On CR-Structure and F-Structure Satisfying $F^7+F=0$

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ABSTRACT

In this paper, we have studied a relationship between CR-structure and F-structure satisfying $F^7+F=0$. Nijenhuis tensor, integrability conditions and metric F-structure have also been discussed.

Keywords: *Projection operators, distributions, Nijenhuis tensor, integrability conditions and CR-structure.*

1. INTRODUCTION:

Let F be a non zero tensor of type (1,1) and of class C^∞, which is defined on n dimensional manifold M such that

(1.1) $$F^7+F=0$$

Let rank of F is r which is constnat everywhere we define the operators on M as

(1.2) $$l=-F^6, \qquad m=F^6+I$$

where I denote the identity operator.

Theorem (1.1) Let M be an F-structure manifold satisfying (1.1), then

(1.3) (a) $l+m=I$ (b) $l^2=l$

(c) $m^2=m$ (d) $lm=ml=0$

Proof: From (1.1) and (1.2), we get the results.

Thus for the tensor field $F\neq 0$ satisfying (1.1) there exist complementary distribution D_l and D_m corresonding to the projection operators l and m respectively. Then $Dim\ D_l=r$, $\dim\ D_m=n-r$

Theorem (1.2): Let M be an F-structure manifold satisfying (1.1) then

(1.4) (a) $lF=Fl=F, \quad mF=Fm=0$

(b) $F^6l=-l, \qquad F^6m=0$

' **Proof:** from (1.1) and (1.2), we get the results

From (1.4) (b), we observe that F^3 acts as an almost complex structure on D_l and as a null operator on D_m.

Theorem (1.3) Let M be an F-structure manifold satisfying (1.1), define

(1.5) $p=m+F^{3,}\ q=m-F^3,$ then

(1.6) (a) $pq = I,$

(b) $p^2 = q^2 = m - l$

Proof: from (1.2), (1.3) (a), (1.3) (c) and (1.5) we get (1.6) (a) from (1.2), (1.3) (c), (1.4) (a) and (1.5) we get (1.6) (b)

Theorem (1.4): Let *M* be an F-structure manifold satisfying (1.1). Define

(1.7) $c = l + F^3, \quad d = l - F^3$ then

(1.8) $cl^n = l^n c = cl = lc = c, dl^n = l^n d = dl = ld = d$

Proof: From (1.2) and (1.7) we get cl=c, then by (1.3) (b) and (1.4) (a), we get the results.

2. **NIJENHUIS TENSOR:**

Definition (2.1) Let *X* and *Y* be two vector fields on a F-structure manifold *M* satisfying (1.1) then their Lie bracket [*X,Y*] is defined as

(2.1) $X,Y = XY - YX$ and Nijenhuis tensor $N \; X,Y$ of *F* as

(2.2) $N \; X,Y = FX,FY - F \; FX,Y - F \; X,FY + F^2 \; X,Y$

Theorem (2.1)): Let *M* be an integrable F-structure manifold *M* satisfying (1.1), then

(2.3) $-F^5 \quad FX,FY + F^2 \; X,Y = l \quad X,FY + F^2 \; X,Y$

Proof: From (2.2),

$$N \; X,Y = FX,FY - F \; FX,Y - F \; X,FY + F^2 \; X,Y$$

as *M* is integrable $\therefore N \; X,Y = 0$ we have,

(2.4) $FX,FY + F^2 \; X,Y = F \quad FX,Y + X,FY$

Operating by $-F^5$ on both sides of (2.4) and using (1.2)

$$-F^5 \quad FX,FY + F^2 \; X,Y = l \quad FX,Y + X,FY$$

Theorem (2.2) Let *M* be an F-structure manifold satisfying (1.1), then

(2.5) (a) $mN \; X,Y = m \; FX,FY$

(b) $mN \; F^5 X,Y = m\left[F^6 X,FY\right]$

Proof: From (2.2) and (1.4) (a), we get (2.5) (a) Now replacing *X* by F^5X in (2.5) (a), we get (2.5) (b).

Theorem (2.3) Let *M* be an F-structure manifold satisfying (1.1), then the following conditions are all equivalent

(2.6) (a) $m \, N \; X,Y = 0$ (b) $m \; FX,FY = 0$

(c) $m\,N\;\; F^5X,Y \;\; = 0$ (d) $m\left[F^6X,FY\right]=0$

(e) $m\left[F^6lX,FY\right]=0$

Proof: from Theorem (2.2), (1.4) (a), (1.4) (b), we get the results.

Theorem (2.4) Let M be an F-structure manifold satisfying (1.1). Let us define

(2.7) $\underset{l}{N}\;\; X,Y \;\; = lX,lY \;\; -l\;\; lX,Y \;\; -l\;\; X,lY \;\; +l^2\;\; X,Y$

(2.8) $\underset{m}{N}\;\; X,Y \;\; = mX,mY \;\; -m\;\; mX,Y \;\; -m\;\; X,mY \;\; +m^2\;\; X,Y$ then

(2.9) (a) $\underset{l}{N}\;\; mX,mY \;\; =l\;\; mX,mY$

(b) $\underset{m}{N}\;\; lX,lY \;\; =m\;\; lX,lY$ (c) $\underset{l}{N}\;\; lX,mY \;\; =\underset{m}{N}\;\; mX,lY \;\; =0$

Proof: Using (1.3 (b), (c), (d) in (2.7) and (2.8), we get the results.

3. CR-STRUCTURE:

Definition (3.1) Let $T_c\;\; M$ denotes the complexified tangent bundle of the differentiable manifold *M*.

A CR-structure on *M* is a complex sub-bundle *H* of $T_c\;\; M$ such that

(3.1) (a) $H_p \cap \tilde{H}_p = \;\; 0$

(b) H is involutive that is $X,Y \in H \Rightarrow \;\; X,Y \;\; \in H$ for complex vector fields *X* and *Y*.

For the integrable F-structure satisfying (1.1) rank $F \;\; = r = 2m$ on *M*. we define

(3.2) $H_p = \;\; X-\sqrt{-1}FX : X \in X \;\; D_l$

where $X \;\; D_l$ is the $F \;\; D_m$ module of all differentiable sections of D_l.

Theorem (3.1) If *P* and *Q* are two elements of *H*, then

(3.3) $P,Q \;\; = \;\; X,Y \;\; - \;\; FX,FY \;\; -\sqrt{-1}\;\; -1 \;\;\; FX,Y \;\; + \;\; X,FY$

Proof: Defining $P = X-\sqrt{-1}\;\; -1\;\; FX, Q = Y-\sqrt{-1}\;\; -1\;\; FY$ and simplifying, we get (3.3)

Theorem (3.2) for $X,Y, \in X \;\; D_l$

(3.4) $l \;\;\; FX,Y \;\; + \;\; X,FY \;\;\; = \;\; FX,Y \;\; + \;\; X,FY$

Proof: Using (1.4) (a) and (2.1), we get the result as

(3.5) $l\ (FX,Y) + (X,FY) = l\ (FXY - YFX + XFY - FYX)$

$$= FXY - YFX + XFY - FYX$$

$$= (FX,Y) + (X,FY)$$

Theorem (3.3) The integrable F-structure satisfying (1.1) on *M* defines a CR-structure *H* on it such that

(3.6) $R_e\ (H) = D_l$

Proof since $(X,FY),\ (FX,Y) \in X(D_l)$ then from (3.3), (3.4), we get

(3.7) $l\ (P,Q) = (P,Q) \Rightarrow (P,Q) \in X(D_l)$

Thus *F* structure satisfying (1.1), defines a CR-structure on *M*.

Definition (3.2) Let $\tilde{K}$ be the complementary distribution of $R_e\ (H)$ to *TM*. We define a morphism

$F : TM \longrightarrow TM,$ given by

$F\ (X) = 0, \forall\ X \in X(\tilde{K})$ such that

(3.8) $F\ (X) = \frac{1}{2}\sqrt{-1}\ (-1)\ (P - \tilde{P})$

where $P = X + \sqrt{-1}\ (-1)\ Y \in X(H_p)$ and $\tilde{P}$ is complex conjugate of P.

Corollary (3.1): From (3.8) we get

(3.9) $F^2\ X = -X$

Theorem (3.4): If *M* has CR structure then $F^7+F=0$ and consequently *F* structure satisfying (1.), is defined on *M* s.t. D_l and D_m concide with $R_e(H)$ and $\tilde{K}$ respectively

Proof: $F^3\ (X) = -F\ (X),\ F^4\ (X) = X,\ F^5\ (X) = F\ (X),\ F^6\ (X) = -X$

$F^7\ (X) = -F\ (X)\ \therefore F^7 + F = 0$

4. **METRIC F-STRUCTURE:**

We now assume that the Riemannian metric tensor g is s.t.

(4.1) $'F\ (X,Y) = g\ (FX,Y)$ is symmetric. That is

(4.2) $g\ (FX,Y) = g\ (X,FY)$

Theorem (4.1) *g* satisfied the relation

(4.3) $g\ (F^5X, F^2Y) = {}^1F\ (X,Y) + 'm\ (X,Y)$ where

(4.4) $`m \; X,Y \; = g \; mX, FY$

Proof: In (4.2) replacing *X* and *Y* by F^5X and *FY* respectively

(4.5) $g \; F^6X, FY \; = g \; F^5, F^2Y$ using (1.2) in (4.5)

(4.6) $g \;\; m - I \;\; X, FY \; = g \; F^5X, F^2Y$

$$g \; mX - X, FY \; = g \; F^5X, F^2Y$$

$$g \; mX, FY \; - g \; X, FY \; = g \; F^5X, F^2Y$$

(4.7) $g \; F^5X, F^2Y \; = - g \; FY, X \; + g \; mX, FY$

using (4.1) and (4.4) in (4.7)

$$g \; F^5X, F^2Y \; = - `F \; Y, X \; + `m \; X, Y$$

$$g \; F^5X, F^2Y \; = - `F \; X, Y \; + `m \; X, Y$$

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