r > 0$, suppose that f preserves distance r; i.e., for all $x, y \in X$ with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$.Then r is called a concervative (or preserved) distance for the mapping f. Let $(X, \|.\|)$ and $(Y, \|.\|)$ be normed spaces. A mapping $L: X \to Y$ is called an *isometry* if

$\|L(x) - L(y)\| = \|x - y\|$ for all $x, y \in X$.

2. Stability of Isometric Euler-Lagrange Additive Mapping in Generalized Quasi-Banach Spaces

Throughout this section, assume that A is a generalized quasi-normed vector space with generalized quasi-norm $\|.\|$ and that B is a generalized

Banach space with generalized quasi-norm$\|.\|$. Let$K$ be modulus of concavity of$\|.\|$.

For a given mapping $f: A \to B$, we define

$$Df(x_1, x_2, \dots x_d) =$$

$$\sum_{i=1}^{d} r_i f\left(\sum_{j=1}^{d} r_j (x_i - x_j)\right) + \left(\sum_{i=1}^{d} r_i\right) f\left(\sum_{i=1}^{d} r_i x_i\right) - \left(\sum_{i=1}^{d} r_i\right) \sum_{i=1}^{d} r_i f(x_i)$$

for all $x_1, x_2 \dots x_d \in A$.We set $M = \sum_{i=1}^{d} r_i$.

We prove the Generalized Hyers-Ulam stability of isometry in generalized quasi-Banach spaces for functional equation $D f(x_1, x_2, \dots x_d) = 0$.

Theorem 2.1Let $r < 1$ and θ be non-negative real numbers and $f: A \to B$ be a mapping such that

$$\|Df(x_1, x_2, \dots x_d)\| \le \theta \sum_{j=1}^{d} \|x_j\|^r$$

$$|\|f(x)\| - \|x\|| \le d\theta \|x\|^r$$

for all $x, x_1, x_2 \dots x_d \in A$.

Then there exits a unique isometric Euler Lagrange additive mapping $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{d\theta K}{M^2 - M^{r+1}} \|x\|^r \text{ for all } x \in A. \quad (7)$$

Proof:

Let us assume that and $x_1, x_2 \dots x_d = x$ in (5).

Then we get,

$$\|M f(Mx) - M^2 f(x)\| \le d\theta \|x\|^r \text{ for all } x \in A. (8)$$

So,

$$\left\|f(x) - \frac{1}{M} f(Mx)\right\| \le \frac{d\theta}{M^2} \|x^r\| \quad \text{for all } x \in A. \quad (9)$$

Hence,

$$\left\|\frac{f(M^l x)}{M^l} - \frac{f(M^m x)}{M^m}\right\| \le \sum_{j=l}^{m-1} \left\|\frac{f(M^j x)}{M^j} - \frac{f(M^{j+1} x)}{M^{j+1}}\right\|$$

$$\left\|\frac{f(M^l x)}{M^l} - \frac{f(M^m x)}{M^m}\right\| \le \frac{d\theta}{M^2} \sum_{j=l}^{m-1} \frac{M^{rj}}{M^j} \|x\|^r \quad (10)$$

for all non-negative integers m and l with $m > l$ and for all $x \in A$. From this it follows that the sequence $\left\{\frac{f(M^n x)}{M^n}\right\}$ is a Cauchy sequence for all $x \in A$.Since B is complete, the sequence $\left\{\frac{f(M^n x)}{M^n}\right\}$ converges. Thus one can define the mapping $H: A \to B$ by

$$H(x) = \lim_{n\to\infty} \frac{f(M^n x)}{M^n} \text{ for all } x \in A.$$

By (5),$\|Df(x_1, x_2, \dots x_d)\|$

(5)

$$\le \lim_{n\to\infty} \frac{1}{M^n} \left\| \sum_{i=1}^{d} r_i f\left(\sum_{j=1}^{d} r_j (M^n x_i - M^n x_j)\right) + \left(\sum_{i=1}^{d} r_i\right) f\left(\sum_{i=1}^{d} r_i M^n x_i\right) - \left(\sum_{i=1}^{d} r_i\right) \sum_{i=1}^{d} r_i f(M^n x_i) \right\|$$

(6)

$$\le \lim_{n\to\infty} \frac{M^{nr}\theta}{M^n} \sum_{j=1}^{d} \|x_j\|^r = 0 \text{for all } x_1, x_2 \dots x_d \in A.$$

Hence,$\sum_{i=1}^{d} r_i H\left(\sum_{j=1}^{d} r_j (x_i - x_j)\right) + \left(\sum_{i=1}^{d} r_i\right) H\left(\sum_{i=1}^{d} r_i x_i\right) = \left(\sum_{i=1}^{d} r_i\right) \sum_{i=1}^{d} r_i H(x_i)$for all $x_1, x_2 \dots x_d \in A$.

So the mapping $H : A \to B$ is Cauchy additive.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (10) we get (7).

Now, let $H' : A \to B$ be another Cauchy additive mapping satisfying (7).

Then we have,

$$\|\mathrm{H}(\mathrm{x}) - H'(x)\| = \frac{1}{M^n} \|H(M^n x) - H'(M^n x)\|$$

$$\leq \frac{K}{M^n}(\|H(M^n x) - f(M^n x)\| + \|H'(M^n x) - f(M^n x)\|)$$

$$\leq \frac{2d\theta\, K^2 M^{nr}}{(M^2 - M^{r+1})M^n}\|x\|^r$$

which tends to zero as $n \to \infty$ for all $x \in A$.

So we can conclude that,

$\mathrm{H(x)} = H'(x)$ for all $x \in A$.

This proves the uniqueness of H′.

Iffollowsfrom(6)

$$\left|\left\|\frac{1}{M^n} f(M^n x)\right\| - \|x\|\right| = \lim_{n\to\infty}\frac{1}{M^n}|\|f(M^n x)\| - \|M^n x\||$$

$$= \frac{1}{M^n} d\theta\, \|(M^n x)\|^r$$

$$= \frac{M^{nr}}{M^n} d\theta\, \|x\|^r$$

which tends to zero as $n \to \infty$ for all $x \in A$. So,

$$\|H(x)\| = \lim_{n\to\infty}\frac{1}{M^n}\|f(M^n x)\|$$

$$= \|x\| \quad \text{for all } x \in A.$$

Since $H: A \to B$ is additive.

$$\|H(x) - H(y)\| = \|H(x-y)\| = \|x - y\|$$

for all $x, y \in A$ as desired.

Theorem 2.2Let $r > 1$ and θ be non-negative real numbers and $f: A \to B$ be a mapping satisfying (5) and (6). Then there exits a unique isometric Euler Lagrange additive mapping such that

$\|f(x) - H(x)\| \leq \frac{d\theta\, K}{M^{r+1} - M^2}\|x\|^r$ for all $x \in A$. (11)

Proof:

It follows from (8) that

$\|M f(Mx) - M^2 f(x)\|\, d\theta\, \|x\|^r$ for all $x \in A$. (12)

$$\left\|f(x) - M f\frac{x}{M}\right\| \leq \frac{d\theta}{M^2}\|x\|^r \tag{13}$$

$$\left\|M^l f\left(\frac{x}{M^l}\right) - M^m f\left(\frac{x}{M^m}\right)\right\| \leq \sum_{j=l}^{m-1}\left\|M^j f\left(\frac{x}{M^j}\right) - M^{j+1} f\left(\frac{x}{M^{j+1}}\right)\right\|$$

$$\left\|M^l f\left(\frac{x}{M^l}\right) - M^m f\left(\frac{x}{M^m}\right)\right\| \leq \frac{K\, d\theta}{M^{r+1}}\sum_{j=l}^{m-1}\frac{M^j}{M^{jr}}\|x\|^r \tag{14}$$

for all non-negative integers m and l with $l < m$ and for all $x \in A$. It follows that from (14) that the sequence $\left\{M^n f(\frac{x}{M^n})\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{M^n f(\frac{x}{M^n})\right\}$ converges. Thus one can define the mapping $\mathrm{H}: \mathrm{A} \to \mathrm{B}$ by

$\mathrm{H(x)} = \lim_{n\to\infty} M^n \mathrm{f}(\frac{x}{M^n})$ for all $x \in A$.

By (5),

$$\|\mathrm{D}\, \mathrm{f}(x_1, x_2 \ldots x_d)\| \leq \lim_{n\to\infty} M^n \left\|\sum_{i=1}^{d} r_i f\left(\sum_{j=1}^{d} r_j\left(\frac{x_i}{M^n} - \frac{x_j}{M^n}\right)\right) + \left(\sum_{i=1}^{d} r_i\right) f\left(\sum_{i=1}^{d} r_i\frac{x_i}{M^n}\right) - \left(\sum_{i=1}^{d} r_i\right)\sum_{i=1}^{d} r_i\, f\left(\frac{x_j}{M^n}\right)\right\|$$

$$\leq \lim_{n\to\infty}\frac{M^n\,\theta}{M^n\, r}\sum_{j=1}^{d}\|x_j\|^r$$

$= 0$for all $x_1, x_2 \ldots x_d \in A$.

Hence,
$\sum_{i=1}^{d} r_i H\left(\sum_{j=1}^{d} r_j\left(x_i - x_j\right)\right) + \left(\sum_{i=1}^{d} r_i\right) H\left(\sum_{i=1}^{d} r_i x_i\right) = \left(\sum_{i=1}^{d} r_i\right)\sum_{i=1}^{d} r_i H(x_i)$for all$x_1, x_2 \ldots x_d \in A$.

So the mapping H : A → B is Cauchy additive.

Moreover letting $l = 0$ and passing the limit $m \to \infty$ in (14) we get (11).Now, let $\mathrm{H}': A \to B$ be another Cauchy additive mapping satisfying (11).

Then we have

$$\|H(x) - H'(x)\| = M^n \left\| H\left(\frac{x}{M^n}\right) - H'(\frac{x}{M^n}) \right\|$$

$$\leq M^n \left[\left\| H\left(\frac{x}{M^n}\right) - f(\frac{x}{M^n}) \right\| - \left\| H'\left(\frac{x}{M^n}\right) - f(\frac{x}{M^n}) \right\| \right]$$

$$\leq \frac{2M^n K^2\, d\theta}{(M^{r+1} - M)M^{nr}} \|x\|^r$$

which tends to zero $n \to \infty$ for all $x \in A$.

So, we can conclude that,

$H(x) = H'(x)$ for all $x \in A$.

This proves the uniqueness of H'.

$$\left| \left\| M^n f \frac{x}{M^n} \right\| - \|x\| \right| = \lim_{n\to\infty} M^n \left| \left\| f \frac{x}{M^n} \right\| - \left\| \frac{x}{M^n} \right\| \right|$$

$$\leq M^n \left[d\theta \left\| \frac{x}{M^n} \right\|^r \right]$$

$\leq \frac{M^n}{M^{nr}} d\theta \|x\|^r$ for all $x \in A$.

Which tends to zero as $n \to \infty$, for all $x \in A$.

$$H(x) = \lim_{n\to\infty} \left\| M^n f \frac{x}{M^n} \right\|$$

$= \|x\|$ for all $x \in A$.

Since $H: A \to B$ is additive.

$$\|H(x) - H(y)\| = \|H(x - y)\|$$

$$= \|(x - y)\|$$

for all $x, y \in A$ as desired.

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