Analytical Solution of 2D Dual-Phase-Lagging Heat Conduction Model

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Abstract—*The heat transport at microscale is vital important in the field of micro-technology. In this paper heat transport in a two-dimensional thin plate based on dual-phase-lagging (DPL) heat conduction model is investigated. The solution was obtained with the help of superposition techniques and solution structure theorem. . The whole analysis is presented in dimensionless form. A numerical example of particular interest has been studied and discussed in details.*

Keywords— *DPL heat conduction model, Superposition technique, Solution structure theorem, Analytical solution.*

I. Introduction

Cattaneo [1] and Vernotte [2] removed the deficiency [3]-[6] occurs in the classical heat conduction equation based on Fourier's law and independently proposed a modified version of heat conduction equation by adding a relaxation term to represent the lagging behavior of energy transport within the solid, which takes the form

$$\tau_q \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T \qquad (1)$$

where k is the thermal conductivity of medium and τ_q is a material property called the relaxation time. This model characterizes the combined diffusion and wave like behavior of heat conduction and predicts a finite speed

$$c = \left(\frac{k}{\rho c_b \tau_q}\right)^{\frac{1}{2}} \qquad (2)$$

for heat propagation [7], where ρ is the density and c_b is the specific heat capacity. This model addresses short time scale effects over a spatial macroscale. Detailed reviews of thermal relaxation in wave theory of heat propagation were performed by Joseph and Preziosi [8] and Ozisik and Tzou [9]. The natural extension of CV model is

$$\mathbf{q}(\mathbf{r}, t+\tau_q) = -k\nabla T(\mathbf{r}, t) \qquad (3)$$

which is called the single-phase-lagging (SPL) heat conduction model [10]-[14]. According to SPL heat conduction model, there is a finite built-up time τ_q for onset of heat flux at $\mathbf{r}$, after a temperature gradient is imposed there i.e. τ_q represents the time lag needed to establish the heat flux (the result) when a temperature gradient (the cause) is suddenly imposed.

Many new simulation models such as phonon-electron interaction in metal films [15]-[16], phonon scattering in dielectric crystals [17], insulators and semi conductors [18]-[20], have recently been developed in order to study the mechanisms of heat conduction in microscale and or nanoscale that cannot be described by Fourier's law. To describe micro-structural interactions a further modification of SPL model gives the dual-phase-lagging (DPL) model [21],

$$\mathbf{q}(\mathbf{r}, t+\tau_q) = -k\nabla T(\mathbf{r}, t+\tau_T) \qquad (4)$$

where τ_T is the phase lag of temperature gradient and τ_q is the phase lag of the heat flux vector. It allows either the temperature gradient (cause) to precede the heat flux vector (effect) or vice-versa.

Due to the complexity of the DPL heat conduction model, temperature solutions can be obtained analytically for limited engineering applications [22]. The most popular solution methodology has resorted to either finite-difference or finite-element methods. Only a few simple cases can be solved analytically. Some of the most popular analytical solutions are the method of Laplace transformation method [23]-[25], separation of variable method [26], [27], the Green function method [28], the integral equation method [29], and variational iteration method [30].

Recently Lam and Fong [31] and Lam [32] conducted studies by employing the superposition technique along with solution structure theorems for the analysis of the CV hyperbolic heat conduction equation and one dimensional DPL heat conduction model. The temperature profile inside a one-dimensional region was obtained in the form of a series solution. The method is relatively simple and requires only a basic background in applied mathematics. However, it was noted that solution structure theorems concentrated only on physical problems subjected to homogeneous boundary conditions. It was pointed out that there is a way to solve problems with non-homogeneous boundary conditions by performing appropriate functional transformations, namely by using auxiliary functions.

The purpose of this study is to apply solution structure theorems to study two dimensional DPL heat conduction in a finite plate subjected to homogeneous boundary conditions. The DPL heat conduction equation is solved using the superposition principle in conjunction with solution structure theorems. The outline of the paper is as follows. DPL heat conduction model is given in section 2. Section 3 deals solution of dual-phase-lagging heat conduction model. Section 4 contains result and discussion. Conclusion is given in section 6.

II. 2D DPL Heat Conduction Model

Tzou [21] overcome the deficiency by including the cause-and-effect of the temperature gradient and heat flux relationship and proposed the dual-phase-lag model, (4). Using Taylor series expansion, the first order approximation of (4) gives

$$\mathbf{q}+\tau_q\frac{\partial \mathbf{q}}{\partial t}=-k\left\{\nabla T+\tau_T\frac{\partial}{\partial t}(\nabla T)\right\} \qquad (5)$$

Equation (5) is called the dual-phase-lagging constitutive equation which corresponds to the particular cases where $\tau_q>0$ and $\tau_T=0$ i.e. CV model. If $\tau_q=\tau_T$ (not necessarily equal to zero), response between temperature gradient and heat flux is instantaneous and in this case (5) is identical with the classical Fourier law [33]. It may be also noted that while the classical Fourier law is macroscopic in both space and time and (5) is microscopic in both space and time. Taking divergence on both sides of (5), we get

$$\nabla\mathbf{q}+\tau_q\frac{\partial \nabla\mathbf{q}}{\partial t}=-k\left\{\nabla^2 T+\tau_T\frac{\partial}{\partial t}(\nabla^2 T)\right\} \qquad (6)$$

Introducing the energy conservation equation [33] to the (6), we get

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_q}{\alpha}\frac{\partial^2 T}{\partial t^2}=\nabla^2 T+\tau_T\frac{\partial}{\partial t}(\nabla^2 T)+\frac{1}{k}\left(g^*+\tau_q\frac{\partial g^*}{\partial t}\right)$$

The two dimensional form of the above is as follows

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_q}{\alpha}\frac{\partial^2 T}{\partial t^2}=\frac{\partial^2 T}{\partial x_1^2}+\frac{\partial^2 T}{\partial y_1^2}+\tau_T\left(\frac{\partial^3 T}{\partial x_1^2\partial t}+\frac{\partial^3 T}{\partial y_1^2\partial t}\right)+\frac{1}{k}\left(g^*+\tau_q\frac{\partial g^*}{\partial t}\right)$$

By introducing dimensionless parameters

$$\theta=\frac{kcT}{\alpha f_r},x=\frac{cx_1}{2\alpha},y=\frac{cy_1}{2\alpha},g=\frac{4\alpha g^*}{cf_r},F_0=\left(\frac{c^2}{2\alpha}\right)t,$$

above can be expressed in dimensionless form as

$$2\frac{\partial\theta}{\partial F_0}+\frac{\partial^2\theta}{\partial F_0^2}=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+B\left(\frac{\partial^3\theta}{\partial x^2\partial F_0}+\frac{\partial^3\theta}{\partial y^2\partial F_0}\right)+\left(g+\frac{1}{2}\frac{\partial g}{\partial F_0}\right) \qquad (7)$$

where Fourier number F_0 represent dimensionless time and $B=(\tau_T/2\tau_q)$. In present study, an isotropic thin plate, $0\le x,y\le 1$, with uniform thickness and constant thermophysical properties is assumed and initial and boundary conditions for heat conduction in thin film are

$$\theta(x,y,0)=\theta_2,\ \frac{\partial\theta(x,y,0)}{\partial F_0}=\theta_3 \qquad (7a)$$

$$\frac{\partial\theta(0,y,F_0)}{\partial x}=0,\frac{\partial\theta(1,y,F_0)}{\partial x}=0 \qquad (7b)$$

$$\frac{\partial\theta(x,0,F_0)}{\partial y}=0,\frac{\partial\theta(x,1,F_0)}{\partial y}=0 \qquad (7c)$$

III. Solution

A. *Method of Superposition*

The superposition technique can be applied to solve linear heat transfer problem with non-homogeneous term [7], [34], [35]. With the application of superposition principle, the original problem (7) can be divided into three sub-problems by setting initial conditions and internal heat source $(G(x,y,F_0))$ as (1) $G=\theta_2=0$, (2) $G=\theta_3=0$, and (3) $\theta_2=\theta_3=0$. Solution to these sub-problems are designated as S_1,S_2 and S_3. Therefore, the general solution of the original hyperbolic DPL heat conduction model is $S=S_1+S_2+S_3$.

B. *Solution Structure Theorem*

With the help of solution structure theorem [7], once the solution of sub-problem (1) is known, solution of sub-problems (2) and (3) can be obtained as follows

$$S_2=\left(2+\frac{\partial}{\partial F_0}\right)\mathrm{F}(x,y,F_0,\theta_2)+B\mathrm{F}(x,y,F_0,\lambda_{m,n}\theta_2)$$

$$S_3=\int_0^{F_0}\mathrm{F}(x,y,F_0,-\tau G(x,y,\tau))d\tau$$

where $\mathrm{F}(x,y,F_0,\theta_3)$ be the solution of sub-problem (1).

C. *Solution of 2D-DPL Heat Conduction Model*

This section only devoted to the solution of the sub-problem (1) of DPL heat conduction model. For the given initial and boundary conditions, one can write solution to the governing equation by using Fourier series as

$$\theta(x,y,F_0)=\sum_{m,n}^{\infty}\theta_{m,n}(F_0)Cos(\lambda_m x)Cos(\lambda_n y) \quad (8)$$

By substituting above (8) into (7) and after some manipulation we get following

$$\frac{\partial^2\theta_{m,n}}{\partial F_0^2}+(2+B\lambda_{m,n})\frac{\partial\theta_{m,n}}{\partial F_0}+\lambda_{m,n}\theta_{m,n}=0 \quad (9)$$

The Solution of above takes the form

$$\theta_{m,n}(F_0)=e^{\alpha_{m,n}F_0}\left\{a_{m,n}Sin(\beta_{m,n}F_0)+b_{m,n}Cos(\beta_{m,n}F_0)\right\} \quad (10)$$

where $\alpha_{m,n}$ and $\beta_{m,n}$ are defined as follows

$$\alpha_{m,n}=-\left(2+B\lambda_{m,n}\right)/2,\ \beta_{m,n}=\sqrt{\lambda_{m,n}-\left(2+B\lambda_{m,n}\right)^2/4}$$

and $\lambda_{m,n}=\lambda_m^2+\lambda_n^2;\lambda_m=m\pi,\lambda_n=n\pi$.

By substituting above (10) into (8) solution of the problem can be expressed as follows

$$\theta(x,y,F_0)=\sum_{m,n}^{\infty}e^{\alpha_{m,n}F_0}\left\{a_{m,n}Sin(\beta_{m,n}F_0)+b_{m,n}Cos(\beta_{m,n}F_0)\right\}\times Cos(\lambda_m x)Cos(\lambda_n y) \quad (11)$$

Now to find the coefficients $a_{m,n}$ and $b_{m,n}$ we consider initial conditions $\theta_2=0$ and $\theta_3=Sin(xy)$. Then $b_{m,n}=0$ and $a_{m,n}$ may be obtained as

$$a_{m,n}=\frac{2}{\beta_{m,n}}\int_0^1\int_0^1 Sin(xy)Cos(\lambda_m x)Cos(\lambda_n y)dxdy.$$

Hence the solution of the problem is complete for $m,n>0$. Since the solution contains Cosine terms at the end of (11), therefore for $m,n=0$ there is also a solution of the problem. For $m,n=0$, (9) becomes

$$\frac{\partial^2\theta_0}{\partial F_0^2}+2\frac{\partial\theta_0}{\partial F_0}=0$$

With the application of initial conditions, solution of above is

$$\theta_0(x,y,F_0)=\frac{1}{2}(1-e^{-F_0})Sin(x,y) \quad (12)$$

Thus the final solution of the two dimensional DPL heat conduction model is

$$\theta(x,y,F_0)=\theta_{m,n}(x,y,F_0)+\theta_0(x,y,F_0)$$
$$=\frac{1}{2}(1-e^{-F_0})Sin(x,y)+2\times$$
$$\sum_{m,n=1}^{\infty}\frac{e^{\alpha_{m,n}F_0}}{\beta_{m,n}}\int_0^1\int_0^1 Sin(\xi,\psi)Cos(\lambda_m\xi)Cos(\lambda_n\psi)d\xi d\psi\times Sin(\beta_{m,n}F_0)Cos(\lambda_m x)Cos(\lambda_n y) \quad (13)$$

IV. Results and Discussion

This section presents complete analysis of thermal wave propagation and observes the effect of $B=\left(\tau_T/2\tau_q\right)$ and Fourier number $\left(F_0\right)$. The figures presented in this study, only the parameters whose values different from the reference value are indicated. The selected reference values include $\theta_2=0,\theta_3=Sin(xy),G=0$.

Figs. 1-3 present the spatial temperature profile for various B at fixed Fourier number $F_0=1.0$. The dimensionless temperature firstly increases with B and when B greater than 0.5 then further increase of temperature is ceases as shown in Fig. 3.

Figs. 4-7 show the spatial temperature profile for various F_0 at fixed $B=0.1$. As Fourier number increases the dimensionless temperature increases as Fourier number is a measure of rate of heat conduction with the heat storage in a given volume element. Larger the Fourier number, deeper is the penetration of heat into the body over a given period of time.

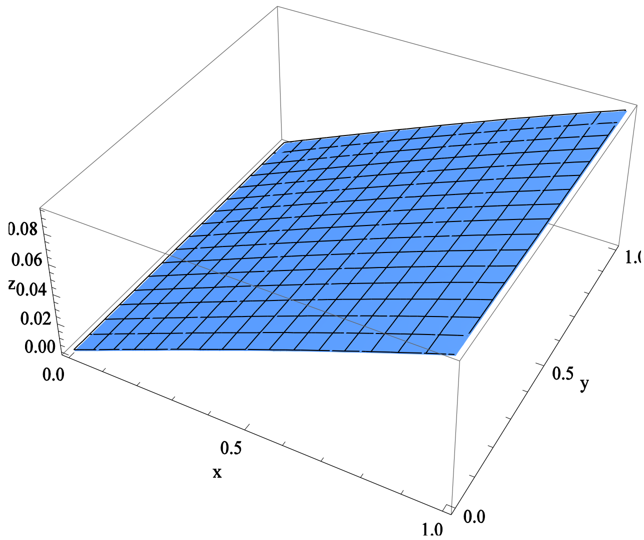

Fig. 1 Spatial temperature profile at $F_0=1.0, B=0.1$.

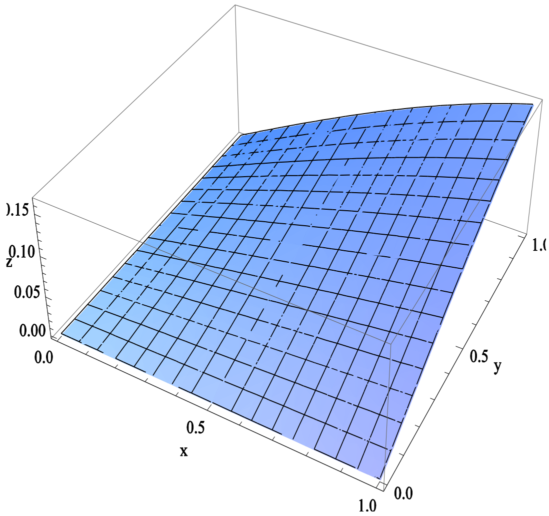

Fig. 2 Spatial temperature profile at $F_0 = 1.0, B = 0.7$.

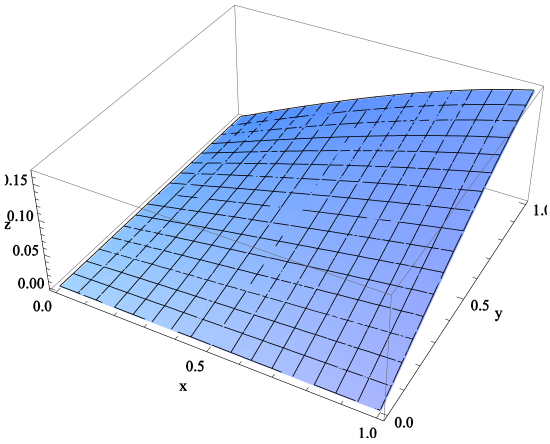

Fig. 3 Spatial temperature profile at $F_0 = 1.0, B = 1.0$.

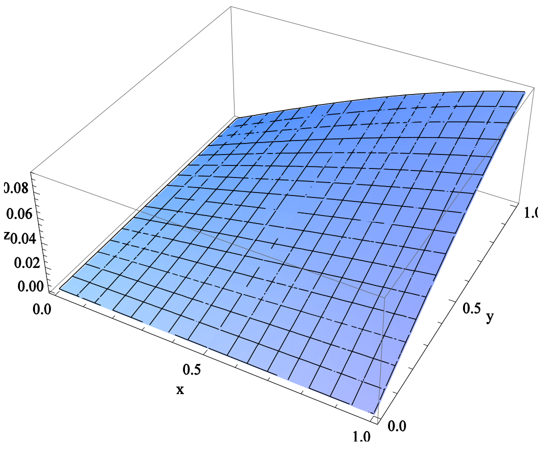

Fig. 4 Spatial temperature profile at $F_0 = 0.5, B = 0.5$.

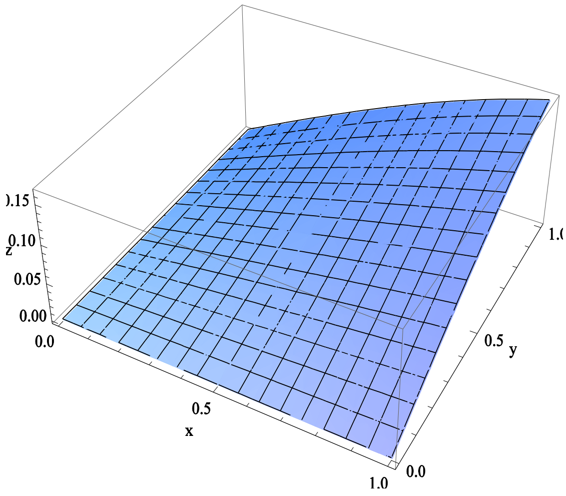

Fig. 5 Spatial temperature profile at $F_0 = 1.0, B = 0.5$.

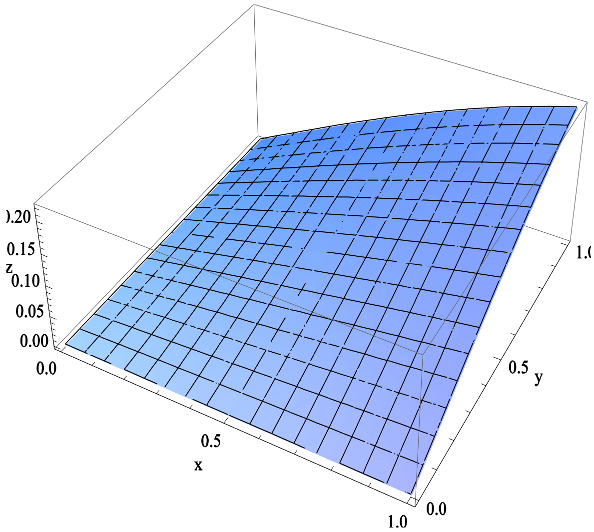

Fig. 6 Spatial temperature profile at $F_0 = 1.5, B = 0.5$.

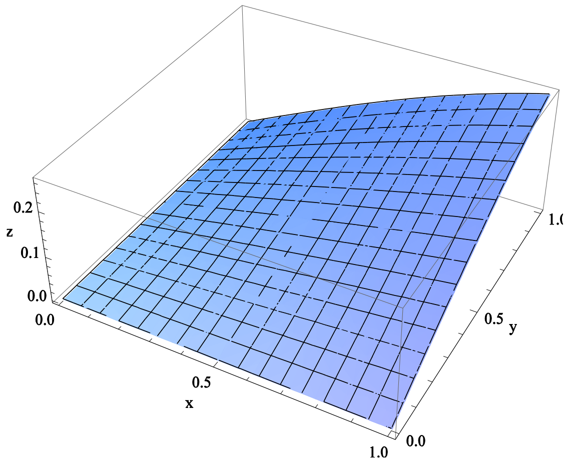

Fig. 7 Spatial temperature profile at $F_0 = 1.0, B = 0.1$.

V. Conclusions

The mathematical model describing heat transfer in a thin plate based on dual-phase-lagging heat conduction is solved by superposition technique. The solution was obtained by utilizing superposition technique, structure theorem and Fourier series expansion. The effect of Fourier number and B on temperature profile has been observed. The temperature increases with increase of B and F_0 but when B is greater than 0.5 then further increase of temperature ceases.

This technique is very applicable for solving non-homogeneous partial differential equation under most generalized boundary conditions and may be applicable for solving the higher dimensional DPL heat conduction model of general body.

VI. Nomenclature

c	Thermal wave propagation speed (m/s)
c_b	Specific heat capacity $(J/kg.K)$
f_r	Reference heat flux $(\mathbf{q}/\mathbf{q}^*)$
F_0	Fourier number $(c^2 t/2\alpha)$
g^*	Internal heat source (W/m^3)
g	Dimensionless internal heat source $(4\alpha g^*/cf_r)$
k	Thermal conductivity $(W/m.K)$
$\mathbf{q}^*$	Dimensionless heat flux $(\mathbf{q}/f_r)$
$\mathbf{r}$	Position vector
t	Time (s)
T	Temperature (K)
ΔT	Temperature gradient (K/m)
x	Dimensionless spatial coordinate $(cx_1/2\alpha)$
y	Dimensionless spatial coordinate $(cy_1/2\alpha)$
x_1, y_1	Spatial coordinate (m)
α	Thermal diffusivity (m^2/s)
θ	Dimensionless Temperature $(kcT/\alpha f_r)$
ρ	Density (kg/m^3)
τ_q	Phase-lag of heat flux (s)
τ_T	Phase-lag of temperature gradient (s)

Acknowledgment

Authors are thankful to DST-CIMS, BHU for providing necessary facilities during the manuscript.

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