Right Ideals of Prime Rings with Left Generalized Derivations

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Abstract: In this paper we present some results concerning derivations of a prime ring R to the left generalized derivations associated with a derivation d of $R$and anonzero right ideal U of R which is semi prime as a ring. We proved that f is a left generalized derivation of a prime ring R, U is a non-commutative right ideal of R and$f([x,y]) = 0$ or $f([x,y]) = \pm[x,y]$, f acts as a homomorphism or anti- homomorphism or $[x,f(x)]$=0 or $[f(x),f(y)] = [x,y]$ or$f(U) \subseteq Z$, then there exists a Martindale ring of quotients i.e. $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$. And also proved $d(z) \neq 0$and $[a,f(x)] \epsilon Z$, for all $x \epsilon U$, then $a \epsilon Z$.

Key words: Prime ring, Semi prime ring, Derivation,Generalized derivation, Left generalized derivation.

Introduction: The study of thecommutativity of prime rings with derivations was initiated by E.C.Posner [11].Recently, M.Bresar[4] defined generalized derivation of rings. Hvala[9] studied the properties of generalized derivations in prime rings .Golbasi[7] extended some well known results concerning derivations of prime rings to the generalized derivations and a nonzero left ideal of a prime ring which is semi-prime as a ring . In this paper weextend some results concerning derivations of prime ring R to the left generalized derivations associated with a derivation d of R and a non zero right ideal U of R which is semi prime as a ring .

Throughout out this paper, R will be prime ring with characteristic different from two and U a nonzero right ideal of R which is semiprime as a ring, Z the multiplicative center of R, $Q_r^{(R)}$ the right Martindale ring of quotients C the extended centroid and $R_C = RC$ the central closure.For any $x,y \in R$, the symbol $[x,y]$ will represent the commutator $xy - yx$. Recall that prime if $aRb = (0)$ implise that $a = 0$ or $b = 0$ and semi prime if $aRa = (0)$ implies that $a = 0$.An additive map d from R to R is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$. Let $f: R \to R$ is a right generalized derivation if there exists a derivation d of R such that$f(xy) = f(x)y + xd(y)$ holds for all $x,y \in R$. If f is a left generalized derivation if there exists a derivation d of R such that$f(xy) = d(x)y + xf(y)$ holds for all $x,y \in R$. If f is a generalized derivation of R associated with d if it is both left and right generalized derivation of R.

To prove the main results we require the following lemmas

Lemma 1: [6,lemma 1] Let R be a prme ring and U a nonzero right ideal of R which is semiprime as a ring. If $aU = 0$ $(Ua = 0)$ for $a \in R$, then $a = 0$.

Lemma 2:[9,Lemma2]Left $f: R \to R_C$ be an additive map satisfying $f(xy) = xf(y)$ for all $x,y \in R$.Then there exists $q \in Q_r(R_C)$ such that$f(x) = qx$,for all $x \in R$.

Lemma 3: **[**11, lemma2.3]Let$R$ be prime ring and U a nonzero right ideal of R which is semiprime as a ring. If d is a derivation of R such that $d(U) = 0$,then $d = 0$.

Theorem 1: Let R be a prime ring,U a nonzero right ideal of R which is semiprime as a ring and f a left generalized derivation of R.If U is non commutative and $f([x,y]) = 0$ for all $x,y \in U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.

Proof: Let $f([x,y]) = 0$ for all$x,y \epsilon U$. (1)

Substitute yx for $x$in $f([x,y]) = 0$, we get

$$f([yx,y]) = 0$$

$\Rightarrow f(y[x,y]) = 0,$

$d(y)[x,y] + yf([x,y]) = 0.$

From (1), we get

$d(y)[x,y] = 0$,for all $x, y \epsilon U$.(2)

Substitute xr for x in equation (2), we get

$d(y)[xr,y] = 0,$

$d(y)[x,y]r + d(y)x[r,y] = 0$,

From (2) the first summand is zero, it is clear that

$d(y)x[r,y] = 0$, for all $x, y \epsilon U, r \epsilon R$.

Writing$xs, s \epsilon R$, in place of $x$in this equation, we get

$d(y)xs[r,y] = 0$, for all$x, y \epsilon U, r \epsilon R$.

Since R is prime ring, we have

$d(y)U = 0$or$[r,y] = 0$,for all$y \epsilon U, r \epsilon R$.

$\Rightarrow d(x)U = 0[r,x] = 0$or,for all $x \epsilon U, r \epsilon R$.

By lemma 1, we get either $x \epsilon Z$ or $d(x) = 0$ for all$x \epsilon U$.

Let $A = \{x \epsilon U \backslash x \epsilon z\}$ and $B = \{x \epsilon U \backslash d(x) = 0\}$. Then A and $B$are two additive subgroups of $(U, +)$ such that$U = A \cup B$. However, a group cannot be the union of proper sub groups. Hence either $U = A$ or $U = B$.

If $U = A$then $U \subset Z$, and so U is commutative, which contradicts the hypothesis.

So we must have $d(x) = 0$,for all $x \in U$. By lemma3, we get $d = 0$.

Hence, there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$,for all $x \in R$, by lemma2.

Theorem 2: Let$R$ be a prime ring,U a nonzero right ideal of R which is semiprime as a ring and f a left generalized derivation of R. If U is non commutative and $f([x,y]) = \pm[x,y]$, for all$x, y \epsilon U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.

Proof: Assume that $f([x,y]) = \pm[x,y]$, for all$x, y \epsilon U$. (3)

Replacing x by yx in above equation, we get

$$f([yx,y]) = \pm[yx,y],$$

$f(y[x,y]) = \pm y[x,y]$,

$d(y)[x,y] + yf([x,y]) = \pm y[x,y],$

From equation (3), we get

$d(y)[x, y] = 0$,for all $x, y \epsilon U$. (4)

Substitute xr for x in equation (4), we get

$d(y)[xr, y] = 0$,

$d(y)[x, y]r + d(y)x[r, y] = 0$,

From (4) the first summand is zero, it is clear that

$d(y)x[r, y] = 0$, for all $x, y \epsilon U, r \epsilon R$.

Writing$xs, s \epsilon R$, in place of x in this equation, we get

$d(y)xs[r, y] = 0$, for all$x, y \epsilon U, r \epsilon R$.

Since R is prime ring, we have

$d(y)U = 0$or $[r, y] = 0$,for all $y \epsilon U, r \epsilon R$.

$\Rightarrow d(x)U = 0$or$[r, x] = 0$,for all $x \epsilon U, r \epsilon R$.

By lemma 1, we get either $x \epsilon Z$ or $d(x) = 0$ for all$x \epsilon U$.

Let $A = \{x \epsilon U \backslash x \epsilon z\}$ and $B = \{x \epsilon U \backslash d(x) = 0\}$. Then A and $B$are two additive subgroups of $(U, +)$ such that $U = A \cup B$. However, a group cannot be the union of proper sub groups. Hence either $U = A$ or $U = B$.

If $U = A$ then $U \subset Z$, and so U is commutative, which contradicts the hypothesis.

So we must have $d(x) = 0$, for all $x \in U$. By lemma 3, we get $d = 0$.

Hence, there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$, by lemma 2.

Corollary 1: Let R be a prime ring, U a non zero right ideal of R which is semi prime as a ring and f a left generalized derivation of R.If U is non commutative and $f(xy) = \pm xy$, for all $x, y \epsilon U$, then there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$.

Theorem 3: Let$R$ be a prime ring,U a nonzero right ideal of R which is semi prime as a ring and f a left generalized derivation of R. If f acts as a homomorphism or anti homomorphism on U, then there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$.

Proof: Assume that f acts as a homomorphism on U.

Then , $f(xy) = f(x)f(y) = d(x)y + xf(y)$ for all$x, y \epsilon U$. (5)

Replacing y by zy, $z \epsilon U$, in the second equality in (5), we have

$f(x)f(zy) = d(x)zy + xf(zy) = d(x)zy + xf(z)f(y)$.(6)

Since f is a homomorphism. On the other hand, we have

$f(x)f(zy) = f(x)f(z)f(y) = f(xz)f(y) = \big(d(x)z + xf(z)\big)f(y)$. (7)

From (6) & (7), we have

$d(x)z(y - f(y)) = 0$ for all $x, y, z \epsilon U$.

Replacing z by $zr, r \epsilon R$, in the above equation, we arrive at

$d(x)zr\big(y - f(y)\big) = 0$ for all $x, y, z \epsilon U, r \epsilon R$.

Since R is prime ring, we have either f is the identity map on U or, $d(U)U = 0$.

Suppose that $f(x) = x$, for all $x \epsilon U$.

Then $f(xy) = xy$,

$$d(x)y + xf(y) = xy,$$

$d(x)y + xy = xy$,

And so, $d(x)y = 0$, for all $x, y \epsilon U$.

Hence we conclude that $d = 0$ by lemma 1. Thus, there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$,by lemma 2.

Now assume that f acts as an anti homomorphism on U.

Then $f(xy) = f(y)f(x) = d(x)y + xf(y)$, for all $x, y \epsilon U$. (8)

Replacing y by xy in the first equation (8), we get

$f(x(xy)) = d(x)xy + xf(xy) = d(x)xy + xf(y)f(x)$.(9)

The second equation (8), we get

$f(xy)f(x) = \big(d(x)y + xf(y)\big)f(x) = d(x)yf(x) + xf(y)f(x)$. (10)

From equation (9) & (10), we get

$d(x)yf(x) = d(x)xy$, for all $x, y \epsilon U$.(11)

Replacing y by $yr, r \epsilon R$,in (11), to get

$d(x)yrf(x) = d(x)xyr = d(x)yf(x)r$.

That is,

$d(x)y[f(x), r] = 0$,for all $x, y \epsilon U$,$r \epsilon R$. (12)

Again writing y by ys, $s \epsilon R$, we have either $d(x)U = 0$ or $[f(x), r] = 0$, for all $x \epsilon U, r \epsilon R$.

According to Brauer's Trick and lemma1, we conclude that $f(U) \subset Z$ or $d(U) = 0$.

In the second case, the proof is complete. The first case gives that f acts as a homomorphism on U. Thus, there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$.

Theorem 4: Let R be a prime ring with characteristic different from two, U anonzero right ideal of Rwhich is semiprime as a ring, and fa leftgeneralized derivation of R. If U is noncommutative and $[x, f(x)] = 0$, for all $x \epsilon U$, then there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$.

Proof: Let $[x, f(x)] = 0$, for all $x \epsilon U$. (13)

Linearization of (13) gives that

$$[x + y, f(x + y)] = [x, f(x)] + [x, f(y)] + [y, f(x)] + [y, f(y)] = 0$$

Using (13) gives that

$[x, f(y)] + [y, f(x)] = 0$, for all $x, y \epsilon U$. (14)

Writing yx instead of x in equation (14), we get

$$[yx, f(y)] + [y, f(yx)] = 0,$$

$y[x, f(y)] + [y, d(y)x + yf(x)] = 0$,

$y[x, f(y)] + [y, d(y)]x + d(y)[y, x] + y[y, f(x)] = 0$,

From (14), we get

$[y, d(y)]x + d(y)[y, x] = 0$for all $x, y \epsilon U$. (15)

Writing zx instead of x in equation (15), and using this equation, we obtain that

$[y, d(y)]zx + d(y)[y, zx] = 0$,

$[y, d(y)]zx + d(y)[y, z]x + d(y)z[y, x] = 0$,

$([y, d(y)]z + d(y)[y, z])x + d(y)z[y, x] = 0$,

$d(y)z[y, x] = 0$, for all $x, y, z \epsilon U$ (16)

Replacing $z$by zr,$r \epsilon R$, in (16), we get

$$d(y)zr[y, x] = 0$$

Since R is prime, we get

$d(y)U = 0$ or $[y, x] = 0$, for all $x, y \epsilon U$.

By lemma 2.1, we have either $d(y) = 0$ or $[x, y] = 0$, for all $x \epsilon U$.

By a standard argument one of these must be held for all $x \epsilon U$. The second result cannot hold since U is non commutative, so the first possibility gives $d(U) = 0$, and hence$d = 0$.

The proof may be completed by using lemma 2.

Theorem5: Let R be a prime ring with characteristic different from two, U a nonzero left ideal of R which is semiprime as a ring, and f a left generalized derivation of R .If U is noncommutative, $d(Z) \neq 0$ and$[f(x), f(y)] = [x, y]$,for all $x, y \epsilon U$, then there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$.

Proof: Let $[f(x), f(y)] = [x, y]$, for all $x, y \epsilon U$.(17)

Taking yx instead of x in the equation (17), we get

$[f(yx), f(y)] = [yx, y]$,

$[d(y)x + yf(x), f(y)] = y[x, y]$,

$$[d(y), f(y)]x + d(y)[x, f(y)] + [y, f(y)]f(x) + y[f(x), f(y)] = y[x, y]$$

From (17), we get

$[d(y), f(y)]x + d(y)[x, f(y)] + [y, f(y)]f(x) = 0$for all $x, y \epsilon U$. (18)

Replacing $x$by $cx = xc$, where $c \epsilon Z$,and using (18), we arrive at

$[d(y), f(y)]xc + d(y)[xc, f(y)] + [y, f(y)]f(xc) = 0$,

$$[d(y), f(y)]xc + d(y)[x, f(y)]c + [y, f(y)]d(c)x + [y, f(y)]cf(x) = 0$$

Since $cf(x) = f(x)c$ so

$$\big([d(y), f(y)]x + d(y)[x, f(y)] + [y, f(y)]f(x)\big)c + [y, f(y)]d(c)x = 0$$

$[y, f(y)]d(c)x = 0$for all $x, y \epsilon U$.

Since $0 \neq d(c) \epsilon Z and U$ is a nonzero right ideal of R, we have

$[y, f(y)] = 0$, for all $x \epsilon U$.

The proof is now completed using theorem 4

Theorem 6: Let R be a prime ring with characteristic different from two, U a nonzero left ideal of R wich is semiprime as a ring, and f a left generalized derivation of R. If U is noncommutativeand $f(U) \subseteq Z$, then there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$.

Proof: Since $f(U) \subseteq Z$, then

$[f(x), y] = 0$, for all $x, y \epsilon U$.

Taking yx instead of $x$in the above equation, we have

$0 = [f(yx), y] = [d(y)x + yf(x), y]$,

$[d(y)x, y] + [yf(x), y] = 0$,

$[d(y), y]x + d(y)[x, y] + y[f(x), y] = 0$.

Since $f(U) \subseteq Z$, then

$[d(y), y]x + d(y)[x, y] = 0$.

Expanding this equation we conclude that

$$d(y)yx - yd(y)x + d(y)xy - d(y)yx = 0,$$

$d(y)xy = yd(y)x$for all $x, y \epsilon U$. (19)

Writing zx instead of x in equation (19), and using (19) we get

$d(y)zxy = yd(y)zx = d(y)zyx$,

$d(y)z[x, y] = 0$, for all $x, y, z \epsilon U$

Taking $zr, r \epsilon R$ in the place of z in the above equation and using the fact $R$is prime, we conclude that $d(y) = 0, or[x, y] = 0$, for all $x, y \epsilon U$. By the standard argument, we have either that U is commutative or $d = 0$.

Since U is not commutative, the proof is complete.

Theorem 7: Let R be a prime ring with characteristic different from two, U a non zero right ideal of R wich is semiprime as a ring, and f a left generalized derivation of R and $a \epsilon R$. If U is non commutative, $d(z) \neq 0$ and $[a, f(x)] \epsilon Z$ for all $x \epsilon U$, then $a \epsilon Z$.

Proof: Since $d(z) \neq 0$, there exists $c \in Z$ such that $d(c) \neq 0$.

Furthermore, since$d$ is derivation, it is clear that$d(c) \in Z$.

Replacing x by $xc = cx$ in the hypothesis, we have

$Z \ni [a, f(cx)] = [a, d(c)x + cf(x)]$,

$d(c)[a, x] + c[a, f(x)] \in Z$.

The second term lies in Z, we get

$d(c)[a, x] \in Z$, for all $x \in U$.

Thus we obtain that $[a, x] \in Z$, for all $x \in U$, and so

$\big[[a, x], r\big] = 0$, for all $x \in U, r \in R$. (20)

Taking x^2 instead of x and using (20), we get

$0 = \big[[a, x]x + x[a, x], r\big] = 2\big[[a, x]x, r\big]$, for all $x \in U, r \in R$ .since clear $R \neq 2$ and $[a, x][x, r] = 0$, for all $x \in U, r \in R$, and so

$[a, x] = 0$or$[x, r] = 0$, for all $x \in U, r \in R$.

Let $A = \{x \in U \backslash [a, x] = 0\}$and $B = \{x \in U \backslash x \in Z\}$. Then A and B are additive sub groups of $(U, +)$ such that $U = A \cup B$. By Brauer's trick, either $U = A$or $U = B$.

Since U is noncommutative, we have $U = A$. Hence$[a, U] = 0$, and so $a \in Z$.

Corollary2: Let R be a prime ring with characteristic different from two, U a non zero right ideal of R which is semiprime as a ring, and f a left generalized derivation of R. If U is non commutative, $d(z) \neq 0$ and $[f(U), f(U)] \subseteq Z$, then there exists $q \in Q_r(R_C)$ such that, $f(x) = qx$, for all $x \in R$.

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