Numerical Study of the Optimal Control of time-varying Singular Systems via Adomian Decomposition Method

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Abstract — *In this article, the problem of optimal control of time-varying singular systems with quadratic performance index has been studied via Adomian Decomposition Method (ADM). The results obtained via RK-Butcher algorithm (RKB)[10] and ADM are compared with the exact solutions of the time-varying optimal control of linear singular systems. It is observed that the results obtained using ADM is closer to the true solution of the problem. Error graphs for the simulated results and exact solutions are presented in graphical form to highlight the efficiency of the ADM. This ADM can be easily implemented in a digital computer and the solution can be obtained for any length of time.*

Keywords — *Optimal control, Time-varying singular systems, Single-term Haar wavelet series, Adomian Decomposition Method.*

I. Introduction

Optimal control of singular systems arises in many industrial processes in the steel industry, oil industry etc. The problem of optimal control of singular systems has invoked immense interest, especially among the researchers in the field of computational mathematics to study the existing problems in the field of control theory and to compute the value of the control vector numerically which controls the state vector.[3]

Many models that enter into this framework can be found in practice and, in particular, in the existing literature. Among these we can mention: Chen and Hsiao [4], Chen and Shih [5], applied Walsh series to study the problem of optimal control of time-invariant and time-varying linear systems. It is to be noted that from the study of past literature that Cobb [6] and Pandolfi [9] seems to have been the first authors to consider the optimal regulator problem of continuous time singular systems. Both of them used state feed backs and their results were derived by the aid of Ricatti-type matrix equations.

K. Balachandran and K. Murugesan [2] derived optimal control of singular systems via single-term Walsh series. C.F. Chen and C.H. Hsiao [3] Walsh series analysis in optimal control in detail. W.L. Chen and Y.P. Shih [5] analysis and optimum control of time varying linear systems via Walsh functions. H. Maurer [7] gave numerical solution of singular control problems using multiple shooting techniques. H.J. Oberle [8] gave the numerical computation of singular control functions in trajectory optimization problems. H.J. Pesch [11] highlight a practical guide to the solution of real-life optimal control problems. M. Razzaghi and H. Marzban [12] derived optimal control of singular systems via piecewise linear polynomial functions. The above all the systems investigated in the cited articles.

In this paper we developed numerical methods for addressing optimal control of time-varying singular systems by an application of the Adomian Decomposition Method which was studied by Sekar and team of his researchers [13-17]. Recently, Park *et al*. [10] discussed the optimal control of time-varying singular systems using RKB. In this paper, the same optimal control of time-varying singular systems was considered (discussed by Park *et al.* [10]) but present a different approach using the Adomian Decomposition Method with more accuracy for optimal control of time-varying singular systems.

II. Adomian Decomposition Method

Suppose k is a positive integer and $f_1, f_2, \dots, f_k$ are k real continuous functions defined on some domain G. To obtain k differentiable functions $y_1, y_2, \dots, y_k$ defined on the interval I such that $(t, y_1(t), y_2(t), \dots, y_k(t)) \in G$ for $t \in I$.

Let us consider the problems in the following system of ordinary differential equations:

$$\frac{dy_i(t)}{dt} = f_i(t, y_1(t), y_2(t), \dots, y_k(t)) \ ,$$

$$y_i(t)\,|_{t=0} = \beta_i \quad (1)$$

where β_i is a specified constant vector, $y_i(t)$ is the solution vector for $i = 1, 2, \dots, k$. In the decomposition method, (1) is approximated by the operators in the form: $Ly_i(t) = f_i(t, y_1(t), y_2(t), \dots, y_k(t))$ where L is the first order operator defined by $L = d/dt$ and $i = 1, 2, \dots, k$. Assuming the inverse operator of L is L^{-1} which is invertible and denoted by $L^{-1}(.) = \int_{t_0}^{t} (.)\, dt$, then applying L^{-1} to $L\, y_i(t)$ yields

$$L^{-1}Ly_i(t) = L^{-1}f_i(t, y_1(t), y_2(t), \ldots, y_k(t))$$

where $i = 1, 2, \ldots, k$. Thus

$$y_i(t) = y_i(t_0) + L^{-1}f_i(t, y_1(t), y_2(t), \ldots, y_k(t)).$$

Hence the decomposition method consists of representing $y_i(t)$ in the decomposition series form given by

$$y_i(t) = \sum_{n=0}^{\infty} f_{i,n}(t, y_1(t), y_2(t), \ldots, y_k(t))$$

where the components $y_{i,n}$, $n \geq 1$ and $i = 1, 2, \ldots, k$ can be computed readily in a recursive manner. Then the series solution is obtained as

$$y_i(t) = y_{i,0}(t) + \sum_{n=1}^{\infty} \{L^{-1}f_{i,n}(t, y_1(t), y_2(t), \ldots, y_k(t))\}.$$

For a detailed explanation of decomposition method and a general formula of Adomian polynomials, we refer reader to [Adomian 1].

III. Optimal Control of Time-Varying Singular Systems

Considered the linear time-varying singular system

$$K(t)\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$x(0) = x_0$$

(2)

where $K(t)$ is an $n \times n$ singular matrix, and $A(t)$ and $B(t)$ are $n \times n$ and $n \times p$ constant matrices, respectively. The elements (not necessarily all the elements) of the matrices $K(t)$, $A(t)$ and $B(t)$ are time dependent, $x(t)$ is an n-component state vector and $u(t)$ is the p-control input vector.

Assuming that $\det(sK - A) \neq 0,\ B = \begin{bmatrix} 0 \\ I_p \end{bmatrix},\ K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ (3)

Where $K_1 = \begin{bmatrix} I_{n-p} & 0 \end{bmatrix}$

Now the problem can be stated as follows: Given the initial state $x(0-) = x_0$ find a control vector $u(t)$ that generates a state $x(t)$ such that $x(t_f) = x_f$, where t_f is a prescribed time and x_f is a fixed vector, and minimizes the cost functional

$$J = \int_0^{t_f} L(x, u)dt$$

(4)

where $L = \frac{1}{2}(x^TQx + u^TRu)$, Q and R denote given real symmetric constant matrices. In case the initial state $x(0-)$ is not known, the method developed by E1-Tohami *et al.* [51] may be used to reconstruct the state. It has been proved by Lovass-Nagy *et al.* [89], which the problem of finding an optimal control reduces to the solution of a two-point boundary value problem.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

x_1 is $(n-p)\times 1$ and x_2 is $p \times 1$, $K_2 = \begin{bmatrix} K_{21} & K_{22} \end{bmatrix}$, where K_{21} is $p \times (n-p)$ and K_{22} is $p \times p$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},\ A_1 = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix},$$
$$A_2 = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix}$$

where $A_{11}, A_{12}, A_{21}, A_{22}$ are respectively $(n-p)\times(n-p), (n-p)\times p, p\times(n-p), p\times p.$

Further take $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T$ where Q_1 and Q_2 are $(n-p)\times n$ and $p \times n$ respectively. Then we have the following equations (Lovass-Nagy *et al.* [89])

$$\frac{dx_1}{dt} = A_{11}x_1 + A_{12}x_2$$

(5)

$$K_{21}\frac{dx_1}{dt} + K_{22}\frac{dx_2}{dt} - A_{21}x_1 - A_{22}x_2 = u$$

(6)

$$\frac{dp_1}{dt} = -A_{11}^Tp_1 + K_{21}^TR\frac{du}{dt} + A_{21}^TRu - Q_1x$$

(7)

$$A_{12}^Tp_1 = K_{22}^TR\frac{du}{dt} + A_{22}^TRu - Q_2x$$

(8)

where $p = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^T$ is the co-state vector corresponding to equations (2). The optimal state and optimal control can be calculated from equations (5)-(8).

The governing equations for determining $u(t)$ and $x(t)$ for the time-varying optimal control problem can be obtained using the set of equations (5)-(8). It should be noted that these governing equations may not suit all types of time-varying optimal control problem. Hence, it is necessary to investigate further to derive the governing equations exclusively (a generalized form) for the time-varying optimal control problem of the form in equation (2).

IV. Formulation of Time-Varying Optimal Control Problem

Rearranging the equations (5)-(8), we have the following system.

$$\begin{bmatrix} I & 0 & 0 & 0 \\ K_{21} & K_{22} & 0 & 0 \\ 0 & 0 & K_{22}^TR & 0 \\ 0 & 0 & -K_{21}^TR & I \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{u} \\ \dot{p}_1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & I & 0 \\ Q_{12}^T & Q_{22} & -A_{22}R & A_{12}^T \\ -Q_{11} & -Q_{12} & A_{21}^TR & -A_{11}^T \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ u \\ p_1 \end{bmatrix}$$

which can be written in the form $K(t)\dot{y}(t) = M(t)y(t)$

Where

$$K(t)=\begin{bmatrix} I & 0 & 0 & 0 \\ k_{21} & K_{22} & 0 & 0 \\ 0 & 0 & K_{22}^T R & 0 \\ 0 & 0 & -K_{21}^T R & I \end{bmatrix},$$

$$M(t)=\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & I & 0 \\ Q_{12}^T & Q_{22} & -A_{22}R & A_{12}^T \\ -Q_{11} & -Q_{12} & A_{21}^T R & -A_{11}^T \end{bmatrix}$$

and $y=\begin{bmatrix} x_1 & x_2 & u & p_1 \end{bmatrix}^T$

where the matrix *K(t)* is singular and some of the elements are time dependent and so it is called as time-varying "singular systems" or "descriptor systems" or "generalized state space systems" and it can not be written in the standard form.

V. EXAMPLE FOR OPTIMAL CONTROL OF TIME-VARYING LINEAR SINGULAR SYSTEMS

The following time-varying linear singular system is considered

$$\begin{bmatrix} 1 & 0 \\ -t & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 1+t & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

(9)

with initial condition

$$x(0)=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the performance index

$$j=\frac{1}{2}\int_0^{t_f}\left(x^T x+u^2\right)d$$

The objective is to determine an optimal control $u(t)$ that will drive the system from an admissible initial state $x(0-)=x_0$ to some desired final state x_f in a given time t_f and minimizing the above cost functional (performance index).

The exact solution of the system (9) is

$$x_1(t)=\exp(-t)$$

$$x_2(t)=\exp(-t)+\sin(t)$$

(10)

and the optimal control is

$$u(t)=\sin(t)$$

(11)

Time t	Approximate solution $x_1(t)$-values				
	Exact Solutions	RKB Solutions	RKB Error	ADM Solutions	ADM Error
0.00	1.000000	1.000000	0	1.000000	0
0.25	0.778801	0.778801	0.00002	0.778801	2E-07
0.50	0.606531	0.606531	0.00007	0.606531	7E-07
0.75	0.472367	0.472367	0.00009	0.472367	9E-07
1.00	0.367879	0.367879	0.00014	0.367879	0.000014
1.25	0.286504	0.286504	0.00017	0.286504	0.000017
1.50	0.223130	0.223130	0.00019	0.223130	0.000019
1.75	0.173774	0.173774	0.00022	0.173774	0.000022
2.00	0.135335	0.135335	0.00026	0.135335	0.000026

Table 1 Solutions for time-varying system for various values of "$x_1(t)$".

The simulation results and the exact solutions of the state vector $x(t)$ are calculated using ADM and (10) and is presented in Table 1-2 along with the solution obtained using RKB method. The corresponding optimal control $u(t)$ is calculated using ADM and (11) and the results are presented in Table 3.

Time t	Approximate solution $x_2(t)$-values				
	Exact Solutions	RKB Solutions	RKB Error	ADM Solutions	ADM Error
0.00	1.000000	1.000000	0	1.000000	0
0.25	1.026204	1.026204	0.000002	1.026204	2E-08
0.50	1.085956	1.085956	0.000007	1.085956	7E-08
0.75	1.154005	1.154005	0.000009	1.154005	0.000009
1.00	1.209350	1.209350	0.000014	1.209350	0.0000014
1.25	1.235489	1.235489	0.000017	1.235489	0.0000017
1.50	1.220625	1.220625	0.000019	1.220625	0.0000019
1.75	1.157759	1.157759	0.000022	1.157759	0.0000022
2.00	1.044632	1.044632	0.000026	1.044632	0.0000026

Table 2 Solutions for time-varying system for various values of "$x_2(t)$".

Time t	Approximate solution $u(t)$-values				
	Exact Solutions	RKB Solutions	RKB Error	ADM Solutions	ADM Error
0.00	0.000000	0.000000	0	0.000000	0
0.25	0.247404	0.247404	4.00E-06	0.247404	4.00E-08
0.50	0.479426	0.479426	6.00E-06	0.479426	6.00E-08
0.75	0.681639	0.681639	3.00E-05	0.681639	3.00E-07
1.00	0.841471	0.841471	4.00E-05	0.841471	4.00E-07
1.25	0.948984	0.948984	5.00E-05	0.948984	5.00E-07
1.50	0.997495	0.997495	6.00E-05	0.997495	6.00E-07
1.75	0.983986	0.983986	7.00E-05	0.983986	7.00E-07
2.00	0.909297	0.909297	8.00E-05	0.909297	8.00E-07

Table 3 Solutions for time-varying system for various values of "$u(t)$".

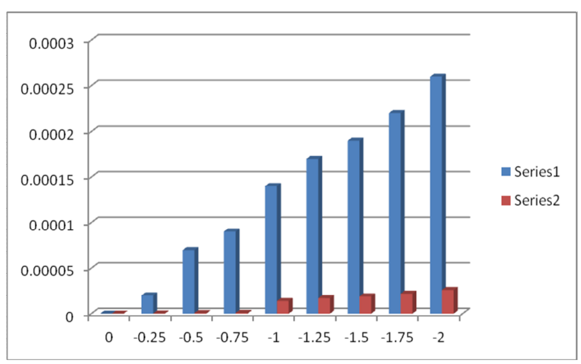

Fig. 1 Error graph for the state $x_1(t)$

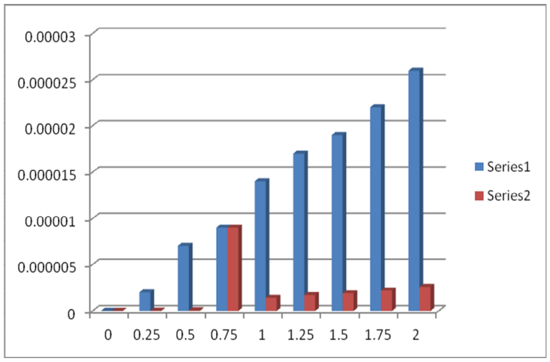

Fig. 2 Error graph for the state $x_2(t)$

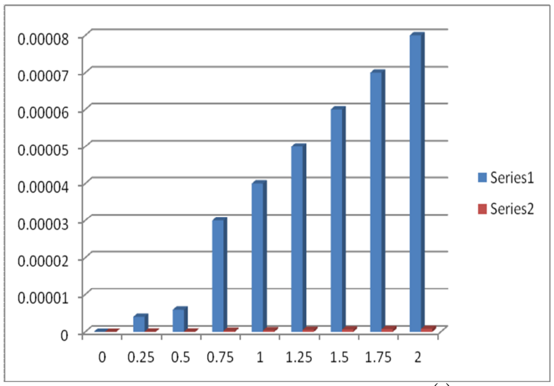

Fig. 3 Error graph for the control input $u(t)$

VI. CONCLUSIONS

The results obtained for the time-varying optimal control of linear singular systems with quadratic performance index show that the ADM works well for finding the state vector $x(t)$ and the control input vector $u(t)$. Table 1-3 and Fig. 1-3 shows that, for most of the time intervals, the absolute error is less (almost zero) with the ADM than with the RKB method, which yields a small error compared with the exact solutions of the problem. Hence the ADM method is more suitable for studying the harmonic oscillators.

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