

# On a Class of Equity Models for the Valuation of the European Call Options

<sup>1\*</sup>Fadugba Sunday Emmanuel, <sup>2</sup>Ajayi Adedoyin Olayinka

<sup>1\*,2</sup>Department of Mathematical Sciences,  
Ekiti State University, Ado Ekiti, Nigeria

## ABSTRACT

*This paper presents binomial and the Leisen-Reimer models for the valuation of the European call option. Binomial model is an iterative solution that models the price evolution over the whole option validity period. This model is a powerful technique that can be used to solve the Black-Scholes partial differential equation and other complex option-pricing models that require solutions of stochastic differential equations. Leisen-Reimer model is a technique that modifies the parameters of the binomial tree to minimize the oscillating behaviour of the value function. We compare the call prices via the two equity models in the context of the Black-Scholes model. The numerical result shows how convergence changes as the number of steps in the binomial calculation increases as we vary the underlying asset price. We also observe that the Leisen-Reimer model removes the oscillation and produces estimates close to the analytic option valuation formula “the Black-Scholes model” using only a small number of steps.*

## 1.0 INTRODUCTION

The Black-Scholes model [1] seems to have dominated option pricing, but it is not the only popular model, the Cox-Ross-Rubinstein (CRR) “Binomial” model is also popular. The binomial models were first suggested by [2] in their paper titled “Option Pricing: A Simplified Approach” in 1979 which assumes that stock price movements are composed of a large number of small binomial movements. Binomial models, which describe the asset price dynamics of the continuous-time model in the limit, serve for approximate valuation of options, especially where formulas cannot be derived analytically due to properties of the considered option type [4]. These models can accommodate complex option pricing problems that require solutions of stochastic differential equations. The binomial

option-pricing model (two-state option-pricing model) is mathematically simple and it is based on the assumption of no arbitrage. The assumption of no arbitrage implies that all risk-free investments earn the risk-free rate of return of investment but yield positive returns. It is the activity of many individuals operating within the context of financial markets that, in fact, upholds these conditions. The activities of arbitrageurs or speculators are often maligned in the media, but their activities insure that our financial market work. They insure that financial assets such as options are priced within a narrow tolerance of their theoretical values [5]. Leisen and Reimer [4] modify the parameters of the binomial tree to minimize the oscillating behaviour of the value function. On the accuracy of binomial model for the valuation of standard options with dividend yield in the context of Black-Scholes model was considered by [7].

In this paper we consider some equity models for the valuation of the European call options. The rest of the paper is organized as follows. Section 2 presents the valuation of the European call options via binomial and the Leisen-Reimer models. In Section 3, we present numerical example and discussion of results. Section 4 concludes the paper.

## 2.0 EQUITY MODELS FOR THE VALUATION OF THE EUROPEAN CALL OPTIONS

### 2.1 BINOMIAL OPTION MODEL

This is defined as an iterative solution that models the price evolution over the whole option validity period. For some types of options such as the American options, using an iterative model is the only choice since there is no known closed form solution that predicts price over time.

Cox-Ross-Rubinstein (CRR) found a better stock movement model other than the geometric Brownian motion model applied by Black-Scholes model.

We divide the life time  $[0, T]$  of the option into  $N$  time subinterval of length  $\Delta t$ , where

$$T = N \delta t \quad (1)$$

Suppose that  $S_0$  is the stock price at the beginning of a given period. Then the binomial model of price movements assumes that at the end of each time period, the price will either go up to  $S_0u$  with probability  $p$  or down to  $S_0d$  with probability  $(1 - p)$  where  $u$  and  $d$  are the up and down factors with  $d < 1 < u$ . We recall that by the principle of risk neutral valuation, the expected return from all the traded options is the risk-free interest rate. We can value future cash flows by discounting their expected values at the risk-free interest rate. The parameters  $u, d$  and  $p$  satisfy the conditions for the risk-neutral valuation and lognormal distribution of the stock price and we have the expected stock at time  $T$  as  $E(S_T)$ . An explicit expression for  $E(S_T)$  is obtained as follows:

Construct a portfolio comprising a long position in  $\Delta$  units of the underlying asset price and a short position when  $N = 1$ . We calculate the value of  $\Delta$  that makes the portfolio riskless. If there is an up movement in the stock price, the value of the portfolio at the end of the life of option is  $S_0u\Delta - f_u$  and if there is a down movement in the stock price, the value of the portfolio at the end of the life of option becomes  $S_0d\Delta - f_d$ . Since the last two expressions are equal, then we have that

$$\begin{aligned} S_0u\Delta - f_u &= S_0d\Delta - f_d \\ S_0u\Delta - S_0d\Delta &= f_u - f_d \\ S_0\Delta(u - d) &= f_u - f_d \\ \Delta &= \frac{f_u - f_d}{S_0(u - d)} \quad (2) \end{aligned}$$

In the above case, the portfolio is riskless and must earn the risk-free interest rate. (2) Shows that  $\Delta$  is the ratio of the change in the option price to the change in the stock price as we move between the nodes. If we denote the risk-free interest rate by  $r$ , the present value of the portfolio is  $(S_0u\Delta - f_u)e^{-r\delta t}$ . The cost of setting up the portfolio is  $S_0\Delta - f$ , it follows that,

$$\begin{aligned} S_0\Delta - f &= (S_0u\Delta - f_u)e^{-r\delta t} \\ f &= S_0\Delta - (S_0u\Delta - f_u)e^{-r\delta t} \quad (3) \end{aligned}$$

Substituting (2) into (3) and simplifying, then (3)

becomes

$$f = e^{-r\delta t} (pf_u - (1 - p)f_d) \quad (4)$$

$$p = \frac{e^{r\delta t} - d}{u - d} \quad (5)$$

For  $N = 1 \Rightarrow \delta t = T$ , then we have a one-step binomial model. Equations (4) and (5) become respectively

$$f = e^{-rT} (pf_u - (1 - p)f_d) \quad (6)$$

$$p = \frac{e^{rT} - d}{u - d} \quad (7)$$

Equations (6) and (7) enable an option to be priced using a one-step binomial model.

Although we do not need to make any assumptions about the probabilities of the up and down movements in order to obtain (4).

The expression  $pf_u - (1 - p)f_d$  is the expected payoff from the option. With this interpretation of  $p$ , (4) then states that the value of the option today is its expected future value discounted at the risk-free rate. For the expected return from the stock when the probability of an up movement is assumed to be  $p$ , the expected stock price,  $E(S_T)$  at time  $T$  is given by

$$E(S_T) = pS_0u + (1 - p)S_0d = pS_0u + S_0d - pS_0d$$

Therefore,

$$E(S_T) = pS_0(u - d) + S_0d \quad (8)$$

Substituting (7) into (8), yields

$$\left. \begin{aligned} E(S_T) &= \left( \frac{e^{rT} - d}{u - d} \right) S_0(u - d) + S_0d \\ &= S_0e^{r\delta t} - S_0d + S_0d = S_0e^{r\delta t} \end{aligned} \right\} \quad (9)$$

Now,

$$pS_0(u - d) + S_0d = S_0e^{r\delta t}$$

$$pu + (1 - p)d = e^{r\delta t}$$

Therefore,

$$e^{r\delta t} = pu + (1 - p)d \quad (10)$$

When constructing a binomial tree to represent the movements in a stock price we choose the parameters  $u$  and  $d$  to match the volatility of a stock price.

The return on the asset price  $S_0$  in a small interval  $\delta t$  of time is

$$\delta S_0 = \mu S_0 \delta t + \sigma S_0 \delta W_t \quad (11)$$

where  $\mu$  = Mean return per unit time,  $\sigma$  = Volatility of the asset price,  $W_t$  = Standard Brownian motion and  $\delta W_t = W_{t+\delta t} - W_t$ . Neglecting powers of  $\delta t$  of order two and above, it follows from (10) that the variance of the return is

$$E\left(\frac{\delta S_0}{S_0}\right)^2 - \left(E\left(\frac{\delta S_0}{S_0}\right)\right)^2 = \sigma^2 \delta t \quad (12)$$

For the one period binomial model, we have that the variance of the return of the asset price in the interval  $\delta t$  as

$$pu^2 + (1-p)d^2 = (pu + (1-p)d)^2$$

To match the stock price volatility with the tree's parameters, we must therefore have that

$$pu^2 + (1-p)d^2 = (pu + (1-p)d)^2 = \sigma^2 \delta t \quad (12)$$

Substituting (7) into (12), we have that

$$e^{2\mu} (u + d) - ud - e^{2\mu} = \sigma^2 \delta t$$

When terms in  $\delta^2 t$  and higher powers of  $\delta t$  are ignored, one solution to this equation is

$$\left. \begin{aligned} u &= e^{\sigma\sqrt{\delta t}} \\ d &= e^{-\sigma\sqrt{\delta t}} \end{aligned} \right\} \quad (13)$$

The probability  $p$  obtained in (7) is called the risk neutral probability. It is the probability of an upward movement of the stock price that ensures that all bets are fair, that is it ensures that there is no arbitrage. Hence(10) follows from the assumption of the risk-neutral valuation.

### 2.1.1 COX-ROSS-RUBINSTEIN MODEL

The Cox-Ross-Rubinstein model contains the Black-Scholes analytical formula as the limiting case as the number of steps tends to infinity [3].

We state the following result for the valuation of European call option using CRR binomial model.

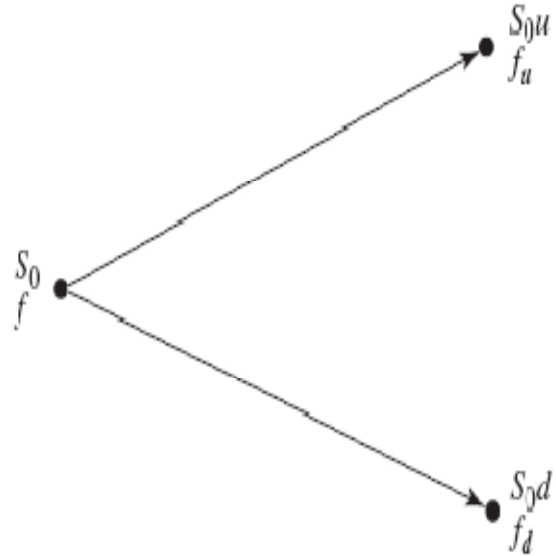
**Theorem 1:** *The probability of a least  $m$  success in  $N$  independent trials, each resulting in a success with probability  $p$  and in a failure with probability  $q$  is given by*

$$\Phi(v, N, p) = \sum_{j=v}^N {}^N C_j p^j (1-p)^{N-j}$$

Let  $p' = R^{-1}pu$  and  $q' = R^{-1}(1-p)d$ , then it follows that

$$f = S_0 \Phi(v, N, p') - Ke^{-rt} \Phi(v, N, p)$$

**Proof:** After one time period, the stock price can move up to  $S_0u$  with probability  $p$  or down to  $S_0d$  with probability  $(1-p)$  as shown in the Fig. 1 below.



**FIGURE I: BINOMIAL TREE FOR THE RESPECTIVE ASSET AND CALL PRICE IN ONE-PERIOD MODEL**

Therefore the corresponding value of the call option at the first time movement  $\delta t$  is given by [6]

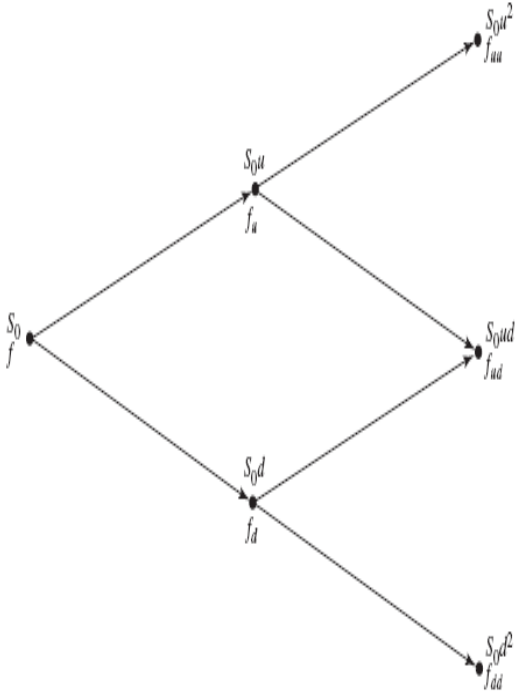
$$\left. \begin{aligned} f_u &= \max(S_0u - K, 0) \\ f_d &= \max(S_0d - K, 0) \end{aligned} \right\} \quad (14)$$

where  $f_u$  and  $f_d$  are the values of the call option after upward and downward movements respectively.

We need to derive a formula to calculate the fair price of the option. The risk neutral call option price at the present time is

$$f = e^{-r\delta t} (pf_u + (1-p)f_d) \quad (15)$$

To extend the binomial model to two periods, let  $f_{uu}$  denote the call value at time  $2\delta t$  for two consecutive upward stock movements,  $f_{ud}$  for one upward and one downward movement and  $f_{dd}$  for two consecutive downward movement of the stock price as shown in the Figure II below.



**FIGURE II: BINOMIAL TREE FOR THE RESPECTIVE ASSET AND CALL PRICE IN A TWO-PERIOD MODEL**

Then we have

$$\left. \begin{aligned} f_{uu} &= \max(S_0uu - K, 0) \\ f_{ud} &= \max(S_0ud - K, 0) \\ f_{dd} &= \max(S_0dd - K, 0) \end{aligned} \right\} \quad (16)$$

The values of the call options at time  $\delta t$  are

$$\left. \begin{aligned} f_u &= e^{-r\delta t} (pf_{uu} + (1-p)f_{ud}) \\ f_d &= e^{-r\delta t} (pf_{ud} + (1-p)f_{dd}) \end{aligned} \right\} \quad (17)$$

Substituting (17) into (15), we have:

$$f = e^{-2r\delta t} (p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}) \quad (18)$$

Equation (18) is called the current call value using time  $2\delta t$ , where the numbers  $p^2$ ,  $2p(1-p)$  and  $(1-p)^2$  are the risk neutral probabilities that the underlying asset prices  $S_{uu}$ ,  $S_{0ud}$  and  $S_{0dd}$  respectively are attained.

We generate the result in (18) to value an option at  $T = N\delta t$  as

$$f = e^{-rN\delta t} \sum_{j=0}^N \binom{N}{j} p^j (1-p)^{N-j} f_{u^j d^{N-j}} \quad (19)$$

where

$$f_{u^j d^{N-j}} = \max(S_0 u^j d^{N-j} - K, 0) \text{ and}$$

$$\binom{N}{j} = \frac{N!}{(N-j)! j!} \text{ is the binomial coefficient. We}$$

assume that  $v$  is the smallest integer for which the option's intrinsic value in (19) is greater than zero. This implies that  $S_0 u^v d^{N-v} \geq K$ . Then (19) can be expressed as

$$\left. \begin{aligned} f &= S_0 e^{-rN\delta t} \sum_{j=v}^N \binom{N}{j} p^j (1-p)^{N-j} u^j d^{N-j} \\ &\quad - K e^{-rN\delta t} \sum_{j=v}^N \binom{N}{j} p^j (1-p)^{N-j} \end{aligned} \right\} \quad (20)$$

Equation (20) gives the present value of the European call option.

The term  $e^{-rN\delta t}$  is the discounting factor that reduces  $f$  to its present value. The first term of

$$\binom{N}{j} p^j (1-p)^{N-j} \text{ is the binomial probability of } j$$

upward movements to occur after the first  $N$  trading periods and  $S_0 u^j d^{N-j}$  is the corresponding value of the asset after  $j$  upward move of the stock price.

The second term is the present value of the option's strike price. Putting  $R = e^{r\delta t}$ , in the first term in (20), we obtain

$$\left. \begin{aligned} f &= S_0 R^{-N} \sum_{j=m}^N \binom{N}{j} p^j (1-p)^{N-j} u^j d^{N-j} \\ &\quad - K e^{-rN\delta t} \sum_{j=m}^N \binom{N}{j} p^j (1-p)^{N-j} \end{aligned} \right\}$$

Therefore,

$$\left. \begin{aligned} f &= S_0 \sum_{j=v}^N \binom{N}{j} (R^{-1}pu)^j (R^{-1}(1-p)d)^{N-j} \\ &\quad - K e^{-rN\delta t} \sum_{j=v}^N \binom{N}{j} p^j (1-p)^{N-j} \end{aligned} \right\} \quad (21)$$

Now, let  $\Phi(v, N, p)$  denotes the binomial distribution function given by

$$\Phi(v, N, p) = \sum_{j=v}^N \binom{N}{j} p^j (1-p)^{N-j} \quad (22)$$

Equation (22) is the probability of at least  $m$  success in  $N$  independent trials, each resulting in a

success with probability  $p$  and in a failure with probability  $(1 - p)$ . Then let  $p' = R^{-1}pu$  and  $(1 - p') = R^{-1}(1 - p)d$ . Consequently, it follows from (21) that

$$f = S_0\Phi(v, N, p') - Ke^{-rt}\Phi(v, N, p) \quad (23)$$

The model in (23) was developed by Cox-Ross-Rubinstein [2] for the valuation of European call option. The corresponding value of the European put option can be obtained using call-put parity of the form  $C_E + Ke^{-rt} = P_E + S_0$ .

## 2.2 THE LEISEN-REIMER MODEL

As the convergence of the binomial tree based value to the limit is not monotone but rather oscillatory, the

$$p(v) = \left(\frac{b}{a}\right)^2 \sqrt[3]{\frac{(9a-1)(9b-1) + 3v\sqrt{a(9b-1)^2 + b(9a-1)^2} - 9abv^2}{(9b-1)^2 - 9bv^2}} \quad (24)$$

with  $a = n - j, b = j + 1$  and  $v$  as input values for the standard normal distribution one uses in the Black-Scholes formula.

**Peizer-Pratt-Inversion formula 1** ( $n = 2j + 1$ )

$$p(v) = \frac{1}{2} + \text{sign}(v) \frac{1}{2} \sqrt{1 - \exp\left[-\left(\frac{v}{n + \frac{1}{3}}\right)^2 \binom{n + \frac{1}{6}}{\frac{1}{3}}\right]} \quad (25)$$

**Peizer-Pratt-Inversion formula 2** ( $n = 2j + 1$ )

$$p(v) = \frac{1}{2} + \text{sign}(v) \frac{1}{2} \sqrt{1 - \exp\left[-\left(\frac{v}{n + \frac{1}{3} + \frac{1}{10(n+1)}}\right)^2 \binom{n + \frac{1}{6}}{\frac{1}{3}}\right]} \quad (26)$$

Then the model parameters are defined by

$$u = e^{(r_d - r_f)\Delta t} \frac{p(d_+)}{p(d_-)} \quad (27)$$

$$d = \frac{e^{(r_d - r_f)\Delta t} - p(d_-)u}{1 - p(d_-)} \quad (28)$$

goal here is to achieve maximum precision with a minimum number of time steps  $N$ . However, one cannot expect that decreasing the step size  $\Delta T = \frac{T}{N}$  will yield a more precise value when using the methods by [4] and [8]. Leisen and Reimer [4] developed a model in which the parameters  $p, u$  and  $d$  of the binomial tree can be altered in order to get a better convergence. Instead of choosing the parameters  $p, u$  and  $d$  to get convergence to the normal distribution, Leisen-Reimer suggest the use of normal approximations to determine the binomial distribution  $B(n, p)$ . In particular, they suggest the following three inversion formulae to replace  $p$  (probability of an up move) by  $p(d_-)$  [9].

**Camp -Paulson-Inversion formula** (for arbitrary  $n$ )

$$d_+ = \frac{\ln\left(\frac{S_0}{K}\right) + (r_d - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad (29)$$

$$d_- = \frac{\ln\left(\frac{S_0}{K}\right) + (r_d - r_f - 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad (30)$$

Using this method, Leisen and Reimer observe much better convergence behaviour.

### 3.0 NUMERICAL EXPERIMENTS

We consider a European call option, with an exercise price of \$30 on March 1, 2014. The option expires on November 1, 2014. Assume that the underlying stock pays no dividends. The stock is trading at \$25 and has a volatility of 0.35 per annum. The annualized continuously compounded risk-free rate is 0.0111 per annum. The convergence of the Cox-Ross-Rubinstein model and the Leisen-Reimer model in the context of the Black-Scholes model for the call options out-of-the-money (OTM) and at-the-money (ATM) are shown in the Figures III and IV respectively. Figures III and IV can be obtained using Matlab codes.

### 3.1 DISCUSSION OF RESULTS

Figures III and IV analyze the effect of the number of periods on the price of the options out-of-the-money

### REFERENCES

- [1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, 81, pp. 637-654, 1973.
- [2] J. Cox, S. Ross and M. Rubinstein, "Option pricing: A simplified approach," *Journal of Financial Economics*, 7, pp. 229-263, 1979.
- [3] J. Hull, Options, "Futures and other derivatives," Pearson Education Inc. Fifth Edition, Prentice Hall, New Jersey, 2003.
- [4] D. Leisen, and M. Reimer, "Binomial models for option valuation - Examining and improving Convergence," *Applied Mathematical Finance*, 3, pp. 319-346, 1996.
- [5] C.R. Nwozo and S. E. Fadugba, "Some numerical methods for options valuation," *Communications in Mathematical*

(OTM) and at-the-money (ATM) respectively. Figures III and IV show how convergence changes as the number of steps in the binomial calculation increases as we vary the underlying asset price. We also observe that the Leisen-Reimer model removes the oscillation and produces estimates close to the analytic option valuation formula "the Black-Scholes model" using only a small number of steps.

### 4.0 CONCLUSION

Binomial model is a powerful technique that can be used to solve the Black-Scholes and other complex option-pricing models that require solutions of stochastic differential equations. Leisen-Reimer model is an alternative technique for the valuation of the European power call options. This model is more accurate, converges smoothly to the Black-Scholes solution and converges faster than its counterpart "Cox-Ross-Rubinstein" binomial model.

*Finance*, 1, pp. 57-74, 2012.

- [6] C.R. Nwozo and S. E. Fadugba, "Monte Carlo method for pricing some path dependent options," *International Journal of Applied Mathematics*, 25, pp. 763-778, 2012.
- [7] C.R. Nwozo and S.E. Fadugba, "On the accuracy of binomial model for the valuation of standard options with dividend yield in the context of Black-Scholes model", *IAENG International Journal of Applied Mathematics*, 44, pp. 33-44, 2014
- [8] Y. Tian, "A Modified Lattice Approach to Option Pricing", *Journal of Future Markets*, 13, pp. 563-577, 1993.
- [9] Wallner C. and Wystup C. "Efficient Computation of Option Price Sensitivities for Options of American Style," Business School of Finance and Management, Centre for Practical Quantitative Finance, pp. 1-31, 2004



LIST OF FIGURES

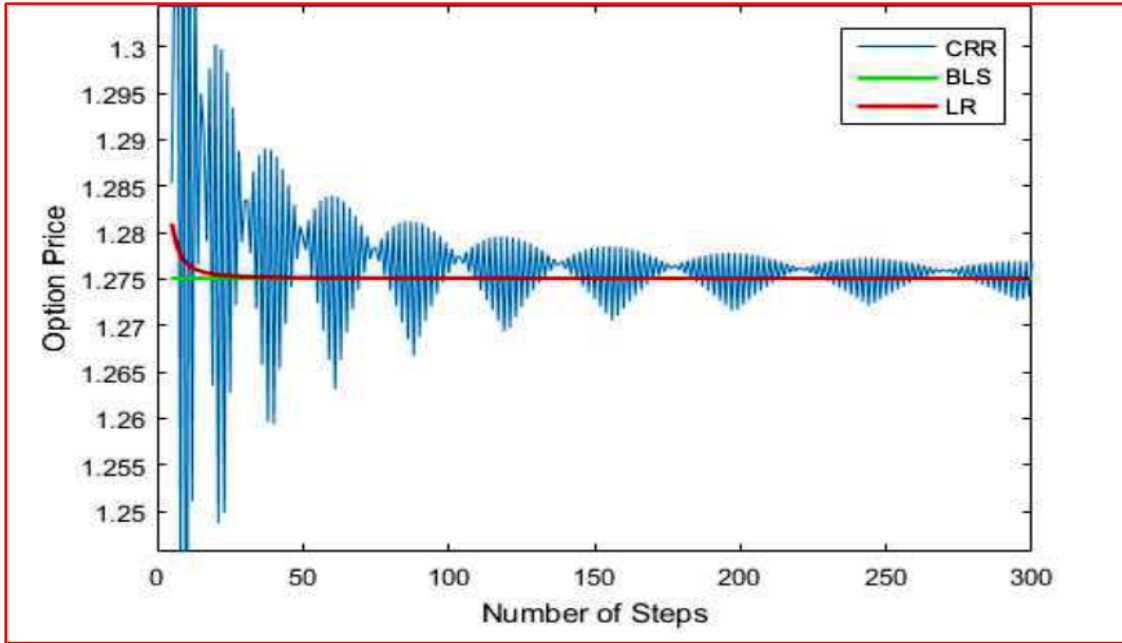


FIGURE III: CONVERGENCE OF COX-ROSS-RUBINSTEIN MODEL AND THE LEISEN-REIMER MODEL TO A BLACK-SCHOLES PRICE FOR THE VALUATION OF THE EUROPEAN CALL OPTION OUT -OF-THE-MONEY (OTM) WITH  $K = \$30$  and  $S_0 = \$25$

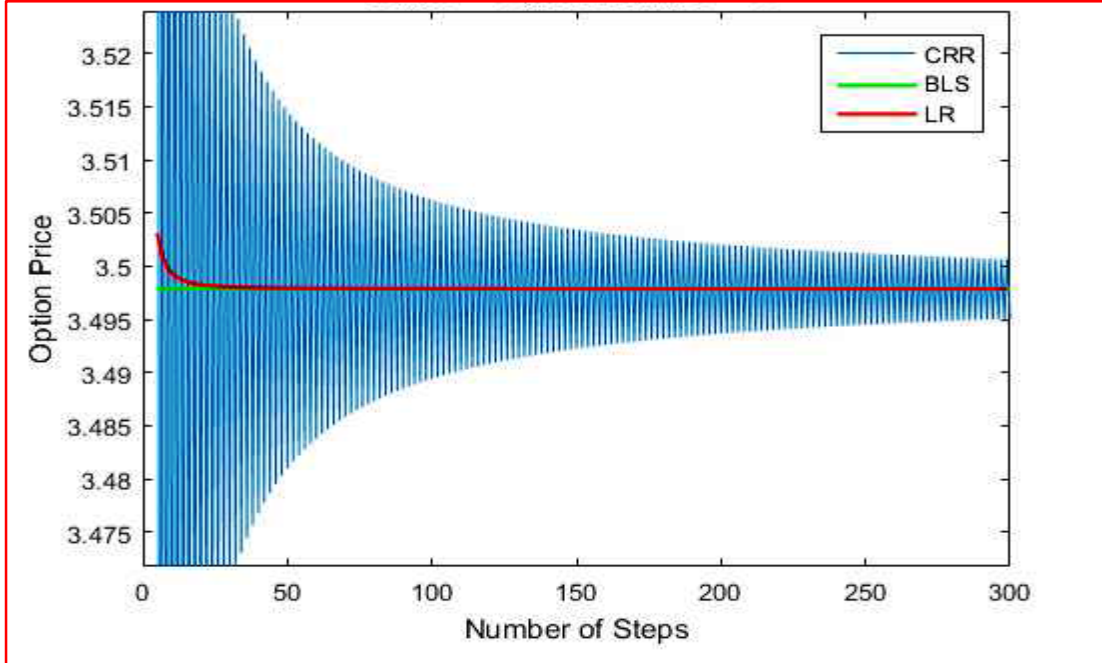


FIGURE IV: CONVERGENCE OF COX-ROSS-RUBINSTEIN MODEL AND THE LEISEN-REIMER MODEL TO A BLACK-SCHOLES PRICE FOR THE VALUATION OF THE EUROPEAN CALL OPTION AT -THE-MONEY (ATM) WITH  $K = \$30$  and  $S_0 = \$30$